(a) $u_t = ku_x$.
(b) $u_t = uu_x$.
(c) $u_t = u\sin(u)u_x$.

1.6 Solve (1.34), and compute the average time for which the broker holds the stock. Analyze the result in light of the financial interpretation of the parameters (m_1, m_2, k).

1.7 (a) Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Write the equation in the coordinates $s = x, t = x - y$.
(b) Find the general solution of the equation.
(c) Consider the equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$. Write it in the coordinates $s = x + y$, $t = 2x$.

2
First-order equations

2.1 Introduction

A first-order PDE for an unknown function $u(x_1, x_2, \ldots, x_n)$ has the following general form:

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}) = 0, \tag{2.1}$$

where F is a given function of $2n + 1$ variables. First-order equations appear in a variety of physical and engineering processes, such as the transport of material in a fluid flow and propagation of wavefronts in optics. Nevertheless they appear less frequently than second-order equations. For simplicity we shall limit the presentation in this chapter to functions in two variables. The reason for this is not just to simplify the algebra. As we shall soon observe, the solution method is based on the geometrical interpretation of u as a surface in an $(n + 1)$-dimensional space. The results will be generalized to equations in any number of variables in Chapter 9.

We thus consider a surface in $\mathbb{R}^3$ whose graph is given by $u(x, y)$. The surface satisfies an equation of the form

$$F(x, y, u, u_x, u_y) = 0. \tag{2.2}$$

Equation (2.2) is still quite general. In many practical situations we deal with equations with a special structure that simplifies the solution process. Therefore we shall progress from very simple equations to more complex ones. There is a common thread to all types of equations – the geometrical approach. The basic idea is that since $u(x, y)$ is a surface in $\mathbb{R}^3$, and since the normal to the surface is given by the vector $(u_x, u_y, -1)$, the PDE (2.2) can be considered as an equation relating the surface to its normal (or alternatively its tangent plane). Indeed the main solution method will be a direct construction of the solution surface.

2.2 Quasilinear equations

We consider first a special class of nonlinear equations where the nonlinearity is confined to the unknown function u. The derivatives of u appear in the equation linearly. Such equations are called *quasilinear*. The general form of a quasilinear equation is

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{2.3}$$

An important special case of quasilinear equations is that of *linear* equations:

$$a(x, y)u_x + b(x, y)u_y = c_0(x, y)u + c_1(x, y), \tag{2.4}$$

where a, b, c_0, c_1 are given functions of (x, y).

Before developing the general theory for quasilinear equations, let us warm up with a simple example.

Example 2.1

$$u_x = c_0 u + c_1. \tag{2.5}$$

In this example we set $a = 1, b = 0$, c_0 is a constant, and $c_1 = c_1(x, y)$. Since (2.5) contains no derivative with respect to the y variable, we can regard this variable as a parameter. Recall from the theory of ODEs that in order to obtain a unique solution we must supply an additional condition. We saw in Chapter 1 that there are many ways to supply additional conditions to a PDE. The natural condition for a first-order PDE is a curve lying on the solution surface. We shall refer to such a condition as an *initial condition*, and the problem will be called an *initial value problem* or a *Cauchy problem* in honor of the French mathematician Augustin Louis Cauchy (1789–1857). For example, we can supplement (2.5) with the initial condition

$$u(0, y) = y. \tag{2.6}$$

Since we are actually dealing with an ODE, the solution is immediate:

$$u(x, y) = e^{c_0 x}\left[\int_0^x e^{-c_0\xi}c_1(\xi, y)d\xi + y\right]. \tag{2.7}$$

A basic approach for solving the general case is to seek special variables in which the equation is simplified (actually, similar to (2.5)). Before doing so, let us draw a few conclusions from this simple example.

(1) Notice that we integrated along the x direction (see Figure 2.1) from each point on the y axis where the initial condition was given, i.e. we actually solved an infinite set of ODEs.

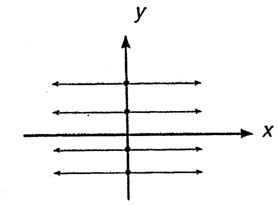

Figure 2.1 Integration of (2.5).

(2) Is there always a solution to (2.5) and an initial condition? At a first sight the answer seems positive; we can write a general solution for (2.5) in the form

$$u(x, y) = e^{c_0 x}\left[\int_0^x e^{-c_0\xi}c_1(\xi, y)d\xi + T(y)\right], \tag{2.8}$$

where the function $T(y)$ is determined by the initial condition. There are examples, however, where such a function does not exist at all! For instance, consider the special case of (2.5) in which $c_1 \equiv 0$. The solution (2.8) now becomes $u(x, y) = e^{c_0 x}T(y)$. Replace the initial condition (2.6) with the condition

$$u(x, 0) = 2x. \tag{2.9}$$

Now $T(y)$ must satisfy $T(0) = 2xe^{-c_0 x}$, which is of course impossible.

(3) We have seen so far an example in which a problem had a unique solution, and an example where there was no solution at all. It turns out that an equation might have infinitely many solutions. To demonstrate this possibility, let us return to the last example, and replace the initial condition (2.6) by

$$u(x, 0) = 2e^{c_0 x}. \tag{2.10}$$

Now $T(y)$ should satisfy $T(0) = 2$. Thus every function $T(y)$ satisfying $T(0) = 2$ will provide a solution for the equation together with the initial condition. Therefore, (2.5) with $c_1 = 0$ has infinitely many solutions under the initial condition (2.10).

We conclude from Example 2.1 that the solution process must include the step of checking for existence and uniqueness. This is an example of the well-posedness issue that was introduced in Chapter 1.

2.3 The method of characteristics

We solve first-order PDEs by the method of *characteristics*. This method was developed in the middle of the nineteenth century by Hamilton. Hamilton investigated the propagation of light. He sought to derive the rules governing this propagation from a purely geometric theory, akin to Euclidean geometry. Hamilton was well aware of the wave theory of light, which was proposed by the Dutch physicist Christian Huygens (1629–1695) and advanced early in the nineteenth century by the English scientist Thomas Young (1773–1829) and the French physicist Augustin Fresnel (1788–1827). Yet, he chose to base his theory on the principle of least time that was proposed in 1657 by the French scientist (and lawyer!) Pierre de Fermat (1601–1665). Fermat proposed a unified principle, according to which light rays travel from a point A to a point B in an orbit that takes the least amount of time. Hamilton showed that this principle can serve as a foundation of a dynamical theory of rays. He thus derived an axiomatic theory that provided equations of motion for light rays. The main building block in the theory is a function that completely characterizes any given optical medium. Hamilton called it the *characteristic function*. He showed that Fermat's principle implies that his characteristic function must satisfy a certain first-order nonlinear PDE. Hamilton's characteristic function and characteristic equation are now called the *eikonal* function and eikonal equation after the Greek word $\epsilon\iota\kappa\omega\nu$ (or $\epsilon\iota\kappa o\nu$) which means "an image".

Hamilton discovered that the eikonal equation can be solved by integrating it along special curves that he called *characteristics*. Furthermore, he showed that in a uniform medium, these curves are exactly the straight light rays whose existence has been assumed since ancient times. In 1911 it was shown by the German physicists Arnold Sommerfeld (1868–1951) and Carl Runge (1856–1927) that the eikonal equation, proposed by Hamilton from his geometric theory, can be derived as a small wavelength limit of the wave equation, as was shown in Chapter 1. Notice that although the eikonal equation is of first order, it is in fact fully nonlinear and not quasilinear. We shall treat it separately later.

We shall first develop the method of characteristics heuristically. Later we shall present a precise theorem that guarantees that, under suitable assumptions, the equation together with its associated condition has a unique solution. The characteristics method is based on 'knitting' the solution surface with a one-parameter family of curves that intersect a given curve in space. Consider the general linear equation (2.4), and write the initial condition parametrically:

$$\Gamma = \Gamma(s) = (x_0(s), y_0(s), u_0(s)), \quad s \in I = (\alpha, \beta). \tag{2.11}$$

The curve Γ will be called the *initial curve*.

The linear equation (2.4) can be rewritten as

$$(a, b, c_0u + c_1) \cdot (u_x, u_y, -1) = 0. \tag{2.12}$$

Since $(u_x, u_y, -1)$ is normal to the surface u, the vector $(a, b, c_0u + c_1)$ is in the

tangent plane. Hence, the system of equations

$$\begin{aligned} \frac{dx}{dt}(t) &= a(x(t), y(t)), \\ \frac{dy}{dt}(t) &= b(x(t), y(t)), \\ \frac{du}{dt}(t) &= c(x(t), y(t))u(t) + c_1(x(t), y(t)) \end{aligned} \tag{2.13}$$

defines spatial curves lying on the solution surface (conditioned so that the curves start on the surface). This is a system of first-order ODEs. They are called the *system of characteristic equations* or, for short, the *characteristic equations*. The solutions are called *characteristic curves* of the equation. Notice that equations (2.13) are autonomous, i.e. there is no explicit dependence upon the parameter t. In order to determine a characteristic curve we need an initial condition. We shall require the initial point to lie on the initial curve Γ. Since each curve $(x(t), y(t), u(t))$ emanates from a different point $\Gamma(s)$, we shall explicitly write the curves in the form $(x(t,s), y(t,s), u(t,s))$. The initial conditions are written as:

$$x(0,s) = x_0(s), \quad y(0,s) = y_0(s), \quad u(0,s) = u_0(s). \tag{2.14}$$

Notice that we selected the parameter t such that the characteristic curve is located on Γ when $t = 0$. One may, of course, select any other parameterization. We also notice that, in general, the parameterization $(x(t,s), y(t,s), u(t,s))$ represents a surface in $\mathbb{R}^3$.

One can readily verify that the method of characteristics applies to the quasilinear equation (2.3) as well. Namely, each point on the initial curve Γ is a starting point for a characteristic curve. The characteristic equations are now

$$\begin{aligned} x_t(t) &= a(x, y, u), \\ y_t(t) &= b(x, y, u), \\ u_t(t) &= c(x, y, u), \end{aligned} \tag{2.15}$$

supplemented by the initial condition

$$x(0,s) = x_0(s), \quad y(0,s) = y_0(s), \quad u(0,s) = u_0(s). \tag{2.16}$$

The problem consisting of (2.3) and initial conditions (2.16) is called the *Cauchy problem* for quasilinear equations.

The main difference between the characteristic equations (2.13) derived for the linear equation, and the set (2.15) is that in the former case the first two equations of (2.13) are independent of the third equation and of the initial conditions. We shall observe later the special role played by the projection of the characteristic curves on the (x, y) plane. Therefore, we write (for the linear case) the equation for this

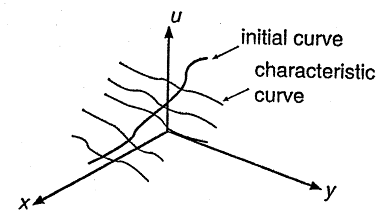

Figure 2.2 Sketch of the method of characteristics.

projection separately:

$$x_t = a(x, y), \quad y_t = b(x, y). \tag{2.17}$$

In the quasilinear case, this uncoupling of the characteristic equations is no longer possible, since the coefficients a and b depend upon u. We also point out that in the linear case, the equation for u is always linear, and thus it is guaranteed to have a global solution (provided that the solutions $x(t)$ and $y(t)$ exist globally).

To summarize the preliminary presentation of the method of characteristics, let us consult Figure 2.2. In the first step we identify the initial curve Γ. In the second step we select a point s on Γ and solve the characteristic equations (2.13) (or (2.15)), using the point we selected on Γ as an initial point. After performing these steps for all points on Γ we obtain a portion of the solution surface (also called the *integral surface*) that consists of the union of the characteristic curves. Philosophically speaking, one might say that the characteristic curves take with them an initial piece of information from Γ, and propagate it with them. Furthermore, each characteristic curve propagates independently of the other characteristic curves.

Let us demonstrate the method for a very simple case.

Example 2.2 Solve the equation

$$u_x + u_y = 2$$

subject to the initial condition $u(x, 0) = x^2$.

The characteristic equations and the parametric initial conditions are

$$x_t(t, s) = 1, \quad y_t(t, s) = 1, \quad u_t(t, s) = 2,$$
$$x(0, s) = s, \quad y(0, s) = 0, \quad u(0, s) = s^2.$$

It is a simple matter to solve for the characteristic curves:

$$x(t, s) = t + f_1(s), \quad y(t, s) = t + f_2(s), \quad u(t, s) = 2t + f_3(s).$$

Upon substituting into the initial conditions, we find

$$x(t, s) = t + s, \quad y(t, s) = t, \quad u(t, s) = 2t + s^2.$$

We have thus obtained a parametric representation of the integral surface. To find an explicit representation of the surface u as a function of x and y we need to invert the transformation $(x(t, s), y(t, s))$, and to express it in the form $(t(x, y), s(x, y))$, namely, we have to solve for (t, s) as functions of (x, y). In the current example the inversion is easy to perform:

$$t = y, \quad s = x - y.$$

Thus the explicit representation of the integral surface is given by

$$u(x, y) = 2y + (x - y)^2.$$

This simple example might lead us to think that each initial value problem for a first-order PDE possesses a unique solution. But we have already seen that this is not the case. What, therefore, are the obstacles we might face? Is (2.3) equipped with initial conditions (2.14) well-posed? For simplicity we shall discuss in this chapter two aspects of well-posedness: existence and uniqueness. Thus the question is whether there exists a unique integral surface for (2.3) that contains the initial curve.

(1) Notice that even if the PDE is linear, the characteristic equations are nonlinear! We know from the theory of ODEs that in general one can only establish local existence of a unique solution (assuming that the coefficients of the equation are smooth functions). In other words, the solutions of nonlinear ODEs might develop singularities within a short distance from the initial point even if the equation is very smooth. It follows that one can expect at most a local existence theorem for a first-order PDE, even if the PDE is linear.
(2) The parametric representation of the integral surface might hide further difficulties. We shall demonstrate this in the sequel by obtaining naive-looking parametric representations of singular surfaces. The difficulty lies in the inversion of the transformation from the plane (t, s) to the plane (x, y). Recall that the implicit function theorem implies that such a transformation is invertible if the Jacobian $J = \partial(x, y)/\partial(t, s) \neq 0$. But we observe that while the dependence of the characteristic curves on the variable t is derived from the PDE itself, the dependence on the variable s is derived from the initial condition. Since the equation and the initial condition do not depend upon each other, it follows that for any given equation there exist initial curves for which the Jacobian vanishes, and the implicit function theorem does not hold.

The functional problem we just described has an important geometrical interpretation. An explicit computation of the Jacobian at points located on the initial curve Γ, using the characteristic equations, gives

$$J = \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} = \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} = (y_0)_s a - (x_0)_s b, \tag{2.18}$$

where $(x_0)_s = \mathrm{d}x_0/\mathrm{d}s$. Thus the Jacobian vanishes at some point if and only if the vectors (a, b) and $((x_0)_s, (y_0)_s)$ are linearly dependent. Hence the geometrical meaning of a vanishing Jacobian is that the projection of Γ on the (x, y) plane is tangent at this point to the projection of the characteristic curve on that plane. As a rule, in order for a first-order quasilinear PDE to have a unique solution near the initial curve, we must have $J \neq 0$. This condition is called the *transversality condition*.

(3) So far we have discussed local problems. One can also encounter global problems. For example, a characteristic curve might intersect the initial curve more than once. Since the characteristic equation is well-posed for a single initial condition, then in such a situation the solution will, in general, develop a singularity. We can think about this situation in the following way. Recall that a characteristic curve 'carries' with it along its orbit a charge of information from its intersection point with Γ. If a characteristic curve intersects Γ more than once, these two 'information charges' might be in conflict.
A similar global problem is the intersection of the projection on the (x, y) plane of different characteristic curves with each other. Such an intersection is problematic for the same reason as the intersection of a characteristic curve with the initial curve. Each characteristic curve carries with it a different information charge, and a conflict might arise at such an intersection.

(4) Another potential problem relates to a lack of uniqueness of the solution to the characteristic equation. We should not worry about this possibility if the coefficients of the equations are smooth (Lipschitz continuous, to be precise). But when considering a nonsmooth problem, we should pay attention to this issue. We shall demonstrate such a case below.

In Section 2.5 we shall formulate and prove a precise theorem (Theorem 2.10) that will include all the problems discussed above. Before doing so, let us examine a few examples.

2.4 Examples of the characteristics method

Example 2.3 Solve the equation $u_x = 1$ subject to the initial condition $u(0, y) = g(y)$.

The characteristic equations and the associated initial conditions are given by

$$x_t = 1, \quad y_t = 0, \quad u_t = 1, \tag{2.19}$$

$$x(0, s) = 0, \quad y(0, s) = s, \quad u(0, s) = g(s), \tag{2.20}$$

respectively. The parametric integral surface is $(x(t, s), y(t, s), u(t, s)) = (t, s, t + g(s))$. It is easy to deduce from here the explicit solution $u(x, y) = x + g(y)$.

On the other hand, if we keep the equation unchanged, but modify the initial conditions into $u(x, 0) = h(x)$, the picture changes dramatically. In this case the parametric solution is

$$(x(t, s), y(t, s), u(t, s)) = (t + s, 0, t + h(s)).$$

Now, however, the transformation $(x(t, s), y(t, s))$ cannot be inverted. Geometrically speaking, the reason is simple: the projection of the initial curve is precisely the x axis, but this is also the projection of a characteristic curve. In the special case where $h(x) = x + c$ for some constant c, we obtain $u(t, s) = s + t + c$. Then it is not necessary to invert the mapping $(x(t, s), y(t, s))$, since we find at once $u = x + c + f(y)$ for every differentiable function $f(y)$ that vanishes at the origin. But for any other choice of h the problem has no solution at all.

We note that for the initial conditions $u(x, 0) = h(x)$ we could have foreseen the problem through a direct computation of the Jacobian:

$$J = \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0. \tag{2.21}$$

Whenever the Jacobian vanishes along an interval (like in the example we are considering), the problem will, in general, have no solution at all. If a solution does exist, we shall see that this implies the existence of infinitely many solutions.

Because of the special role played by the projection of the characteristic curves on the (x, y) plane we shall use the term *characteristics* to denote them for short. There are several ways to compute the characteristics. One of them is to solve the full characteristic equations, and then to project the solution on the (x, y) plane. We note that the projection of a characteristic curve is given by the condition $s =$ constant. Substituting this condition into the equation $s = s(x, y)$ determines an explicit equation for the characteristics. An alternative method is valid whenever the PDE is linear. The linearity implies that the first two characteristic equations are independent of u. Thus they can be solved directly for the characteristics themselves without solving first for the parametric integral surface. Furthermore, since the characteristic equations are autonomous (i.e. they do not explicitly include the variable t), it follows that the equations for the characteristics can be written simply as the first-order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x, y)}{a(x, y)}.$$

Example 2.4 The current example will be useful for us in Chapter 3, where we shall need to solve linear equations of the form

$$a(x, y)u_x + b(x, y)u_y = 0. \tag{2.22}$$

The equations for the characteristic curves

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{du}{dt} = 0$$

imply at once that the solution u is constant on the characteristics that are determined by

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \tag{2.23}$$

For instance, when $a = 1, b = \sqrt{-x}$ (see Example 3.7) we obtain that u is constant along the lines $\frac{3}{2}y + (-x)^{3/2} = \text{constant}$.

Example 2.5 Solve the equation $u_x + u_y + u = 1$, subject to the initial condition

$$u = \sin x, \quad \text{on} \quad y = x + x^2, \quad x > 0.$$

The characteristic equations and the associated initial conditions are given by

$$x_t = 1, \quad y_t = 1, \quad u_t + u = 1, \tag{2.24}$$

$$x(0, s) = s, \quad y(0, s) = s + s^2, \quad u(0, s) = \sin s, \tag{2.25}$$

respectively. Let us compute first the Jacobian along the initial curve:

$$J = \begin{vmatrix} 1 & 1 \\ 1 & 1 + 2s \end{vmatrix} = 2s. \tag{2.26}$$

Thus we anticipate a unique solution at each point where $s \neq 0$. Since we are limited to the regime $x > 0$ we indeed expect a unique solution.

The parametric integral surface is given by

$$(x(t, s), y(t, s), u(t, s)) = (s + t, s + s^2 + t, 1 - (1 - \sin s)e^{-t}).$$

In order to invert the mapping $(x(t, s), y(t, s))$, we substitute the equation for x into the equation for y to obtain $s = (y - x)^{1/2}$. The sign of the square root was selected according to the condition $x > 0$. Now it is easy to find $t = x - (y - x)^{\frac{1}{2}}$, whence the explicit representation of the integral surface

$$u(x, y) = 1 - [1 - \sin(y - x)^{\frac{1}{2}}]e^{-x+(y-x)^{\frac{1}{2}}}.$$

Notice that the solution exists only in the domain

$$D = \{(x, y) \mid 0 < x < y\} \cup \{(x, y) \mid x \leq 0 \text{ and } x + x^2 < y\},$$

and in particular it is not differentiable at the origin of the (x, y) plane. To see the geometrical reason for this, consult Figure 2.3. We see that the slope of characteristic passing through the origin equals 1, which is exactly the slope of the projection of

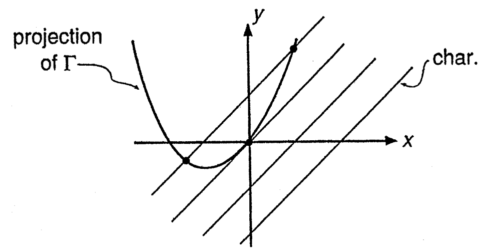

Figure 2.3 The characteristics and projection of Γ for Example 2.5.

the initial curve there. Namely, the transversality condition does not hold there (a fact we already expected from our computation of the Jacobian above). Indeed the violation of the transversality condition led to nonuniqueness of the solution near the curve

$$\{(x, y) \mid x < 0 \quad \text{and} \quad y = x + x^2\},$$

which is manifested in the ambiguity of the sign of the square root.

Example 2.6 Solve the equation $-yu_x + xu_y = u$ subject to the initial condition $u(x, 0) = \psi(x)$.

The characteristic equations and the associated initial conditions are given by

$$x_t = -y, \quad y_t = x, \quad u_t = u, \tag{2.27}$$

$$x(0, s) = s, \quad y(0, s) = 0, \quad u(0, s) = \psi(s). \tag{2.28}$$

Let us examine the transversality condition:

$$J = \begin{vmatrix} 0 & s \\ 1 & 0 \end{vmatrix} = -s. \tag{2.29}$$

Thus we expect a unique solution (at least locally) near each point on the initial curve, except, perhaps, the point $x = 0$.

The solution of the characteristic equations is given by

$$(x(t, s), y(t, s), u(t, s)) = (f_1(s)\cos t + f_2(s)\sin t,\ f_1(s)\sin t - f_2(s)\cos t,\ e^t f_3(s)).$$

Substituting the initial condition into the solution above leads to the parametric integral surface

$$(x(t, s), y(t, s), u(t, s)) = (s\cos t, s\sin t, e^t \psi(s)).$$

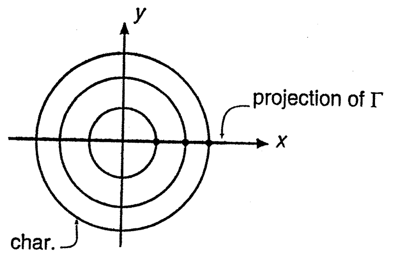

Figure 2.4 The characteristics and projection of Γ for Example 2.6.

Isolating s and t we obtain the explicit representation

$$u(x,y)=\psi(\sqrt{x^2+y^2})\exp\left[\arctan\left(\frac{y}{x}\right)\right].$$

It can be readily verified that the characteristics form a one-parameter family of circles around the origin (see Figure 2.4). Therefore, each one of them intersects the projection of the initial curve (the x axis) twice. We also saw that the Jacobian vanishes at the origin. So how is it that we seem to have obtained a unique solution? The mystery is easily resolved by observing that in choosing the positive sign for the square root in the argument of ψ, we effectively reduced the solution to the ray $\{x > 0\}$. Indeed, in this region a characteristic intersects the projection of the initial curve only once.

Example 2.7 Solve the equation $u_x + 3y^{2/3}u_y = 2$ subject to the initial condition $u(x,1) = 1 + x$.

The characteristic equations and the associated initial conditions are given by

$$x_t = 1,\quad y_t = 3y^{2/3},\quad u_t = 2, \tag{2.30}$$

$$x(0,s) = s,\quad y(0,s) = 1,\quad u(0,s) = 1 + s. \tag{2.31}$$

In this example we expect a unique solution in a neighborhood of the initial curve since the transversality condition holds:

$$J = \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0. \tag{2.32}$$

The parametric integral surface is given by

$$x(t,s) = s + t,\quad y(t,s) = (t+1)^3,\quad u(t,s) = 2t + 1 + s.$$

Before proceeding to compute an explicit solution, let us find the characteristics. For this purpose recall that each characteristic curve passes through a specific s value. Therefore, we isolate t from the equation for x, and substitute it into the expression

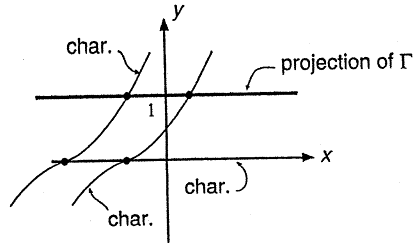

Figure 2.5 Self-intersection of characteristics.

for y. We obtain $y = (x + 1 - s)^3$, and, thus, for each fixed s this is an equation for a characteristic. A number of characteristics and their intersection with the projection of the initial curve $y = 1$ are sketched in Figure 2.5. While the picture indicates no problems, we were not careful enough in solving the characteristic equations, since the function $y^{2/3}$ is not Lipschitz continuous at the origin. Thus the characteristic equations might not have a unique solution there! In fact, it can be easily verified that $y = 0$ is also a solution of $y_t = 3y^{2/3}$. But, as can be seen from Figure 2.5, the well behaved characteristics near the projection of the initial curve $y = 1$ intersect at some point the extra characteristic $y = 0$. Thus we can anticipate irregular behavior near $y = 0$. Inverting the mapping $(x(t,s), y(t,s))$ we obtain

$$t = y^{1/3} - 1,\ s = x + 1 - y^{1/3}.$$

Hence the explicit solution to the PDE is $u(x,y) = x + y^{1/3}$, which is indeed singular on the x axis.

Example 2.8 Solve the equation $(y + u)u_x + yu_y = x - y$ subject to the initial conditions $u(x,1) = 1 + x$.

This is an example of a quasilinear equation. The characteristic equations and the initial data are:

$$\text{(i)}\ x_t = y + u,\quad \text{(ii)}\ y_t = y,\quad \text{(iii)}\ u_t = x - y,$$
$$x(0,s) = s,\quad y(0,s) = 1,\quad u(0,s) = 1 + s.$$

Let us examine the transversality condition. Notice that while u is yet to be found, the transversality condition only involves the values of u *on* the initial curve Γ. It is easy to verify that on Γ we have $a = 2 + s,\ \ b = 1$. It follows that the tangent to the characteristic has a nonzero component in the direction of the y axis. Thus it is nowhere tangent to the projection of the initial curve (the x axis, in this case).

Alternatively, we can compute the Jacobian directly:

$$J = \begin{vmatrix} 2+s & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0. \tag{2.33}$$

We conclude that there exists an integral surface at least at the vicinity of Γ. From the characteristic equation (ii) and the associated initial condition we find $y(t,s) = e^t$. Adding the characteristic equations (i) and (iii) we get $(x+u)_t = x+u$. Therefore, $u + x = (1+2s)e^t$. Returning to (i) we obtain $x(t,s) = (1+s)e^t - e^{-t}$ and $u(t,s) = se^t + e^{-t}$. Observing that $x - y = se^t - e^{-t}$, we finally get $u = 2/y + (x-y)$. The solution is *not* global (it becomes singular on the x axis), but it is well defined near the initial curve.

2.5 The existence and uniqueness theorem

We shall summarize the discussion on linear and quasilinear equations into a general theorem. For this purpose we need the following definition.

Definition 2.9 Consider a quasilinear equation (2.3) with initial conditions (2.16) defining an initial curve for the integral surface. We say that the equation and the initial curve satisfy the *transversality condition* at a point s on Γ, if the characteristic emanating from the projection of $\Gamma(s)$ intersects the projection of Γ nontangentially, i.e.

$$J\,|_{t=0} = x_t(0,s)y_s(0,s) - y_t(0,s)x_s(0,s) = \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} \neq 0.$$

Theorem 2.10 *Assume that the coefficients of the quasilinear equation (2.3) are smooth functions of their variables in a neighborhood of the initial curve (2.16). Assume further that the transversality condition holds at each point s in the interval $(s_0 - 2\delta, s_0 + 2\delta)$ on the initial curve. Then the Cauchy problem (2.3), (2.16) has a unique solution in the neighborhood $(t,s) \in (-\epsilon,\epsilon) \times (s_0 - \delta, s_0 + \delta)$ of the initial curve. If the transversality condition does not hold for an interval of s values, then the Cauchy problem (2.3), (2.16) has either no solution at all, or it has infinitely many solutions.*

Proof The existence and uniqueness theorem for ODEs, applied to (2.15) together with the initial data (2.16), guarantees the existence of a unique characteristic curve for each point on the initial curve. The family of characteristic curves forms a parametric representation of a surface. The transversality condition implies that the parametric representation provides a smooth surface. Let us verify now that the

surface thus constructed indeed satisfies the PDE (2.3). We write

$$\tilde{u} = \tilde{u}(x,y) = u(t(x,y), s(x,y)),$$

and compute

$$a\tilde{u}_x + b\tilde{u}_y = a(u_t t_x + u_s s_x) + b(u_t t_y + u_s s_y) = u_t(at_x + bt_y) + u_s(as_x + bs_y).$$

But the characteristic equations and the chain rule imply

$$1 = t_t = at_x + bt_y, \quad 0 = s_t = as_x + bs_y.$$

Hence $a\tilde{u}_x + b\tilde{u}_y = u_t = c$, i.e. $\tilde{u}$ satisfies (2.3).

To show that there are no further integral surfaces, we prove that the characteristic curves we constructed must lie on an integral surface. Since the characteristic curve starts on the integral surface, we only have to show that it remains there. This is intuitively clear, since the characteristic curve is, by definition, orthogonal at every point to the surface normal. On the other hand, clearly for a curve starting on some surface to leave the surface, its tangent must at some point have a nonzero projection on the normal to the surface. This simple geometrical reasoning can be supported through an explicit computation; for this purpose we write a given integral surface in the form $u = f(x,y)$. Let $(x(t), y(t), u(t))$ be a characteristic curve. We assume $u(0) = f(x(0), y(0))$. Define the function

$$\Psi(t) = u(t) - f(x(t), y(t)).$$

Differentiating by t we write

$$\Psi_t = u_t - f_x(x,y)x_t - f_y(x,y)y_t.$$

Substituting the system (2.15) into the above equations for Ψ_t we obtain

$$\Psi_t = c(x,y,\Psi + f) - f_x(x,y)a(x,y,\Psi + f) - f_y(x,y)b(x,y,\Psi + f). \tag{2.34}$$

But the initial condition implies $\Psi(0) = 0$. It is easy to check (using (2.3)) that $\Psi(t) \equiv 0$ solves the ODE (2.34). Since that equation has smooth coefficients, it has a unique solution. Thus $\Psi \equiv 0$ is the only solution, and the curve $(x(t), y(t), u(t))$ indeed lies on the integral surface. Therefore the integral surface we constructed earlier through the parametric representation induced by the characteristic equations is unique.

When the transversality condition does not hold along an interval of s values, the characteristic there is the same as the projection of Γ. If the solution of the characteristic equation is a curve that is not identical to the initial curve, then the tangent (vector) to the initial curve at some point cannot be at that point tangential to any integral surface. In other words, the initial condition contradicts the equation and thus there can be no solution to the Cauchy problem. If, on the other hand, the characteristic curve agrees with the initial curve at that point, there are infinitely many ways to extend it into a compatible integral surface that contains it. Therefore, in this case we have infinitely many solutions to the Cauchy problem. We now present a method for constructing this family of solutions. Select an arbitrary point $P_0 = (x_0, y_0, u_0)$ on Γ. Construct a new initial curve Γ', passing through P_0, which is not tangent to Γ at P_0. Solve the new Cauchy problem consisting of (2.3) with Γ' as initial curve. Since, by construction, the transversality condition holds now, the first part of the theorem guarantees a unique solution. Since there are infinitely many ways of selecting such an initial curve Γ', we obtain infinitely many solutions. $\square$

The following simple example demonstrates the case where the transversality condition fails along some interval.

Example 2.11 Consider the Cauchy problem

$$u_x + u_y = 1, \quad u(x, x) = x.$$

Show that it has infinitely many solutions.

The transversality condition is violated identically. However the characteristic direction is $(1, 1, 1)$, and so is the direction of the initial curve. Hence, the initial curve is itself a characteristic curve. Thus there exist infinitely many solutions. To find these solutions, set the problem

$$u_x + u_y = 1, \quad u(x, 0) = f(x),$$

for an arbitrary f satisfying $f(0) = 0$. The solution is easily found to be $u(x, y) = y + f(x - y)$.

Notice that the Cauchy problem

$$u_x + u_y = 1, \quad u(x, x) = 1,$$

on the other hand, is not solvable. To see this observe that the transversality condition fails again, but now the initial curve is *not* a characteristic curve. Thus there is no solution.

Remark 2.12 There is one additional possibility not covered by Theorem 2.10. This is the case where the transversality condition does not hold on isolated points (as was indeed the case in some of the preceding examples). It is difficult to formulate universal statements here. Instead, each such case has to be analyzed separately.

2.6 The Lagrange method

First-order quasilinear equations were in fact studied by Lagrange even before Hamilton. Lagrange developed a solution method that is also geometric in nature, albeit less general than Hamilton's method. The main advantage of Lagrange's method is that it provides general solutions for the equation, regardless of the initial data.

Let us reconsider (2.15). The set of all solutions to this system forms a two-parameter set of curves. To justify this assertion, notice that since the system (2.15) is autonomous, it is equivalent to the system

$$y_x = b(x, y, u)/a(x, y, u), \quad u_x = c(x, y, u)/a(x, y, u). \tag{2.35}$$

Since (2.35) is a system of two first-order ODEs in the (y, u) plane, where x is a parameter, it follows that the set of solutions is determined by two initial conditions.

Lagrange assumed that the two-parameter set of solution curves for (2.15) can be represented by the intersection of two families of integral surfaces

$$\psi(x, y, u) = \alpha, \quad \phi(x, y, u) = \beta. \tag{2.36}$$

When we vary the parameters α and β we obtain (through intersecting the surfaces ψ and ϕ) the two-parameter set of curves that are generated by the intersection. Recall that a solution surface of (2.3) passing through an initial curve is obtained from a one-parameter family of curves solving the characteristic equation (2.15). Each such one-parameter subfamily describes a curve in the parameter space (α, β). Since such a curve can be expressed in the form $F(\alpha, \beta) = 0$, it follows that every solution of (2.3) and (2.16) is given by

$$F(\psi(x, y, u), \phi(x, y, u)) = 0. \tag{2.37}$$

Since the surfaces ψ and ϕ were determined by the equation itself, (2.37) defines a general solution to the PDE. When we solve for a particular initial curve, we just have to use this curve to determine the specific functional form of F associated with that initial curve.

We are still left with one "little" problem: how to find the surfaces ψ and ϕ. In the theory of ODEs one solves first-order equations by the method of integration factors. While this method is always feasible in theory, it involves great technical difficulties. In fact, it is possible to find integration factors only in special cases. In a sense, the Lagrange method is a generalization of the integration factor method for ODEs, as we have to find solution surfaces for the two first-order ODEs (2.35). Hence it is not surprising that the method is applicable only in special cases.

We proceed by introducing a method for computing the surfaces ψ and ϕ, and then apply the method to a specific example.

Example 2.13 Recall that by definition, the surfaces $\psi = \alpha,\ \phi = \beta$ contain the characteristic curves. Assume there exist two independent vector fields $\vec{P}_1 = (a_1, b_1, c_1)$ and $\vec{P}_2 = (a_2, b_2, c_2)$ (i.e. they are nowhere tangent to each other) that are both orthogonal to the vector $\vec{P} = (a, b, c)$ (the vector defining the characteristic equations). This means $aa_1 + bb_1 + cc_1 = 0 = aa_2 + bb_2 + cc_2$. Let us assume further that the vector fields $\vec{P}_1$ and $\vec{P}_2$ are exact, i.e. $\nabla \times \vec{P}_1 = 0 = \nabla \times \vec{P}_2$. This implies that there exist two potentials ψ and ϕ satisfying $\nabla\psi = \vec{P}_1$ and $\nabla\phi = \vec{P}_2$. By construction it follows that $\mathrm{d}\psi = \nabla\psi \cdot \vec{P} = 0 = \nabla\phi \cdot \vec{P} = \mathrm{d}\phi$, namely, ψ and ϕ are constant on every characteristic curve, and form the requested two-parameter integral surfaces.

Let us apply this method to find the general solution to the equation

$$-yu_x + xu_y = 0. \tag{2.38}$$

The characteristic equations are

$$x_t = -y, \quad y_t = x, \quad u_t = 0.$$

In this example $\vec{P} = (-y, x, 0)$. It is easy to guess orthogonal vector fields $\vec{P}_1 = (x, y, 0)$ and $\vec{P}_2 = (0, 0, 1)$. The reader can verify that they are indeed exact vector fields. The associated potentials are

$$\psi(x, y, u) = \frac{1}{2}(x^2 + y^2), \quad \phi(x, y, u) = (0, 0, u).$$

Therefore, the general solution of (2.38) is given by

$$F(x^2 + y^2, u) = 0, \tag{2.39}$$

or

$$u = g(x^2 + y^2). \tag{2.40}$$

To find the specific solution of (2.38) that satisfies a given initial condition, we shall use that condition to eliminate g. For example, let us compute the solution of (2.38) satisfying $u(x, 0) = \sin x$ for $x > 0$. Substituting the initial condition into (2.40) yields $g(\xi) = \sin\sqrt{\xi}$; hence $u(x, y) = \sin\sqrt{x^2 + y^2}$ is the required solution.

While the Lagrange method has an advantage over the characteristics method, since it provides a *general* solution to the equation, valid for all initial conditions, it also has a number of disadvantages.

(1) We have already explained that ψ and ϕ can only be found under special circumstances. Many tricks for this purpose have been developed since Lagrange's days, yet, only a limited number of equations can be solved in this way.
(2) It is difficult to deduce from the Lagrange method any potential problems arising from the interaction between the equation and the initial data.
(3) The Lagrange method is limited to quasilinear equations. Its generalization to arbitrary nonlinearities is very difficult. On the other hand, as we shall soon see, the characteristics method can be naturally extended to a method that is applicable to general nonlinear PDEs.

It would be fair to say that the main value of the Lagrange method is historical, and in supplying general solutions to certain canonical equations (such as in the example above).

2.7 Conservation laws and shock waves

The existence theorem for quasilinear equations only guarantees (under suitable conditions) the existence of a local solution. Nevertheless, there are cases of interest where we need to compute the solution of a physical problem beyond the point where the solution breaks down. In this section we shall discuss such a situation. For simplicity we shall perform the analysis in some detail for a canonical prototype of quasilinear equations given by

$$u_y + uu_x = 0. \tag{2.41}$$

This equation plays an important role in hydrodynamics. It models the flow of mass with concentration u, where the speed of the flow depends on the concentration. The variable y has the physical interpretation of time. We shall show that the solutions to this equation often develop a special singularity that is called a *shock wave*. In hydrodynamics the equation is called the *Euler equation* (cf. Chapter 1; the reader may be baffled by now by the multitude of differential equations that are called after Euler. We have to bear in mind that Euler was a highly prolific mathematician who published over 800 papers and books). Towards the end of the section we shall generalize the analysis that is performed for (2.41) to a larger family of equations, and in particular, we shall apply the theory to study traffic flow.

As a warm-up we start with the simple linear equation

$$u_y + cu_x = 0. \tag{2.42}$$

The difference between this equation and (2.41), is that in (2.42) the flow speed is given by the positive constant c. The initial condition

$$u(x, 0) = h(x) \tag{2.43}$$

will be used for both equations. Solving the characteristic equations for the linear equation (2.42) we get

$$(x, y, u) = (s + ct, t, h(s)).$$

Eliminating s and t yields the explicit solution $u = h(x - cy)$. The solution implies that the initial profile does not change; it merely moves with speed c along the positive x axis, namely, we have a fixed wave, moving with a speed c while preserving the initial shape.

Euler's equation (2.41) is solved similarly. The characteristic equations are

$$x_t = u, \quad y_t = 1, \quad u_t = 0,$$

and their solution is

$$(x, y, u) = (s + h(s)t, t, h(s)),$$

where we used the parameterization $x(0, s) = s,\ y(0, s) = 0,\ u(0, s) = h(s)$ for the initial data. Therefore, the solution of the PDE is

$$u = h(x - uy), \tag{2.44}$$

except that this time this solution is actually implicit. In order to analyze this solution further we eliminate the y variable from the equations for the characteristics (the projection of the characteristic curves on the (x, y) plane):

$$x = s + h(s)y. \tag{2.45}$$

The third characteristic equation implies that for each fixed s, i.e. along each characteristic, u preserves its initial value $u = h(s)$. The other characteristic equations imply, then, that the characteristics are straight lines.

Since different characteristics have different slopes that are determined by the initial values of u, they might intersect. Such an intersection has an obvious physical interpretation that can be seen from (2.45): The initial data $h(s)$ determine the speed of the characteristic emanating from a given s. Therefore, if a characteristic leaving the point s_1 has a higher speed than a characteristic leaving the point s_2, and if $s_1 < s_2$, then after some (positive) time the faster characteristic will overtake the slower one.

As we explained above, the solution is not well defined at points where characteristic curves intersect. To see the resulting difficulty from an algebraic perspective, we differentiate the implicit solution with respect to x to get $u_x = h'(1 - yu_x)$, implying

$$u_x = \frac{h'}{1 + yh'}. \tag{2.46}$$

Recalling that physically the variable y stands for time, we consider the ray $y > 0$. We conclude that the solution's derivative blows up at the critical time

$$y_c = -\frac{1}{h'(s)}. \tag{2.47}$$

Hence the classical solution is not defined for $y > y_c$. This conclusion is consistent with the heuristic physical interpretation presented above. Indeed a necessary condition for a singularity formation is that $h'(s) < 0$ at least at one point, such that a faster characteristic will start from a point behind a slower characteristic. If $h(s)$ is never decreasing, there will be no singularity; however, such data are exceptional. Observe that the solution becomes singular at the first time y that satisfies (2.47); such a value is achieved for the value s, where $h'(s)$ is minimal.

Equation (2.41) arises in the investigation of a fundamental physical problem. Thus we cannot end our analysis when a singularity forms. In other words, while the solution becomes singular at the critical time (2.47), the fluid described by the equations keeps flowing unaware of our mathematical troubles! Therefore we must find a means of extending the solution beyond y_c.

Extending singular solutions is not a simple matter. There are several ways to construct such extensions, and we must select a method that conforms with fundamental physical principles. The basic idea is to define a new problem. This new problem is formulated so as to be satisfied by each classical solution of the Euler equation, and such that each continuously differentiable solution of the new problem will satisfy the Euler equation. Yet, the new problem will also have nonsmooth solutions. A solution of the new extended problem is called a *weak solution*, and the new problem itself is called the weak formulation of the original PDE. We shall see that sometimes there exist more than one weak solution, and this will require upgrading the weak formulation to include a selection principle.

We choose to formulate the weak problem by replacing the differential equation with an integral balance. In fact, we have already discussed in Chapter 1 the connection between an integral balance and the associated differential relation emerging from it. We explained that the integral balance is more fundamental, and can only be transformed into a differential relation when the functions involved are sufficiently smooth. To apply the integral balance method we rewrite (2.41) in the form

$$\partial_y u + \frac{1}{2}\frac{\partial}{\partial x}(u^2) = 0, \tag{2.48}$$

and integrate (with respect to x, and for a fixed y) over an arbitrary interval $[a, b]$ to obtain

$$\partial_y \int_a^b u(\xi, y)\mathrm{d}\xi + \frac{1}{2}[u^2(b, y) - u^2(a, y)] = 0. \tag{2.49}$$

It is clear that every solution of the PDE satisfies the integral relation (2.49) as well. Also, since a and b are arbitrary, any function $u \in C^1$ that satisfies (2.49) would also satisfy the PDE. Nevertheless, the integral balance is also well defined for functions not in C^1; actually, (2.49) is even defined for functions with finitely many discontinuities.

We now demonstrate the construction of a weak solution that is a smooth function (continuously differentiable) except for discontinuities along a curve $x = \gamma(y)$. Since the solution is smooth on both sides of γ, it satisfies the equation there. It remains to compute γ. For this purpose we write the weak formulation in the form

$$\partial_y \left[\int_a^{\gamma(y)} u(\xi, y) \mathrm{d}\xi + \int_{\gamma(y)}^b u(\xi, y) \mathrm{d}\xi \right] + \frac{1}{2}[(u^2(b, y) - u^2(a, y)] = 0.$$

Differentiating the integrals with respect to y and using the PDE itself leads to

$$\gamma_y(y)u^- - \gamma_y(y)u^+ - \frac{1}{2}\left[\int_a^{\gamma(y)} (u^2(\xi, y))_\xi \mathrm{d}\xi + \int_{\gamma(y)}^b (u^2(\xi, y))_\xi \mathrm{d}\xi \right]$$
$$+ \frac{1}{2}[u^2(b, y) - u^2(a, y)] = 0.$$

Here we used u^- and u^+ to denote the values of u when we approach the curve γ from the left and from the right, respectively. Performing the integration we obtain

$$\gamma_y(y) = \frac{1}{2}(u^- + u^+), \tag{2.50}$$

namely, the curve γ moves at a speed that is the average of the speeds on the left and right ends of it.

Example 2.14 Consider the Euler equation (2.41) with the initial conditions

$$u(x, 0) = h(x) = \begin{cases} 1 & x \le 0, \\ 1 - x/\alpha & 0 < x < \alpha, \\ 0 & x \ge \alpha. \end{cases} \tag{2.51}$$

Since $h(x)$ is not monotone increasing, the solution will develop a singularity at some finite (positive) time. Formula (2.47) implies $y_c = \alpha$. For all $y < \alpha$ the (smooth) solution is given by

$$u(x, y) = \begin{cases} 1 & x \le y, \\ \dfrac{x - \alpha}{y - \alpha} & y < x < \alpha, \\ 0 & x \ge \alpha. \end{cases} \tag{2.52}$$

After the critical time y_c when the solution becomes singular we need to define a weak solution. We seek a solution with a single discontinuity. Formula (2.50)

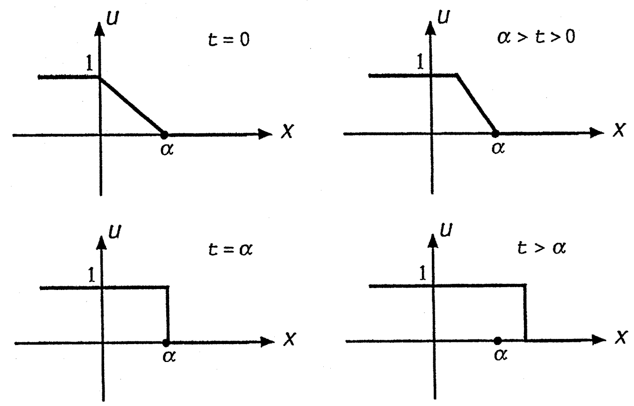

Figure 2.6 Several snapshots in the development of a shock wave.

implies that the discontinuity moves with a speed $\frac{1}{2}$. Therefore the following weak solution is compatible with the integral balance even for $y > \alpha$:

$$u(x, y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y - \alpha), \\ 0 & x > \alpha + \frac{1}{2}(y - \alpha). \end{cases} \tag{2.53}$$

The solution thus constructed has the structure of a moving jump discontinuity. It describes a step function moving at a constant speed. Such a solution is called a *shock wave*. Several snapshots of the formation and propagation of a shock wave are depicted in Figure 2.6.

Strictly speaking, the solution is not continuously differentiable even at time $y = 0$; however, this is a minor complication, since it can be shown that the formula for the classical solution is valid even when the derivative of the initial data has finitely many discontinuities as long as it is bounded.

Example 2.15 We now consider the opposite case where the initial data are increasing:

$$u(x, 0) = \begin{cases} 0 & x \le 0, \\ x/\alpha & 0 < x < \alpha, \\ 1 & x \ge \alpha. \end{cases} \tag{2.54}$$

Since this time $h' \ge 0$, there is no critical (positive) time where the characteristics intersect. On the contrary, the characteristics diverge. This situation is called an *expansion wave*, in contrast to the wave in the previous example which is called a *compression wave*. We use the classical solution formula to obtain

$$u(x,y)=\begin{cases}0 & x\le 0,\\ \dfrac{x}{\alpha+y} & 0<x<\alpha+y,\\ 1 & x\ge\alpha+y.\end{cases} \tag{2.55}$$

It is useful to consider for both examples the limiting case where $\alpha\to 0$. In Example 2.14 the initial data are the same as the shock weak solution (2.53), and therefore this solution is already valid at $y=0$. In contrast, in Example 2.15 the characteristics expand, the singularity is smoothed out at once, and the solution is

$$u(x,y)=\begin{cases}0 & x\le 0,\\ \dfrac{x}{y} & 0<x<y,\\ 1 & x\ge y.\end{cases} \tag{2.56}$$

We notice, though, that we could in principle write in the expansion wave case a weak solution that has a shock wave structure:

$$u(x,y)=\begin{cases}0 & x<\alpha+\frac{1}{2}(y-\alpha),\\ 1 & x>\alpha+\frac{1}{2}(y-\alpha).\end{cases} \tag{2.57}$$

We see that the weak formulation by itself does not have a unique solution! Additional arguments are needed to pick out the correct solution among the several options. In the case we consider here it is intuitively clear that the shock solution (2.57) is not adequate, since slower characteristics starting from the ray $x<0$ cannot overtake the faster characteristics that start from the ray $x>0$. Another, more physical, approach to this problem will be described now.

The theory of weak solutions and shock waves is quite difficult. We therefore present essentially the same ideas we developed above from a somewhat different perspective. Instead of looking at the specific canonical equation (2.48) with general initial conditions, we look at a more general PDE with canonical initial conditions. Specifically we consider the following first-order quasilinear PDE

$$u_y+\frac{\partial}{\partial x}F(u)=0. \tag{2.58}$$

Equations of this kind are called *conservation laws*. To understand the name, let us recall the derivation of the heat equation in Section 1.4.1. The energy balance (1.8) is actually of the form (2.58), where F denotes flux. In the canonical example (2.48), the flux is $F=\frac{1}{2}u^2$, and u is typically interpreted as mass density.

Equation (2.58) is supplemented with the initial condition

$$u(x,0)=\begin{cases}u^- & x<0,\\ u^+ & x>0.\end{cases} \tag{2.59}$$

To write the weak formulation for (2.58), we assume that the solution takes the shape of a shock wave of the form

$$u(x,y)=\begin{cases}u^- & x<\gamma(y),\\ u^+ & x>\gamma(y).\end{cases} \tag{2.60}$$

It remains therefore to find the shock orbit $x=\gamma(y)$. We find γ by integrating (2.58) with respect to x along the interval (x_1,x_2), with $x_1<\gamma,\ x_2>\gamma$. Taking (2.60) into account, we get

$$\frac{\partial}{\partial y}\{[x_2-\gamma(y)]u^+ + [\gamma(y)-x_1]u^-\}=F(u^+)-F(u^-). \tag{2.61}$$

The last equation implies

$$\gamma_y(y)=\frac{F(u^+)-F(u^-)}{u^+-u^-}:=\frac{[F]}{[u]}, \tag{2.62}$$

where we used the notation $[\cdot]$ to denote the change (jump) of a quantity across the shock.

Conservation laws appear in many areas of continuum mechanics (including hydrodynamics, gas dynamics, combustion, etc.), where the jump equation (2.62) is called the *Rankine–Hugoniot* condition.

Notice that in the special case of $F=\frac{1}{2}u^2$ that we considered earlier, the rule (2.62) reduces to (2.50). We also point out that we integrated (2.58) along an arbitrary finite interval, although the values of x_1 and x_2 do not appear in the final conclusion. The reason for introducing this artificial interval is that the integral of (2.58) over the real line is not bounded. Since we interpret u as a density of a physical quantity (such as mass), our model is really artificial, and a realistic model will have to take into account the effects of finite boundaries.

Our analysis of the general conservation law (2.58) is not yet complete. From our study of Example 2.15 we expect that shock would only occur if the characteristics collide. In the case of general conservation laws, this condition is expressed as:

The entropy condition Characteristics must enter the shock curve, and are not allowed to emanate from it.

The motivation for the entropy condition is rooted in gas dynamics and the second law of thermodynamics. In order not to stray too much away from the theory of PDEs, we shall give a heuristic reasoning, based on the interpretation of entropy as minus the amount of "information" stored in a given physical system. We thus phrase the second law of thermodynamics as stating that in a closed system information is only lost as time y increases, and cannot be created. Now, we have shown that characteristics carry with them information on the solution of a first-order

PDE. Therefore the emergence of a characteristic *from* a shock is interpreted as a *creation* of information which should be forbidden. To give the entropy condition an algebraic form, we write the conservation law (2.58) as $u_y + F_u u_x = 0$. Therefore the characteristic speed is given by F_u, and the entropy condition can be expressed as

$$F_u(u^-) > \gamma_y > F_u(u^+). \tag{2.63}$$

Applying this rule to the special case $F(u) = \frac{1}{2}u^2$ and using (2.62) we obtain that the shock solution is valid only if $u^- > u^+$, a conclusion we reached earlier from different considerations.

The theory of conservation laws has a nice application to the real-world problem of traffic flow. We therefore end this section by a qualitative analysis of this problem. Consider the flow of cars along one direction in a road. Although cars are discrete entities, we model them as a continuum, and denote by $u(x, y)$ the car density at a point x and time y.

A great deal of research has been devoted to the question of how to model the flux term $F(u)$. Clearly the flux is very low in bumper-to-bumper traffic, where each car barely moves. It may be surprising at first sight, but the flux is also low when the traffic is very light. In this case drivers tend to drive fast, and maintain a large distance between each other (at least this is what they ought to do when they drive fast...). Therefore the flux, which counts the total number of cars crossing a given point per time, is low. If we assume that a car occupies on average a length of 5 m, then the highest density is $u_b = 200$ cars/km. It was found experimentally that the maximal flux is about $F_{max} = 1500$ cars/hour, and is achieved at a speed of about 35 km/hour (20 miles/hour) (see [22] for a detailed discussion of traffic flow in the current context). Therefore the optimal density (if one wants to maximize the flux) is $u_{max} \sim 43$ cars/km. The concave shape of $F(u)$ is depicted in Figure 2.7.

Let us look at some practical implications of the model. Suppose that at time $y = 0$ there is a traffic jam at some point $x = x_j$. This could be caused by an accident, a red traffic light, a policeman directing the traffic, etc. Assume further that there is a line

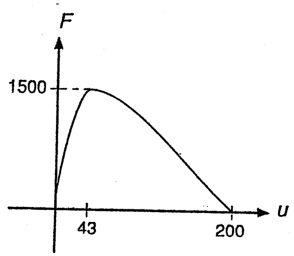

Figure 2.7 The traffic flux F as a function of the density u.

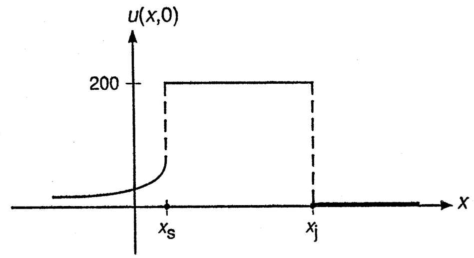

Figure 2.8 The car density at a red traffic light.

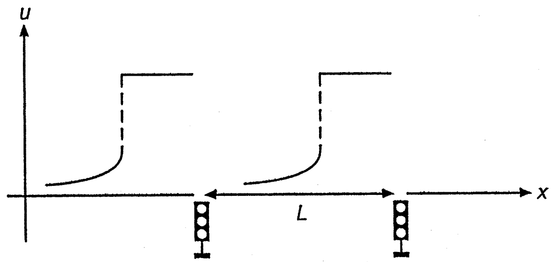

Figure 2.9 Traffic flow through a sequence of traffic lights.

of stationary cars extending from x_s to x_j (with $x_j > x_s$). Cars approach the traffic jam from $x < x_s$. At some point the drivers slow down, as they reach the regime where the car density is greater than u_{max}. Therefore the density u just before x_s is as shown schematically in Figure 2.8. The Rankine–Hugoniot condition (2.62) implies that the shock speed is negative. Although the curve $u(x, 0)$ is increasing, the derivative F_u is now negative, and therefore the entropy condition holds. We conclude that a shock wave will propagate from x_s *backwards*. Indeed as drivers approach a traffic jam there is a stage when they enter the shock and have to decelerate rapidly. The opposite occurs as we leave the point x_j. The density is decreasing and the entropy condition is violated. Therefore we have an expansion wave.

Our analysis can be applied to the design of traffic lights timing. Assume there are several consecutive traffic lights, separated by a distance L (see Figure 2.9). When the traffic approaches one of the traffic lights, a shock propagates backward. To estimate the speed of the shock we assume that behind the shock the density is optimal and so the flux is F_{max}. Then $\gamma_y = -F_{max}/(u_b - u_{max})$. Therefore the time it takes the shock to reach the previous traffic light is

$$T_s = \frac{L(u_b - u_{max})}{F_{max}}.$$

If the red light is maintained over a period exceeding T_s, the high density profile will extend throughout the road, and traffic will come to a complete stop.

2.8 The eikonal equation

Before proceeding to the general nonlinear case, let us analyze in detail the special case of the eikonal equation (see Chapter 1). We shall see that this equation can also be solved by characteristics. The two-dimensional eikonal equation takes the form

$$u_x^2 + u_y^2 = n^2, \tag{2.64}$$

where the surfaces $u = c$ (where c is some constant) are the wavefronts, and n is the refraction index of the medium. The initial conditions are given in the form of an initial curve Γ.

To write the characteristic equations, notice that the eikonal equation can be expressed as

$$(u_x, u_y, n^2) \cdot (u_x, u_y, -1) = 0.$$

Thus the vector (u_x, u_y, n^2) describes a direction tangent to the solution (integral) surface. To verify this argument algebraically, write equations for the x and y components of the characteristic curve, and check that the equation for the u component is consistent with (2.64).

We thus set

$$\frac{dx}{dt} = u_x, \quad \frac{dy}{dt} = u_y, \quad \frac{du}{dt} = n^2. \tag{2.65}$$

Since u_x and u_y are unknown at this stage, we compute

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}(u_x) = u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} = u_{xx}u_x + u_{xy}u_y \\ &= \frac{1}{2}(u_x^2 + u_y^2)_x = \frac{1}{2}(n^2(x, y))_x,\end{aligned} \tag{2.66}$$

and similarly

$$\frac{d^2y}{dt^2} = \frac{1}{2}(n^2(x, y))_y. \tag{2.67}$$

To write the solution of the eikonal equation, notice that it follows from the definition of the characteristic curves that

$$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt} = u_x^2 + u_y^2 = n^2. \tag{2.68}$$

Integrating the last equation leads to a formula that determines u at the point $(x(t), y(t))$ in terms of the initial value of u and the values of the refraction index along the integration path:

$$u(x(t), y(t)) = u(x(0), y(0)) + \int_0^t n^2(x(\tau), y(\tau))d\tau, \tag{2.69}$$

where $(x(t), y(t))$ is a solution of (2.66) and (2.67).

Before solving specific examples, we should clarify an important point regarding the initial conditions for the characteristic equations. Since the original equation (2.65) involves the derivatives of u that are not known at this stage, we eliminated these derivatives by differentiating the characteristic equations once more with respect to the parameter t. Indeed the equations we obtained ((2.66) and (2.67)) no longer depend on u itself; however these are second-order equations! Therefore, it is not enough to provide a single initial condition (such as the initial point of the characteristic curve on the initial curve Γ), but, rather, we must provide the derivatives (x_t, y_t) too. Equivalently, we should provide the vector tangent to the characteristic at the initial point. For this purpose we shall use the fact that the required vector is precisely the gradient (u_x, u_y) of u. From the eikonal equation itself we know that the size of that vector is $n(x, y)$, and from the initial condition we can find its projection at each point of Γ in the direction tangent to Γ. But obviously the size of a planar vector and its projection along a given direction determine the vector uniquely. Hence we obtain the additional initial condition.

Example 2.16 Solve the eikonal equation (2.64) for a medium with a constant refraction index $n = n_0$, and initial condition $u(x, 2x) = 1$.

The physical meaning of the initial condition is that the wavefront is a straight line. The characteristic equations are $d^2x/dt^2 = 0 = d^2y/dt^2$. Thus, the characteristics are straight lines, emanating from the initial line $y = 2x$. Since u is constant on such a line, the gradient of u is orthogonal to it. Hence the second initial condition for the characteristic is

$$\frac{dx}{dt}(0) = \frac{2}{\sqrt{5}}n_0, \quad \frac{dy}{dt}(0) = -\frac{1}{\sqrt{5}}n_0.$$

We thus obtain:

$$x(t, s) = \frac{2}{\sqrt{5}}n_0t + x_0(s), \quad y(t, s) = -\frac{1}{\sqrt{5}}n_0t + y_0(s), \quad u(t, s) = n_0^2t + u_0(s). \tag{2.70}$$

In order to find $x_0(s)$ and $y_0(s)$, we write the initial curve parameterically as $(s, 2s, 1)$. Substituting the initial curve into (2.70) leads to the integral surface

$$(x, y, u) = \left(\frac{2}{\sqrt{5}}n_0 t + s, -\frac{1}{\sqrt{5}}n_0 t + 2s, n_0^2 t + 1\right). \tag{2.71}$$

Eliminating $t = (2x - y)/\sqrt{5}n_0$, we obtain the explicit solution

$$u(x, y) = 1 + \frac{n_0}{\sqrt{5}}(2x - y).$$

The solution we have obtained has a simple physical interpretation: in a homogeneous medium the characteristic curves are straight lines (classical light rays), and an initial planar wavefront propagates in the direction orthogonal to them. Therefore all wavefronts are planar.

Example 2.17 Compute the function $u(x, y)$ satisfying the eikonal equation $u_x^2 + u_y^2 = n^2$ and the initial condition $u(x, 1) = n\sqrt{1 + x^2}$ (n is a constant parameter).

Write the initial conditions parametrically in the form $(x, y, u) = (s, 1, n\sqrt{1 + s^2})$. This condition implies $x_t(0, s) = u_x = ns/\sqrt{1 + s^2}$. Substituting the last expression into the eikonal equation gives $y_t(0, s) = u_y = n/\sqrt{1 + s^2}$. Integrating the characteristic equations we obtain

$$(x(t, s), y(t, s), u(t, s)) = \left(\frac{ns}{\sqrt{1 + s^2}}t + s, \quad \frac{n}{\sqrt{1 + s^2}}t + 1, \quad n(nt + \sqrt{1 + s^2})\right).$$

In order to write an explicit solution, observe the identity

$$x^2 + y^2 = (nt + \sqrt{1 + s^2})^2$$

satisfied by the integral surface. Therefore, the solution is $u = n\sqrt{x^2 + y^2}$. This solution represents a spherical wave starting from a single point at the origin of coordinates.

2.9 General nonlinear equations

The general first-order nonlinear equation takes the form

$$F(x, y, u, u_x, u_y) = 0. \tag{2.72}$$

We shall develop a solution method for such equations. The method is an extension of the method of characteristics. To simplify the presentation we shall use the notation $p = u_x$, $q = u_y$.

Consider a point (x_0, y_0, u_0) on the integral surface. We want to find the slope of a (characteristic) curve on the integral surface passing through this point. In the quasilinear case the equation determined directly the slope of a specific curve on the integral surface. We shall now construct that curve in a somewhat different manner. Let us write for this purpose the equation of the tangent plane to the integral surface through (x_0, y_0, u_0):

$$p(x - x_0) + q(y - y_0) - (u - u_0) = 0. \tag{2.73}$$

Notice that the derivatives p and q at (x_0, y_0, u_0) are not independent. Equation (2.72) imposes the relation

$$F(x_0, y_0, u_0, p, q) = 0. \tag{2.74}$$

The last two equations define a one-parameter family of tangent planes. Such a family spreads out a cone. To honor the French geometer Gaspard Monge (1746–1818) this cone is called the *Monge cone*. The natural candidate for the curve we seek in the tangent plane (defining for us the direction of the characteristic curve) is, therefore, the generator of the Monge cone. To compute the generator we differentiate (2.73) by the parameter p:

$$(x - x_0) + \frac{dq}{dp}(y - y_0) = 0. \tag{2.75}$$

Assume that F is not degenerate, i.e. $F_p^2 + F_q^2$ never vanishes. Without loss of generality assume $F_q \neq 0$. Then it follows from (2.74) and the implicit function theorem that

$$q'(p) = -\frac{F_p}{F_q}. \tag{2.76}$$

Substituting (2.76) into (2.75) we obtain the equation for the Monge cone generator:

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q}. \tag{2.77}$$

Equations (2.73) and (2.77) imply three differential equations for the characteristic curves:

$$\begin{aligned} x_t &= F_p(x, y, u, p, q), \\ y_t &= F_q(x, y, u, p, q), \\ u_t &= pF_p(x, y, u, p, q) + qF_q(x, y, u, p, q). \end{aligned} \tag{2.78}$$

It is easy to verify that equations (2.78) coincide in the quasilinear case with the characteristic equations (2.15). However in the fully nonlinear case the characteristic equations (2.78) do not form a closed system. They contain the hitherto unknown functions p and q. In other words, the characteristic curves carry with them a tangent plane that has to be found as part of the solution. To derive equations for p and q we write

$$p_t = u_{xx}x_t + u_{xy}y_t = u_{xx}F_p + u_{xy}F_q, \tag{2.79}$$

and similarly

$$q_t = u_{yx}F_p + u_{yy}F_q. \tag{2.80}$$

In order to eliminate u_{xx}, u_{xy} and u_{yy}, recall that the equation $F = 0$ holds along the characteristic curves. We therefore obtain upon differentiation

$$F_x + pF_u + p_xF_p + q_xF_q = 0, \tag{2.81}$$
$$F_y + qF_u + p_yF_p + q_yF_q = 0. \tag{2.82}$$

Substituting (2.81) and (2.82) into (2.79) and (2.80) leads to

$$p_t = -F_x - pF_u,$$
$$q_t = -F_y - qF_u.$$

To summarize we write the entire set of characteristic equations:

$$\begin{aligned} x_t &= F_p(x, y, u, p, q), \\ y_t &= F_q(x, y, u, p, q), \\ u_t &= pF_p(x, y, u, p, q) + qF_q(x, y, u, p, q), \\ p_t &= -F_x(x, y, u, p, q) - pF_u(x, y, u, p, q), \\ q_t &= -F_y(x, y, u, p, q) - qF_u(x, y, u, p, q). \end{aligned} \tag{2.83}$$

A simple computation of F_t, using (2.83), indeed verifies that the PDE holds at all points along a characteristic curve. The main addition to the theory we presented earlier for the quasilinear case is that the characteristic curves have been replaced by more complex geometric structures. Since each characteristic curve now drags with it a tangent plane, we call these structures *characteristic strips*, and equations (2.83) are called the *strip equations*.

We are now ready to formulate the general Cauchy problem for first-order PDEs. Consider (2.72) with the initial condition given by an initial curve $\Gamma \in C^1$:

$$x = x_0(s), \;\; y = y_0(s), \;\; u = u_0(s). \tag{2.84}$$

We shall show in the proof of the next theorem how to derive initial conditions also for p and q in order to obtain a complete initial value problem for the system (2.83). We do not expect every Cauchy problem to be solvable. Clearly some form of transversality condition must be imposed. It turns out, however, that slightly more than that is required:

Definition 2.18 Let a point $P_0 = (x_0(s_0), y_0(s_0), u_0(s_0), p_0(s_0), q_0(s_0))$ satisfy the compatibility conditions

$$F(P_0) = 0, \quad u_0'(s_0) = p_0(s_0)x_0'(s_0) + q_0(s_0)y_0'(s_0). \tag{2.85}$$

If, in addition,

$$x_0'(s_0)F_q(P_0) - y_0'(s_0)F_p(P_0) \neq 0 \tag{2.86}$$

is satisfied, then we say that the Cauchy problem (2.72), (2.84) satisfies the *generalized transversality condition* at the point P_0.

Theorem 2.19 *Consider the Cauchy problem (2.72), (2.84). Assume that the generalized transversality condition (2.85)–(2.86) holds at P_0. Then there exists $\varepsilon > 0$ and a unique solution $(x(t, s), y(t, s), u(t, s), p(t, s), q(t, s))$ for the Cauchy problem which is defined for $|s - s_o| + |t| < \varepsilon$. Moreover, the parametric representation defines a smooth integral surface $u = u(x, y)$.*

Proof We start by deriving full initial conditions for the system (2.83). Actually the Cauchy problem has already provided three conditions for (x, y, u):

$$x(0, s) = x_0(s), \;\; y(0, s) = y_0(s), \;\; u(0, s) = u_0(s). \tag{2.87}$$

We are left with the task of finding initial conditions

$$p(0, s) = p_0(s), \;\; q(0, s) = q_0(s), \tag{2.88}$$

for $p(t, s)$ and $q(t, s)$. Clearly $p_0(s)$ and $q_0(s)$ must satisfy at every point s the differential condition

$$u_0'(s) = p_0(s)x_0'(s) + q_0(s)y_0'(s), \tag{2.89}$$

and the equation itself

$$F(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s)) = 0. \tag{2.90}$$

However (2.85) guarantees that these two requirements indeed hold at $s = s_0$. The transversality condition (2.86) ensures that the Jacobian of the system (2.89)–(2.90) (with respect to the variables p_0 and q_0) does not vanish at s_0. Therefore, the implicit function theorem implies that one can derive from (2.89)–(2.90) the required initial conditions for p_0 and q_0. Hence the characteristic equations have a full set of initial conditions in a neighborhood $|s - s_0| < \delta$. Since the system of ODEs (2.83) is well-posed, the existence of a unique smooth solution

$$(x(t,s),\, y(t,s),\, u(t,s),\, p(t,s),\, q(t,s))$$

is guaranteed for (t, s) in a neighborhood of $(0, s_0)$. As in the quasilinear case, one can verify that (2.83) and (2.90) imply that

$$F(x(t,s), y(t,s), u(t,s), p(t,s), q(t,s)) = 0 \qquad \forall\, |s - s_0| < \delta,\ |t| < \varepsilon. \tag{2.91}$$

In order that the parametric representation for (x, y, u) will define a smooth surface $u(x, y)$, we must show that the mapping $(x(t, s), y(t, s))$ can be inverted to a smooth mapping $(t(x, y), s(x, y))$. Such an inversion exists if the Jacobian $J = \partial(x, y)/\partial(t, s)$ does not vanish. But the characteristic equations imply

$$J\,|_{(0,s_0)} = \left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{(0,s_0)} = x_s(0,s_0)y_t(0,s_0) - x_t(0,s_0)y_s(0,s_0) \neq 0, \tag{2.92}$$

where the last inequality follows from the transversality condition (2.86).

We have thus constructed a smooth function $u(x, y)$. Does it satisfy the Cauchy problem for the nonlinear PDE? This requires that the relations

$$u_t(t,s) = p(t,s)x_t(t,s) + q(t,s)y_t(t,s) \tag{2.93}$$

and

$$u_s(t,s) = p(t,s)x_s(t,s) + q(t,s)y_s(t,s) \tag{2.94}$$

would hold. Condition (2.93) is clearly valid since the characteristic equations were, in fact, constructed to satisfy it. The compatibility condition (2.94) holds on the initial curve, i.e. at $t = 0$. It remains, though, to check this condition also for values of t other than zero. We therefore define the auxiliary function

$$R(t,s) = u_s - px_s - qy_s.$$

We have to show that $R(t, s) = 0$. As we have already argued, the initial data for p and q imply

$$R(0,s) = 0.$$

To check that R also vanishes for other values of t we compute

$$\begin{aligned} R_t &= u_{st} - p_t x_s - p x_{st} - q_t y_s - q y_{st} \\ &= \frac{\partial}{\partial s}(u_t - px_t - qy_t) + p_s x_t + q_s y_t - p_t x_s - q_t y_s. \end{aligned}$$

Using (2.93) and the characteristic equations we get

$$\begin{aligned} R_t &= p_s F_p + q_s F_q + x_s(F_x + pF_u) + y_s(F_y + qF_u) \\ &= F_u(px_s + qy_s) + p_s F_p + q_s F_q + x_s F_x + y_s F_y. \end{aligned}$$

Adding and subtracting $F_u u_s$ to the last expression we find

$$R_t = F_s - F_u R = -F_u R,$$

where we used the fact that $F_s = 0$ which follows from (2.91). Since the initial condition for the linear homogeneous ODE $R_t = -F_u R$ is homogeneous too ($R(0, s) = 0$), it follows that $R(t, s) \equiv 0$.

To demonstrate the method we just developed, let us solve again the Cauchy problem from Example 2.16.

Example 2.20 We write the eikonal equation in the form

$$F(x,y,u,p,q) = p^2 + q^2 - n_0^2 = 0. \tag{2.95}$$

Hence the characteristic equations are

$$\begin{aligned} x_t &= 2p, \\ y_t &= 2q, \\ u_t &= 2n_0^2, \\ p_t &= 0, \\ q_t &= 0. \end{aligned} \tag{2.96}$$

The initial conditions for (x, y, u) are given by

$$x(0,s) = s, \quad y(0,s) = 2s, \quad u(0,s) = 1. \tag{2.97}$$

We use (2.89)–(2.90) to derive the initial data for p and q:

$$\begin{aligned} p(0,s) + 2q(0,s) &= 0, \\ p^2(0,s) + q^2(0,s) - n_0^2 &= 0. \end{aligned}$$

Solving these equation we obtain

$$p(0,s) = \frac{2n_0}{\sqrt{5}}, \quad q(0,s) = -\frac{n_0}{\sqrt{5}}. \tag{2.98}$$

It is an easy matter to solve the full characteristic equations:

$$(x,y,u,p,q) = \left(s + \frac{2n_0}{\sqrt{5}}t,\ 2s - \frac{n_0}{\sqrt{5}}t,\ 1 + 2n_0^2 t,\ \frac{2n_0}{\sqrt{5}},\ -\frac{n_0}{\sqrt{5}}\right), \tag{2.99}$$

After eliminating t we finally obtain

$$u(x,y) = 1 + \frac{n_0}{\sqrt{5}}(2x - y).$$

Notice that the parametric representation obtained in the current example is different from the one we derived in Example 2.16, since the parameter t we used here is half the parameter used in Example 2.16.

2.10 Exercises

2.1 Consider the equation $u_x + u_y = 1$, with the initial condition $u(x,0) = f(x)$.
(a) What are the projections of the characteristic curves on the (x,y) plane?
(b) Solve the equation.

2.2 Solve the equation $xu_x + (x+y)u_y = 1$ with the initial conditions $u(1,y) = y$. Is the solution defined everywhere?

2.3 Let p be a real number. Consider the PDEs

$$xu_x + yu_y = pu \qquad -\infty < x < \infty, \quad -\infty < y < \infty.$$

(a) Find the characteristic curves for the equations.
(b) Let $p = 4$. Find an explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$.
(c) Let $p = 2$. Find two solutions that satisfy $u(x,0) = x^2$, for every $x > 0$.
(d) Explain why the result in (c) does not contradict the existence–uniqueness theorem.

2.4 Consider the equation $yu_x - xu_y = 0$ $(y > 0)$. Check for each of the following initial conditions whether the problem is solvable. If it is solvable, find a solution. If it is not, explain why.

(a) $u(x,0) = x^2$.
(b) $u(x,0) = x$.
(c) $u(x,0) = x, \quad x > 0$.

2.5 Let $u(x,y)$ be an integral surface of the equation

$$a(x,y)u_x + b(x,y)u_y + u = 0,$$

where $a(x,y)$ and $b(x,y)$ are positive differentiable functions in the entire plane. Define

$$D = \{(x,y), |x| < 1, |y| < 1\}.$$

(a) Prove that the projection on the (x,y) plane of each characteristic curve passing through a point in D intersects the boundary of D at exactly two points.
(b) Show that if u is positive on the boundary of D, then it is positive at every point in D.

(c) Suppose that u attains a local minimum (maximum) at a point $(x_0, y_0) \in D$. Evaluate $u(x_0, y_0)$.
(d) Denote by m the minimal value of u on the boundary of D. Assume $m > 0$. Show that $u(x,y) \geq m$ for all $(x,y) \in D$.
Remark This is an atypical example of a first-order PDE for which a maximum principle holds true. Maximum principles are important tools in the study of PDEs, and they are valid typically for second-order elliptic and parabolic PDEs (see Chapter 7).

2.6 The equation $xu_x + (x^2 + y)u_y + (y/x - x)u = 1$ is given along with the initial condition $u(1,y) = 0$.
(a) Solve the problem for $x > 0$. Compute $u(3,6)$.
(b) Is the solution defined for the entire ray $x > 0$?

2.7 Solve the Cauchy problem $u_x + u_y = u^2$, $u(x,0) = 1$.

2.8 (a) Solve the equation $xuu_x + yuu_y = u^2 - 1$ for the ray $x > 0$ under the initial condition $u(x, x^2) = x^3$.
(b) Is there a unique solution for the Cauchy problem over the entire real line $-\infty < x < \infty$?

2.9 Consider the equation

$$uu_x + u_y = -\frac{1}{2}u.$$

(a) Show that there is a unique integral surface in a neighborhood of the curve

$$\Gamma_1 = \{(s, 0, \sin s) \mid -\infty < s < \infty\}.$$

(b) Find the parametric representation $x = x(t,s),\ y = y(t,s),\ u = u(t,s)$ of the integral surface S for initial condition of part (a).
(c) Find an integral surface S_1 of the same PDE passing through the initial curve

$$\Gamma_1 = \{(s, s, 0) \mid -\infty < s < \infty\}.$$

(d) Find a parametric representation of the intersection curves of the surfaces S and S_1.
Hint Try to characterize that curve relative to the PDE.

2.10 A river is defined by the domain

$$D = \{(x,y) | \quad |y| < 1, \ -\infty < x < \infty\}.$$

A factory spills a contaminant into the river. The contaminant is further spread and convected by the flow in the river. The velocity field of the fluid in the river is only in the x direction. The concentration of the contaminant at a point (x,y) in the river and at time τ is denoted by $u(x,y,\tau)$. Conservation of matter and momentum implies that u satisfies the first-order PDE

$$u_\tau - (y^2 - 1)u_x = 0.$$

The initial condition is $u(x,y,0) = e^y e^{-x^2}$.

(a) Find the concentration u for all (x, y, τ).
(b) A fish lives near the point $(x, y) = (2, 0)$ at the river. The fish can tolerate contaminant concentration levels up to 0.5. If the concentration exceeds this level, the fish will die at once. Will the fish survive? If yes, explain why. If no, find the time in which the fish will die.
Hint Notice that y appears in the PDE just as a parameter.

2.11 Solve the equation $(y^2 + u)u_x + yu_y = 0$ in the domain $y > 0$, under the initial condition $u = 0$ on the planar curve $x = y^2/2$.

2.12 Solve the equation $u_y + u^2 u_x = 0$ in the ray $x > 0$ under the initial condition $u(x, 0) = \sqrt{x}$. What is the domain of existence of the solution?

2.13 Consider the equation $uu_x + xu_y = 1$, with the initial condition $\left(\frac{1}{2}s^2 + 1, \frac{1}{6}s^3 + s, s\right)$. Find a solution. Are there other solutions? If not, explain why; if there *are* further solutions, find at least two of them, and explain the lack of uniqueness.

2.14 Consider the equation $xu_x + yu_y = 1/\cos u$.
(a) Find a solution to the equation that satisfies the condition $u(s^2, \sin s) = 0$ (you can write down the solution in the implicit form $F(x, y, u) = 0$).
(b) Find some domain of s values for which there exists a unique solution.

2.15 (a) Find a function $u(x, y)$ that solves the Cauchy problem

$$(x + y^2)u_x + yu_y + \left(\frac{x}{y} - y\right)u = 1, \quad u(x, 1) = 0 \quad x \in \mathbb{R}.$$

(b) Check whether the transversality condition holds.
(c) Draw the projections on the (x, y) plane of the initial condition and the characteristic curves emanating from the points $(2, 1, 0)$ and $(0, 1, 0)$.
(d) Is the solution you obtained in (a) defined at the origin $(x, y) = (0, 0)$? Explain your answer in light of the existence–uniqueness theorem.

2.16 Solve the Cauchy problem

$$xu_x + yu_y = -u, \quad u(\cos s, \sin s) = 1 \quad 0 \le s \le \pi.$$

Is the solution defined everywhere?

2.17 Consider the equation

$$xu_x + u_y = 1.$$

(a) Find a characteristic curve passing through the point $(1, 1, 1)$.
(b) Show that there exists a unique integral surface $u(x, y)$ satisfying $u(x, 0) = \sin x$.
(c) Is the solution defined for all x and y?

2.18 Consider the equation $uu_x + u_y = -\frac{1}{2}u$.
(a) Find a solution satisfying $u(x, 2x) = x^2$.
(b) Is the solution unique?

2.19 (a) Find a function $u(x, y)$ that solves the Cauchy problem

$$x^2 u_x + y^2 u_y = u^2, \quad u(x, 2x) = x^2 \quad x \in \mathbb{R}.$$

(b) Check whether the transversality condition holds.

(c) Draw the projections on the (x, y) plane of the initial curve and the characteristic curves that start at the points $(1, 2, 1)$ and $(0, 0, 0)$.
(d) Is the solution you found in part (a) defined for all x and y?

2.20 Consider the equation

$$yu_x - uu_y = x.$$

(a) Write a parametric representation of the characteristic curves.
(b) Solve the Cauchy problem

$$yu_x - uu_y = x, \qquad u(s, s) = -2s \qquad -\infty < s < \infty.$$

(c) Is the following Cauchy problem solvable:

$$yu_x - uu_y = x, \qquad u(s, s) = s \qquad -\infty < s < \infty?$$

(d) Set

$$w_1 = x + y + u, \quad w_2 = x^2 + y^2 + u^2, \quad w_3 = xy + xu + yu.$$

Show that $w_1(w_2 - w_3)$ is constant along each characteristic curve.

2.21 (a) Find a function $u(x, y)$ that solves the Cauchy problem

$$xu_x - yu_y = u + xy, \quad u(x, x) = x^2 \quad 1 \le x \le 2.$$

(b) Check whether the transversality condition holds.
(c) Draw the projections on the (x, y) plane of the initial curve and the characteristic curves emanating from the points $(1, 1, 1)$ and $(2, 2, 4)$.
(d) Is the solution you found in (a) well defined in the entire plane?

2.22 Solve the Cauchy problem $u_x^2 + u_y = 0$, $u(x, 0) = x$.

2.23 Let $u(x, t)$ be the solution to the Cauchy problem

$$u_t + cu_x + u^2 = 0, \quad u(x, 0) = x,$$

where c is a constant, t denotes time, and x denotes a space coordinate.
(a) Solve the problem.
(b) A person leaves the point x_0 at time $t = 0$, and moves in the positive x direction with a velocity c (i.e. the quantity $x - ct$ is fixed for him). Show that if $x_0 > 0$, then the solution as seen by the person approaches zero as $t \to \infty$.
(c) What will be observed by such a person if $x_0 < 0$, or if $x_0 = 0$?

2.24 (a) Solve the problem

$$xu_x - uu_y = y, \qquad u(1, y) = y \qquad -\infty < y < \infty.$$

(b) Is the solution unique? What is the maximal domain where it is defined?

2.25 Find at least five solutions for the Cauchy problem

$$u_x + u_y = 1, \quad u(x,x) = x.$$

2.26 (a) Solve the problem

$$x u_y - y u_x + u = 0,$$
$$u(x,0) = 1 \qquad x > 0.$$

(b) Is the solution unique? What is the maximal domain where it is defined?

2.27 (a) Use the Lagrange method to find a function $u(x, y)$ that solves the problem

$$u u_x + u_y = 1 \tag{2.100}$$
$$u(3x, 0) = -x \quad -\infty < x < \infty. \tag{2.101}$$

(b) Show that the curve $\{(3x, 2, 4 - 3x) | -\infty < x < \infty\}$ is contained in the solution surface $u(x, y)$.
(c) Solve

$$u u_x + u_y = 1$$
$$u(3x, 2) = 4 - 3x \quad -\infty < x < \infty.$$

2.28 Analyze the following problems using the Lagrange method. For each problem determine whether there exists a unique solution, infinitely many solutions or no solution at all. If there is a unique solution, find it; if there are infinitely many solutions, find at least two of them. Present all solutions explicitly.
(a)

$$x u u_x + y u u_y = x^2 + y^2 \quad x > 0, \; y > 0,$$
$$u(x,1) = \sqrt{x^2 + 1}.$$

(b)

$$x u u_x + y u u_y = x^2 + y^2 \quad x > 0, \; y > 0,$$
$$u(x,x) = \sqrt{2}x.$$

2.29 Consider the equation

$$x u_x + (1+y) u_y = x(1+y) + xu.$$

(a) Find the general solution.
(b) Assume an initial condition of the form $u(x, 6x - 1) = \phi(x)$. Find a necessary and sufficient condition for ϕ that guarantees the existence of a solution to the problem. Solve the problem for the appropriate ϕ that you found.
(c) Assume an initial condition of the form $u(-1, y) = \psi(y)$. Find a necessary and sufficient condition for ψ that guarantees the existence of a solution to the problem. Solve the problem for the appropriate ψ that you found.

(d) Explain the differences between (b) and (c).

2.30 (a) Find a compatibility condition for the Cauchy problem

$$u_x^2 + u_y^2 = 1, \qquad u(\cos s, \sin s) = 0 \quad 0 \le s \le 2\pi.$$

(b) Solve the above Cauchy problem.
(c) Is the solution uniquely defined?

3

Second-order linear equations in two independent variables

3.1 Introduction

In this chapter we classify the family of second-order linear equations for functions in two independent variables into three distinct types: hyperbolic (e.g., the wave equation), parabolic (e.g., the heat equation), and elliptic equations (e.g., the Laplace equation). It turns out that solutions of equations of the same type share many exclusive qualitative properties. We show that by a certain change of variables any equation of a particular type can be transformed into a canonical form which is associated with its type.

3.2 Classification

We concentrate in this chapter on second-order linear equations for functions in two independent variables x, y. Such an equation has the form

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \qquad (3.1)$$

where $a, b, \dots, f, g$ are given functions of x, y, and $u(x, y)$ is the unknown function. We introduced the factor 2 in front of the coefficient b for convenience. We assume that the coefficients a, b, c do not vanish simultaneously.

The operator

$$L_0[u] = au_{xx} + 2bu_{xy} + cu_{yy}$$

that consists of the second-(highest-)order terms of the operator L is called the *principal part* of L. It turns out that many fundamental properties of the solutions of (3.1) are determined by its principal part, and, more precisely, by the *sign* of the discriminant $\delta(L) := b^2 - ac$ of the equation. We classify the equation according to the sign of $\delta(L)$.

Definition 3.1 Equation (3.1) is said to be *hyperbolic* at a point (x, y) if

$$\delta(L)(x, y) = b(x, y)^2 - a(x, y)c(x, y) > 0,$$

it is said to be *parabolic* at (x, y) if $\delta(L)(x, y) = 0$, and it is said to be *elliptic* at (x, y) if $\delta(L)(x, y) < 0$.

Let Ω be a domain in $\mathbb{R}^2$ (i.e. Ω is an open connected set). The equation is hyperbolic (resp., parabolic, elliptic) in Ω, if it is hyperbolic (resp., parabolic, elliptic) at all points $(x, y) \in \Omega$.

Definition 3.2 The transformation $(\xi, \eta) = (\xi(x, y), \eta(x, y))$ is called a *change of coordinates* (or a nonsingular transformation) if the Jacobian $J := \xi_x\eta_y - \xi_y\eta_x$ of the transformation does not vanish at any point (x, y).

Lemma 3.3 *The type of a linear second-order PDE in two variables is invariant under a change of coordinates. In other words, the type of the equation is an intrinsic property of the equation and is independent of the particular coordinate system used.*

Proof Let

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \qquad (3.2)$$

and let $(\xi, \eta) = (\xi(x, y), \eta(x, y))$ be a nonsingular transformation. Write $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$. We claim that w is a solution of a second-order equation of the same type. Using the chain rule one finds that

$$\begin{aligned}
u_x &= w_\xi\xi_x + w_\eta\eta_x,\\
u_y &= w_\xi\xi_y + w_\eta\eta_y,\\
u_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi\xi_{xx} + w_\eta\eta_{xx},\\
u_{xy} &= w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy},\\
u_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi\xi_{yy} + w_\eta\eta_{yy}.
\end{aligned}$$

Substituting these formulas into (3.2), we see that w satisfies the following linear equation:

$$\ell[w] := Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G,$$

where the coefficients of the principal part of the linear operator ℓ are given by

$$\begin{aligned}
A(\xi, \eta) &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2,\\
B(\xi, \eta) &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,\\
C(\xi, \eta) &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2.
\end{aligned}$$

Notice that we do not need to compute the coefficients of the lower-order derivatives (D, E, F) since the type of the equation is determined only by its principal part (i.e. by the coefficients of the second-order terms). An elementary calculation shows that these coefficients satisfy the following matrix equation:

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}.$$

Denote by J the Jacobian of the transformation. Taking the determinant of the two sides of the above matrix equation, we find

$$-\delta(\ell) = AC - B^2 = J^2(ac - b^2) = -J^2\delta(L).$$

Therefore, the type of the equation is invariant under nonsingular transformations. □

In Chapter 1 we encountered the three (so called) *fundamental equations of mathematical physics*: the heat equation, the wave equations and the Laplace equation. All of them are linear second-order equations. One can easily verify that the wave equation is hyperbolic, the heat equation is parabolic, and the Laplace equation is elliptic.

We shall show in the next sections that if (3.1) is hyperbolic (resp., parabolic, elliptic) in a domain D, then one can find a coordinate system in which the equation has a simpler form that we call the *canonical form* of the equation. Moreover, in such a case the principal part of the canonical form is equal to the principal part of the fundamental equation of mathematical physics of the same type. This is one of the reasons for studying these fundamental equations.

Definition 3.4 The *canonical form of a hyperbolic equation* is

$$\ell[w] = w_{\xi\eta} + \ell_1[w] = G(\xi, \eta),$$

where ℓ_1 is a first-order linear differential operator, and G is a function.

Similarly, the canonical form of a parabolic equation is

$$\ell[w] = w_{\xi\xi} + \ell_1[w] = G(\xi, \eta),$$

and the canonical form of an elliptic equation is

$$\ell[w] = w_{\xi\xi} + w_{\eta\eta} + \ell_1[w] = G(\xi, \eta).$$

Note that the principal part of the canonical form of a hyperbolic equation is not equal to the wave operator. We shall show in Section 4.2 that a simple (linear) change of coordinates transforms the wave equation into the equation $w_{\xi\eta} = 0$.

3.3 Canonical form of hyperbolic equations

Theorem 3.5 *Suppose that (3.1) is hyperbolic in a domain D. There exists a coordinate system (ξ, η) in which the equation has the canonical form*

$$w_{\xi\eta} + \ell_1[w] = G(\xi, \eta),$$

where $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, ℓ_1 is a first-order linear differential operator, and G is a function which depends on (3.1).

Proof Without loss of generality, we may assume that $a(x, y) \neq 0$ for all $(x, y) \in D$. We need to find two functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that

$$A(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0,$$
$$C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0.$$

The equation that was obtained for the function η is actually the same equation as for ξ; therefore, we need to solve only one equation. It is a first-order equation that is not quasilinear; but as a quadratic form in ξ it is possible to write it as a product of two linear terms

$$\frac{1}{a}\left[a\xi_x + (b - \sqrt{b^2 - ac})\xi_y\right]\left[a\xi_x + (b + \sqrt{b^2 - ac})\xi_y\right] = 0.$$

Therefore, we need to solve the following linear equations:

$$a\xi_x + (b + \sqrt{b^2 - ac})\xi_y = 0, \tag{3.3}$$
$$a\xi_x + (b - \sqrt{b^2 - ac})\xi_y = 0. \tag{3.4}$$

In order to obtain a nonsingular transformation $(\xi(x, y), \eta(x, y))$ we choose ξ to be a solution of (3.3) and η to be a solution of (3.4).

These equations are a special case of Example 2.4. The characteristic equations for (3.3) are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = b + \sqrt{b^2 - ac}, \quad \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0.$$

Therefore, ξ is constant on each characteristic. The characteristics are solutions of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b + \sqrt{b^2 - ac}}{a}. \tag{3.5}$$

The function η is constant on the characteristic determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b - \sqrt{b^2 - ac}}{a}. \tag{3.6}$$

□

Definition 3.6 The solutions of (3.5) and (3.6) are called the two families of the *characteristics* (or *characteristic projections*) of the equation $L[u] = g$.

Example 3.7 Consider the *Tricomi equation:*

$$u_{xx} + xu_{yy} = 0 \qquad x < 0. \tag{3.7}$$

Find a mapping $q=q(x, y), r=r(x, y)$ that transforms the equation into its canonical form, and present the equation in this coordinate system.

The characteristic equations are

$$\frac{\mathrm{d}y_\pm}{\mathrm{d}x} = \pm\sqrt{-x},$$

and their solutions are $\frac{3}{2}y_\pm \pm (-x)^{3/2} =$ constant. Thus, the new independent variables are

$$q(x, y) = \frac{3}{2}y + (-x)^{3/2}, \qquad r(x, y) = \frac{3}{2}y - (-x)^{3/2}.$$

Clearly,

$$q_x = -r_x = -\frac{3}{2}(-x)^{1/2}, \qquad q_y = r_y = \frac{3}{2}.$$

Define $v(q, r) = u(x, y)$. By the chain rule

$$u_x = \frac{-3}{2}(-x)^{1/2}v_q + \frac{3}{2}(-x)^{1/2}v_r, \qquad u_y = \frac{3}{2}(v_q + v_r),$$
$$u_{xx} = -\frac{9}{4}xv_{qq} - \frac{9}{4}xv_{rr} + 2\frac{9}{4}xv_{qr} + \frac{3}{4}(-x)^{-1/2}(v_q - v_r),$$
$$u_{yy} = -\frac{9}{4}(v_{qq} + 2v_{qr} + v_{rr}).$$

Substituting these expressions into the Tricomi equation we obtain

$$u_{xx} + xu_{yy} = -9(q-r)^{2/3}\left[v_{qr} + \frac{v_q - v_r}{6(q-r)}\right] = 0.$$

Example 3.8 Consider the equation

$$u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0. \tag{3.8}$$

Find a coordinate system $s = s(x, y)$, $t = t(x, y)$ that transforms the equation into its canonical form. Show that in this coordinate system the equation has the form $v_{st} = 0$, and find the general solution.

The characteristic equations are

$$\frac{\mathrm{d}y_\pm}{\mathrm{d}x} = -\sin x \pm \sqrt{\sin^2 x + \cos^2 x} = -\sin x \pm 1.$$

Consequently, the solutions are $y_\pm = \cos x \pm x+$ constant. The requested transformation is

$$s(x, y) = \cos x + x - y, \qquad t(x, y) = \cos x - x - y.$$

Consider now the function $v(s, t) = u(x, y)$ and substitute it into (3.8). We get

$$\begin{aligned}&[v_{ss}(-\sin x + 1)^2 + 2v_{st}(-\sin x + 1)(-\sin x - 1) + v_{tt}(-\sin x - 1)^2\\ &+ v_s(-\cos x) + v_t(-\cos x)] - 2\sin x\,[v_{ss}(\sin x - 1) + v_{st}(\sin x - 1)\\ &+ v_{st}(\sin x + 1) + v_{tt}(\sin x + 1)] - \cos^2 x\,[v_{ss} + 2v_{st} + v_{tt}]\\ &- \cos x(-v_s - v_t) = 0.\end{aligned}$$

Thus, $-4v_{st} = 0$, and the canonical form is

$$v_{st} = 0.$$

It is easily checked that its general solution is $v(s, t) = F(s) + G(t)$, for every $F, G \in C^2(\mathbb{R})$. Therefore, the general solution of (3.8) is

$$u(x, y) = F(\cos x + x - y) + G(\cos x - x - y).$$

3.4 Canonical form of parabolic equations

Theorem 3.9 *Suppose that (3.1) is parabolic in a domain D. There exists a coordinate system (ξ, η) where the equation has the canonical form*

$$w_{\xi\xi} + \ell_1[w] = G(\xi, \eta),$$

where $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, ℓ_1 is a first-order linear differential operator, and G is a function which depends on (3.1).

Proof Since $b^2 - ac = 0$, we may assume that $a(x, y) \neq 0$ for all $(x, y) \in D$. We need to find two functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that $B(\xi, \eta) = C(\xi, \eta) = 0$ for all $(x, y) \in D$. It is enough to make $C = 0$, since the parabolicity of the equation will then imply that $B = 0$. Therefore, we need to find a function η that is a solution of the equation

$$C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \frac{1}{a}(a\eta_x + b\eta_y)^2 = 0.$$

From this it follows that η is a solution of the first-order linear equation

$$a\eta_x + b\eta_y = 0. \tag{3.9}$$

Hence, the solution η is constant on each characteristic, i.e., on a curve that is a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}. \tag{3.10}$$

Now, the only constraint on the second independent variable ξ, is that the Jacobian of the transformation should not vanish in D, and we may take any such function ξ. Note that a parabolic equation admits only one family of characteristics while for hyperbolic equations we have two families. □

Example 3.10 Prove that the equation

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0 \tag{3.11}$$

is parabolic and find its canonical form; find the general solution on the half-plane $x > 0$.

We identify $a = x^2$, $2b = -2xy$, $c = y^2$; therefore, $b^2 - ac = x^2y^2 - x^2y^2 = 0$ and the equation is parabolic. The equation for the characteristics is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x},$$

and the solution is $xy =$ constant. Therefore, we define $\eta(x, y) = xy$. The second variable can be simply chosen as $\xi(x, y) = x$. Let $v(\xi, \eta) = u(x, y)$. Substituting the new coordinates ξ and η into (3.11), we obtain

$$x^2(y^2v_{\eta\eta} + 2yv_{\xi\eta} + v_{\xi\xi}) - 2xy(v_\eta + xyv_{\eta\eta} + xv_{\xi\eta})$$
$$+ x^2v_{\eta\eta} + xyv_\eta + xv_\xi + xyv_\xi = 0.$$

Thus,

$$\xi^2v_{\xi\xi} + \xi v_\xi = 0,$$

or $v_{\xi\xi} + (1/\xi)v_\xi = 0$, and this is the desired canonical form.

Setting $w = v_\xi$, we arrive at the first-order ODE $w_\xi + (1/\xi)w = 0$. The solution is $\ln w = -\ln\xi + \tilde{f}(\eta)$, or $w = f(\eta)/\xi$. Hence, v satisfies

$$v = \int v_\xi \mathrm{d}\xi = \int w \mathrm{d}\xi = \int \frac{f(\eta)}{\xi}\mathrm{d}\xi = f(\eta)\ln\xi + g(\eta).$$

Therefore, the general solution $u(x, y)$ of (3.11) is $u(x, y) = f(xy)\ln x + g(xy)$, where $f, g \in C^2(\mathbb{R})$ are arbitrary real functions.

3.5 Canonical form of elliptic equations

The computation of a canonical coordinate system for the elliptic case is somewhat more subtle than in the hyperbolic case or in the parabolic case. Nevertheless,

under the additional assumption that the coefficients of the principal part of the equation are real analytic functions, the procedure for determining the canonical transformation is quite similar to the one for the hyperbolic case.

Definition 3.11 Let D a planar domain. A function $f : D \to \mathbb{R}$ is said to be *real analytic* in D if for each point $(x_0, y_0) \in D$, we have a convergent power series expansion

$$f(x, y) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_{j,k-j}(x - x_0)^j(y - y_0)^{k-j},$$

valid in some neighborhood N of (x_0, y_0).

Theorem 3.12 *Suppose that (3.1) is elliptic in a planar domain D. Assume further that the coefficients a, b, c are real analytic functions in D. Then there exists a coordinate system (ξ, η) in which the equation has the canonical form*

$$w_{\xi\xi} + w_{\eta\eta} + \ell_1[w] = G(\xi, \eta),$$

where ℓ_1 is a first-order linear differential operator, and G is a function which depends on (3.1).

Proof Without loss of generality we may assume that $a(x, y) \neq 0$ for all $(x, y) \in D$. We are looking for two functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ that satisfy the equations

$$A(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2, \tag{3.12}$$
$$B(\xi, \eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0. \tag{3.13}$$

This is a system of two nonlinear first-order equations. The main difficulty in the elliptic case is that (3.12)–(3.13) are coupled. In order to decouple these equations, we shall use the complex plane and the analyticity assumption. We may write the system (3.12)–(3.13) in the following form:

$$a(\xi_x^2 - \eta_x^2) + 2b(\xi_x\xi_y - \eta_x\eta_y) + c(\xi_y^2 - \eta_y^2) = 0, \tag{3.14}$$
$$a\xi_x\mathrm{i}\eta_x + b(\xi_x\mathrm{i}\eta_y + \xi_y\mathrm{i}\eta_x) + c\xi_y\mathrm{i}\eta_y = 0, \tag{3.15}$$

where $\mathrm{i} = \sqrt{-1}$. Define the complex function $\phi = \xi + \mathrm{i}\eta$. The system (3.14)–(3.15) is equivalent to the complex valued equation

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0.$$

Surprisingly, we have arrived at the same equation as in the hyperbolic case. But in the elliptic case the equation does not admit any real solution, or, in other words, elliptic equations do not have characteristics. As in the hyperbolic case, we factor out the above quadratic PDE, and obtain two linear equations, but now these are complex valued differential equations (where x, y are complex variables!). The nontrivial question of the existence and uniqueness of solutions immediately arises. Fortunately, it is known that if the coefficients of these first-order linear equations are real analytic then it is possible to solve them using the same procedure as in the real case. Moreover, the solutions of the two equations are complex conjugates.

So, we need to solve the equations

$$a\phi_x + (b \pm \mathrm{i}\sqrt{ac-b^2})\phi_y = 0. \tag{3.16}$$

As before, the solutions ϕ, ψ are constant on the "characteristics" (which are defined on the complex plane):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \mathrm{i}\sqrt{ac-b^2}}{a}. \tag{3.17}$$

As in the hyperbolic case, the equation in the new coordinates system has the form

$$4v_{\phi\psi} + \cdots = 0.$$

This is still not the elliptic canonical form with real coefficients. We return to our real variables ξ and η using the linear transformation

$$\xi = \operatorname{Re}\phi, \quad \eta = \operatorname{Im}\phi.$$

Since ξ and η are solutions of the system (3.12)–(3.13), it follows that in the variables ξ and η the equation has the canonical form. In Exercise 3.9 the reader will be asked to prove that the Jacobian of the canonical transformations in the elliptic case and in the hyperbolic case do not vanish. □

Example 3.13 Consider the Tricomi equation:

$$u_{xx} + xu_{yy} = 0, \qquad x > 0. \tag{3.18}$$

Find a canonical transformation $q = q(x, y)$, $r = r(x, y)$ and the corresponding canonical form.

The differential equations for the "characteristics" are $\mathrm{d}y/\mathrm{d}x = \pm\sqrt{-x}$, and their solutions are $\frac{3}{2}y \pm \mathrm{i}(x)^{3/2} =$ constant. Therefore, the canonical variables are $q(x, y) = \frac{3}{2}y$ and $r(x, y) = -(x)^{3/2}$. Clearly,

$$q_x = 0,\ q_y = \frac{3}{2} \qquad r_x = -\frac{3}{2}(x)^{1/2},\ r_y = 0.$$

Set $v(q, r) = u(x, y)$. Hence,

$$u_x = -\frac{3}{2}(x)^{1/2}v_r, \qquad u_y = \frac{3}{2}v_q,$$
$$u_{xx} = \frac{9}{4}xv_{rr} - \frac{3}{4}(x)^{-1/2}v_r, \qquad u_{yy} = \frac{9}{4}v_{qq}.$$

Substituting these into the Tricomi equation we obtain the canonical form

$$\frac{1}{x}u_{xx} + u_{yy} = \frac{9}{4}\left(v_{qq} + v_{rr} + \frac{1}{3r}v_r\right) = 0.$$

3.6 Exercises

3.1 Consider the equation

$$u_{xx} - 6u_{xy} + 9u_{yy} = xy^2.$$

(a) Find a coordinates system (s, t) in which the equation has the form:

$$9v_{tt} = \tfrac{1}{3}(s-t)t^2.$$

(b) Find the general solution $u(x, y)$.
(c) Find a solution of the equation which satisfies the initial conditions $u(x, 0) = \sin x$, $u_y(x, 0) = \cos x$ for all $x \in \mathbb{R}$.

3.2 (a) Show that the following equation is hyperbolic:

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0.$$

(b) Find the canonical form of the equation.
(c) Find the general solution $u(x, y)$.
(d) Find a solution $u(x, y)$ that satisfies $u(-x, 2x) = x$ and $u(x, 0) = \sin 2x$.

3.3 Consider the equation

$$u_{xx} + 4u_{xy} + u_x = 0.$$

(a) Bring the equation to a canonical form.
(b) Find the general solution $u(x, y)$ and check by substituting back into the equation that your solution is indeed correct.
(c) Find a specific solution satisfying

$$u(x, 8x) = 0, \qquad u_x(x, 8x) = 4\mathrm{e}^{-2x}.$$

3.4 Consider the equation

$$y^5u_{xx} - yu_{yy} + 2u_y = 0, \qquad y > 0.$$

(a) Find the canonical form of the equation.
(b) Find the general solution $u(x, y)$ of the equation.

(c) Find the solution $u(x, y)$ which satisfies $u(0, y) = 8y^3$, and $u_x(0, y) = 6$, for all $y > 0$.

3.5 Consider the equation

$$xu_{xx} - yu_{yy} + \tfrac{1}{2}(u_x - u_y) = 0.$$

(a) Find the domain where the equation is elliptic, and the domain where it is hyperbolic
(b) For each of the above two domains, find the corresponding canonical transformation.

3.6 Consider the equation

$$u_{xx} + (1 + y^2)^2 u_{yy} - 2y(1 + y^2)u_y = 0.$$

(a) Find the canonical form of the equation.
(b) Find the general solution $u(x, y)$ of the equation.
(c) Find the solution $u(x, y)$ which satisfies $u(x, 0) = g(x)$, and $u_y(x, 0) = f(x)$, where $f, g \in C^2(\mathbb{R})$.
(d) Find the solution $u(x, y)$ for $f(x) = -2x$, and $g(x) = x$.

3.7 Consider the equation

$$u_{xx} + 2u_{xy} + [1 - q(y)]u_{yy} = 0,$$

where

$$q(y) = \begin{cases} -1 & y < -1, \\ 0 & |y| \le 1, \\ 1 & y > 1. \end{cases}$$

(a) Find the domains where the equation is hyperbolic, parabolic, and elliptic.
(b) For each of the above three domains, find the corresponding canonical transformation and the canonical form.
(c) Draw the characteristics for the hyperbolic case.

3.8 Consider the equation

$$4y^2 u_{xx} + 2(1 - y^2)u_{xy} - u_{yy} - \frac{2y}{1 + y^2}(2u_x - u_y) = 0.$$

(a) Find the canonical form of the equation.
(b) Find the general solution $u(x, y)$ of the equation.
(c) Find the solution $u(x, y)$ which satisfies $u(x, 0) = g(x)$, and $u_y(x, 0) = f(x)$, where $f, g \in C^2(\mathbb{R})$ are arbitrary functions.

3.9 (a) Prove that in the hyperbolic case the canonical transformation is nonsingular ($J \neq 0$).
(b) Prove that in the elliptic case the canonical transformation is nonsingular ($J \neq 0$).

3.10 Consider the equation

$$u_{xx} - 2u_{xy} + 4e^y = 0.$$

(a) Find the canonical form of the equation.
(b) Find the solution $u(x, y)$ which satisfies $u(0, y) = f(y)$, and $u_x(0, y) = g(y)$.

3.11 In continuation of Example 3.8, consider the equation

$$u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0.$$

(a) Find a solution of the equation which satisfies $u(0, y) = f(y)$, $u_x(0, y) = g(y)$, where f, g are given functions.
(b) Find conditions on f and g such that the solution $u(x, y)$ of part (a) is a classical solution.

3.12 Consider the equation

$$u_{xx} + yu_{yy} = 0.$$

Find the canonical forms of the equation for the domain where the equation is hyperbolic, and for the domain where it is elliptic.

4

The one-dimensional wave equation

4.1 Introduction

In this chapter we study the one-dimensional wave equation on the real line. The canonical form of the wave equation will be used to show that the Cauchy problem is well-posed. Moreover, we shall derive simple explicit formulas for the solutions. We also discuss some important properties of the solutions of the wave equation which are typical for more general hyperbolic problems as well.

4.2 Canonical form and general solution

The homogeneous wave equation in one (spatial) dimension has the form

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty \le a < x < b \le \infty,\ t > 0, \tag{4.1}$$

where $c \in \mathbb{R}$ is called the *wave speed*, a terminology that will be justified in the discussion below.

To obtain the canonical form of the wave equation, define the new variables

$$\xi = x + ct \qquad \eta = x - ct,$$

and set $w(\xi,\eta)=u(x(\xi,\eta),t(\xi,\eta))$ (see Section 3.3 for the method to obtain this canonical transformation). Using the chain rule for the function $u(x,t) = w(\xi(x,t),\eta(x,t))$, we obtain

$$u_t = w_\xi \xi_t + w_\eta \eta_t = c(w_\xi - w_\eta), \qquad u_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi + w_\eta,$$

and

$$u_{tt} = c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}), \qquad u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}.$$

Hence,

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta} = 0.$$

This is the canonical form for the wave equation. Since $(w_\xi)_\eta = 0$, it follows that $w_\xi = f(\xi)$, and then $w = \int f(\xi)\,d\xi + G(\eta)$. Therefore, the general solution of the equation $w_{\xi\eta} = 0$ has the form

$$w(\xi,\eta) = F(\xi) + G(\eta),$$

where $F, G \in C^2(\mathbb{R})$ are two arbitrary functions. Thus, in the original variables, the general solution of the wave equation is

$$u(x,t) = F(x+ct) + G(x-ct). \tag{4.2}$$

In other words, if u is a solution of the one-dimensional wave equation, then there exist two real functions $F, G \in C^2$ such that (4.2) holds. Conversely, any two functions $F, G \in C^2$ define a solution of the wave equation via formula (4.2).

For a fixed $t_0 > 0$, the graph of the function $G(x - ct_0)$ has the same shape as the graph of the function $G(x)$, except that it is shifted to the right by a distance ct_0. Therefore, the function $G(x - ct)$ represents a wave moving to the right with velocity c, and it is called a *forward wave*. The function $F(x + ct)$ is a wave traveling to the left with the same speed, and it is called a *backward wave*. Indeed c can be called the *wave speed*.

Equation (4.2) demonstrates that any solution of the wave equation is the sum of two such traveling waves. This observation will enable us to obtain graphical representations of the solutions (the graphical method).

We would like to extend the validity of (4.2). Observe that for any two real piecewise continuous functions F, G, (4.2) defines a piecewise continuous function u that is a superposition of a forward wave and a backward wave traveling in opposite directions with speed c. Moreover, it is possible to find two sequences of smooth functions, $\{F_n(s)\}$, $\{G_n(s)\}$, converging at any point to F and G, respectively, which converge uniformly to these functions in any bounded and closed interval that does not contain points of discontinuity. The function

$$u_n(x,t) = F_n(x+ct) + G_n(x-ct)$$

is a proper solution of the wave equation, but the limiting function $u(x,t) = F(x+ct) + G(x-ct)$ is not necessarily twice differentiable, and therefore might not be a solution. We call a function $u(x,t)$ that satisfies (4.2) with piecewise continuous functions F, G a *generalized solution* of the wave equation.

Let us further discuss the general solution (4.2). Consider the (x,t) plane. The following two families of lines

$$x - ct = \text{constant}, \qquad x + ct = \text{constant},$$

are called the *characteristics* of the wave equation (see Section 3.3). For the wave equation, the characteristics are straight lines in the (x,t) plane with slopes $\pm 1/c$.

It turns out that as for first-order PDEs, the "information" is transferred via these curves.

We arrive now at one of the most important properties of the characteristics. Assume that for a fixed time t_0, the solution u is a smooth function except at one point (x_0, t_0). Clearly, either F is not smooth at $x_0 + ct_0$, and/or the function G is not smooth at $x_0 - ct_0$. There are two characteristics that pass through the point (x_0, t_0); these are the lines

$$x - ct = x_0 - ct_0, \quad x + ct = x_0 + ct_0.$$

Consequently, for any time $t_1 \neq t_0$ the solution u is smooth except at one or two points $x_\pm$ that satisfy

$$x_- - ct_1 = x_0 - ct_0, \quad x_+ + ct_1 = x_0 + ct_0.$$

Therefore, the singularities (nonsmoothness) of solutions of the wave equation are traveling only along characteristics. This phenomenon is typical of hyperbolic equations in general: a singularity is not smoothed out; rather it travels at a finite speed. This is in contrast to parabolic and elliptic equations, where, as will be shown in the following chapters, singularities are immediately smoothed out.

Example 4.1 Let $u(x, t)$ be a solution of the wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

which is defined in the whole plane. Assume that u is constant on the line $x = 2 + ct$. Prove that $u_t + cu_x = 0$.

The solution $u(x, t)$ has the form $u(x, t) = F(x + ct) + G(x - ct)$. Since $u(2 + ct, t) =$ constant, it follows that

$$F(2 + 2ct) + G(2) = \text{constant}.$$

Set $s = 2 + 2ct$, we have $F(s) =$ constant. Consequently $u(x, t) = G(x - ct)$. Computing now the expression $u_t + cu_x$, we obtain

$$u_t + cu_x = -cG'(x - ct) + cG'(x - ct) = 0.$$

4.3 The Cauchy problem and d'Alembert's formula

The *Cauchy problem* for the one-dimensional homogeneous wave equation is given by

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < \infty,\ t > 0, \tag{4.3}$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty. \tag{4.4}$$

A solution of this problem can be interpreted as the amplitude of a sound wave propagating in a very long and narrow pipe, which in practice can be considered as a one-dimensional infinite medium. This system also represents the vibration of an infinite (ideal) string. The initial conditions f, g are given functions that represent the amplitude u, and the velocity u_t of the string at time $t = 0$.

A *classical (proper) solution* of the Cauchy problem (4.3)–(4.4) is a function u that is continuously twice differentiable for all $t > 0$, such that u and u_t are continuous in the half-space $t \geq 0$, and such that (4.3)–(4.4) are satisfied. Generally speaking, classical solutions should have the minimal smoothness properties in order to satisfy continuously all the given conditions in the classical sense.

Recall that the general solution of the wave equation is of the form

$$u(x, t) = F(x + ct) + G(x - ct). \tag{4.5}$$

Our aim is to find F and G such that the initial conditions of (4.4) are satisfied. Substituting $t = 0$ into (4.5) we obtain

$$u(x, 0) = F(x) + G(x) = f(x). \tag{4.6}$$

Differentiating (4.5) with respect to t and substituting $t = 0$, we have

$$u_t(x, 0) = cF'(x) - cG'(x) = g(x). \tag{4.7}$$

Integration of (4.7) over the integral $[0, x]$ yields

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(s)\,ds + C, \tag{4.8}$$

where $C = F(0) - G(0)$. Equations (4.6) and (4.8) are two linear algebraic equations for $F(x)$ and $G(x)$. The solution of this system of equations is given by

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{C}{2}, \tag{4.9}$$
$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{C}{2}. \tag{4.10}$$

By substituting these expressions for F and G into the general solution (4.5), we obtain the formula

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, \tag{4.11}$$

which is called *d'Alembert's formula*. Note that sometimes (4.9)–(4.10) are also useful, as they give us explicit formulas for the forward and the backward waves.

The following examples illustrate the use of d'Alembert's formula.

Example 4.2 Consider the Cauchy problem

$$u_{tt} - u_{xx} = 0 \qquad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) = \begin{cases} 0 & -\infty < x < -1, \\ x+1 & -1 \le x \le 0, \\ 1-x & 0 \le x \le 1, \\ 0 & 1 < x < \infty, \end{cases}$$

$$u_t(x,0) = g(x) = \begin{cases} 0 & -\infty < x < -1, \\ 1 & -1 \le x \le 1, \\ 0 & 1 < x < \infty. \end{cases}$$

(a) Evaluate u at the point $(1, \frac{1}{2})$.
(b) Discuss the smoothness of the solution u.

(a) Using d'Alembert's formula, we find that

$$u(1, \tfrac{1}{2}) = \frac{f(\frac{3}{2}) + f(\frac{1}{2})}{2} + \frac{1}{2}\int_{\frac{1}{2}}^{\frac{3}{2}} g(s)\,ds.$$

Since $\frac{3}{2} > 1$ it follows that $f(\frac{3}{2}) = 0$. On the other hand, $0 \le \frac{1}{2} \le 1$; therefore, $f(\frac{1}{2}) = \frac{1}{2}$. Evidently, $\int_{\frac{1}{2}}^{\frac{3}{2}} g(s)ds = \int_{\frac{1}{2}}^{1} 1ds = \frac{1}{2}$. Thus, $u(1, \frac{1}{2}) = \frac{1}{2}$.
(b) The solution is not classical, since $u \notin C^1$. Yet u is a generalized solution of the problem. Note that although g is not continuous, nevertheless the solution u is a continuous function. The singularities of the solution propagate along characteristics that intersect the initial line $t = 0$ at the singularities of the initial conditions. These are exactly the characteristics $x \pm t = -1, 0, 1$. Therefore, the solution is smooth in a neighborhood of the point $(1, \frac{1}{2})$ which does not intersect these characteristics.

Example 4.3 Let $u(x,t)$ be the solution of the Cauchy problem

$$u_{tt} - 9u_{xx} = 0 \qquad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) = \begin{cases} 1 & |x| \le 2, \\ 0 & |x| > 2, \end{cases}$$

$$u_t(x,0) = g(x) = \begin{cases} 1 & |x| \le 2, \\ 0 & |x| > 2. \end{cases}$$

(a) Find $u(0, \frac{1}{6})$.
(b) Discuss the large time behavior of the solution.

(c) Find the maximal value of $u(x,t)$, and the points where this maximum is achieved.
(d) Find all the points where $u \in C^2$.

(a) Since

$$u(x,t) = \frac{f(x+3t) + f(x-3t)}{2} + \frac{1}{6}\int_{x-3t}^{x+3t} g(s)ds,$$

it follows that for $x = 0$ and $t = \frac{1}{6}$, we have

$$u(0, \frac{1}{6}) = \frac{f(\frac{1}{2}) + f(-\frac{1}{2})}{2} + \frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(s)\,ds = \frac{1+1}{2} + \frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1ds = \frac{7}{6}.$$

(b) Fix $\xi \in \mathbb{R}$ and compute $\lim_{t\to\infty} u(\xi, t)$. Clearly,

$$\lim_{t\to\infty} f(\xi + 3t) = 0, \quad \lim_{t\to\infty} f(\xi - 3t) = 0, \quad \lim_{t\to\infty} \int_{\xi-3t}^{\xi+3t} g(s)\,ds = \int_{-2}^{2} 1ds = 4.$$

Therefore, $\lim_{t\to\infty} u(\xi, t) = \frac{2}{3}$.
(c) Recall that for any real functions f, g,

$$\max\{f(x) + g(x)\} \le \max f(x) + \max g(x).$$

It turns out that in our special case there exists a point (x,t), where all the terms in (4.11) attain their maximal value simultaneously, and therefore at such a point the maximum of u is attained.

Indeed, $\max\{f(x+3t)\} = 1$ which is attained on the strip $-2 \le x + 3t \le 2$. Similarly, $\max\{f(x-3t)\} = 1$ which is attained on the strip $-2 \le x - 3t \le 2$, while $\max\{\int_{x-3t}^{x+3t} g(s)\,ds = \int_{-2}^{2} 1ds = 4$, and it is attained on the intersection of the half-planes $x + 3t \ge 2$ and $x - 3t \le -2$. The intersection of all these sets is the set of all points that satisfy the two equations

$$x + 3t = 2,$$
$$x - 3t = -2.$$

This system has a unique solution at $(x,t) = (0, \frac{2}{3})$. Thus, the solution u achieves its maximum at the point $(0, \frac{2}{3})$, where $u(0, \frac{2}{3}) = \frac{5}{3}$.
(d) The initial conditions are smooth except at the points $x = \pm 2$. Therefore, the solution is smooth at all points that are *not* on the characteristics

$$x \pm 3t = -2, \quad x \pm 3t = 2.$$

The function u is a generalized solution that is piecewise continuous for any fixed time $t > 0$.

The well-posedness of the Cauchy problem follows from the d'Alembert formula.

Theorem 4.4 *Fix $T > 0$. The Cauchy problem (4.3)–(4.4) in the domain $-\infty < x < \infty,\ 0 \le t \le T$ is well-posed for $f \in C^2(\mathbb{R}),\ g \in C^1(\mathbb{R})$.*

Proof The existence and uniqueness follow directly from the d'Alembert formula. Indeed, this formula provides us with a solution, and we have shown that any solution of the Cauchy problem is necessarily equal to the d'Alembert solution. Note that from our smoothness assumption ($f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$), it follows that $u \in C^2(\mathbb{R} \times (0,\infty)) \cap C^1(\mathbb{R} \times [0,\infty))$, and therefore, the d'Alembert solution is a classical solution. On the other hand, for $f \in C(\mathbb{R})$ and g that is locally integrable, the d'Alembert solution is a generalized solution.

It remains to prove the stability of the Cauchy problem, i.e. we need to show that small changes in the initial conditions give rise to a small change in the solution. Let u_i be two solutions of the Cauchy problem with initial conditions f_i, g_i, where $i = 1, 2$. Now, if

$$|f_1(x) - f_2(x)| < \delta, \qquad |g_1(x) - g_2(x)| < \delta,$$

for all $x \in \mathbb{R}$, then for all $x \in \mathbb{R}$ and $0 \le t \le T$ we have

$$|u_1(x,t) - u_2(x,t)| \le \frac{|f_1(x+ct) - f_2(x+ct)|}{2} + \frac{|f_1(x-ct) - f_2(x-ct)|}{2}$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} |g_1(s) - g_2(s)|\,\mathrm{d}s < \frac{1}{2}(\delta + \delta) + \frac{1}{2c}2ct\delta \le (1+T)\delta.$$

Therefore, for a given $\varepsilon > 0$, we take $\delta < \varepsilon/(1+T)$. Then for all $x \in \mathbb{R}$ and $0 \le t \le T$ we have

$$|u_1(x,t) - u_2(x,t)| < \varepsilon.$$

□

Remark 4.5 (1) The Cauchy problem is ill-posed on the domain $-\infty < x < \infty$, $t \ge 0$.
(2) The d'Alembert formula is also valid for $-\infty < x < \infty$, $T < t \le 0$, and the Cauchy problem is also well-posed in this domain. The physical interpretation is that the process is reversible.

4.4 Domain of dependence and region of influence

Let us return to the Cauchy problem (4.3)–(4.4), and examine what is the information that actually determines the solution u at a fixed point (x_0, t_0). Consider the (x,t) plane and the two characteristics passing through the point (x_0, t_0):

$$x - ct = x_0 - ct_0, \qquad x + ct = x_0 + ct_0.$$

These straight lines intersect the x axis at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$, respectively. The triangle formed by the these characteristics and the interval $[x_0 - ct_0, x_0 + ct_0]$ is called a *characteristic triangle* (see Figure 4.1).

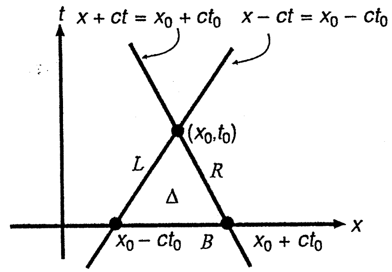

Figure 4.1 Domain of dependence.

By the d'Alembert formula

$$u(x_0, t_0) = \frac{f(x_0 + ct_0) + f(x_0 - ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s. \tag{4.12}$$

Therefore, the value of u at the point (x_0, t_0) is determined by the values of f at the vertices of the characteristic base and by the values of g along this base. Thus, $u(x_0, t_0)$ depends only on the part of the initial data that is given on the interval $[x_0 - ct_0, x_0 + ct_0]$. Therefore, this interval is called *domain of dependence* of u at the point (x_0, t_0). If we change the initial data at points outside this interval, the value of the solution u at the point (x_0, t_0) will not change. Information on a change in the data travels with speed c along the characteristics, and therefore such information is not available for $t \le t_0$ at the point x_0. The change will finally influence the solution at the point x_0 at a later time. Hence, for every point (x,t) in a fixed characteristic triangle, $u(x,t)$ is determined only by the initial data that are given on (part of) the characteristic base (see Figure 4.1). Furthermore, if the initial data are smooth on this base, then the solution is smooth in the whole triangle.

We may ask now the opposite question: which are the points on the half-plane $t > 0$ that are influenced by the initial data on a fixed interval $[a, b]$? The set of all such points is called the *region of influence* of the interval $[a, b]$. It follows from the discussion above that the points of this interval influence the value of the solution u at a point (x_0, t_0) if and only if $[x_0 - ct_0, x_0 + ct_0] \cap [a, b] \neq \emptyset$. Hence the initial data along the interval $[a, b]$ influence only points (x, t) satisfying

$$x - ct \le b, \quad \text{and} \quad x + ct \ge a.$$

These are the points inside the forward (truncated) characteristic cone that is defined by the base $[a, b]$ and the edges $x + ct = a$, $x - ct = b$ (it is the union of the regions I–IV of Figure 4.2).

Assume, for instance, that the initial data f, g vanish outside the interval $[a, b]$. Then the amplitude of the vibrating string is zero at every point outside the influence

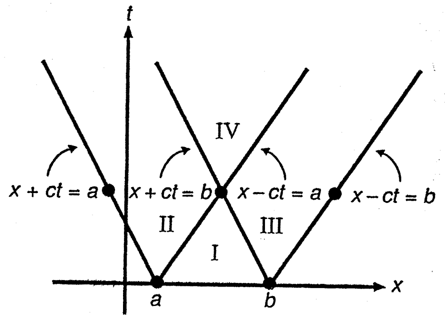

Figure 4.2 Region of influence.

region of this interval. On the other hand, for a fixed point x_0 on the string, the effect of the perturbation (from the zero data) along the interval $[a, b]$ will be felt after a time $t_0 \geq 0$, and eventually, for t large enough, the solution takes the constant value $u(x_0, t) = (1/2c)\int_a^b g(s)\,ds$. This occurs precisely at points (x_0, t) that are inside the cone

$$x_0 - ct \leq a, \quad \text{and} \quad x_0 + ct \geq b,$$

(see region IV in Figure 4.2).

Using these observations, we demonstrate in the following example the so-called *graphical method* for solving the Cauchy problem for the wave equation.

Example 4.6 Consider the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = f(x) = \begin{cases} 2 & |x| \leq a, \\ 0 & |x| > a, \end{cases}$$

$$u_t(x,0) = g(x) = 0 \qquad -\infty < x < \infty.$$

Draw the graphs of the solution $u(x, t)$ at times $t_i = ia/2c$, where $i = 0, 1, 2, 3$. Using d'Alembert's formula, we write the solution u as a sum of backward and forward waves

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}.$$

Since these waves are piecewise constant functions, it is clear that for each t, the solution u is also a piecewise constant function of x with values $u = 0, 1, 2$.

Consider the (x, t) plane. We draw the characteristic lines that pass through the special points on the initial line $t = 0$ where the initial data are not smooth. In

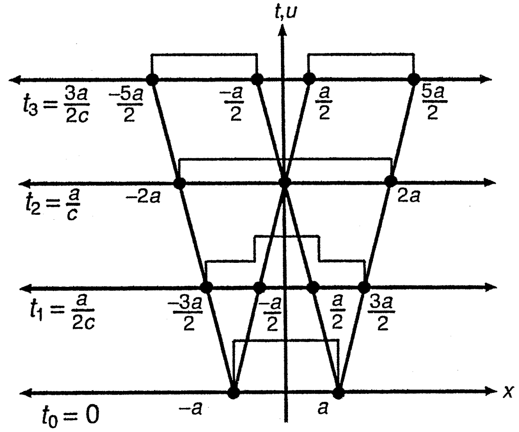

Figure 4.3 The graphical method.

the present problem these are the points $x = \pm a$. We also draw the lines $t = t_i$ that will serve us as the abscissas (x axes) for the graphs of the functions $u(x, t_i)$. Note that the ordinate of the coordinate system is used as the t and the u axes (see Figure 4.3).

Consider the time $t = t_1$. The forward wave has traveled $a/2$ units to the right, and the backward wave has traveled $a/2$ units to the left. The support (the set of points where the function is not zero) of the forward wave at time t_1 is the interval $[-a/2, 3a/2]$, while $[-3a/2, a/2]$ is the support of the backward wave at t_1. Therefore, the support of the solution at t_1 is the interval $[-3a/2, 3a/2]$, i.e. the region of influence of $[-a, a]$ at $t = t_1$. Now, at the intersection of the supports of the two waves (the interval $[-a/2, a/2]$) u takes the value $1 + 1 = 2$, while on the intervals $[-3a/2, -a/2)$, $(a/2, 3a/2]$, where the supports do not intersect, u takes the value 1. Obviously, $u = 0$ at all other points.

Consider the time $t_2 = a/c$. The support of the forward (backward) wave is $[0, 2a]$ ($[-2a, 0]$, respectively). Consequently, the support of the solution u is $[-2a, 2a]$, i.e. the region of influence of $[-a, a]$ at $t = t_2$. The intersection of the supports of the two waves is the point $x = 0$, where u takes the value 2. On the intervals $[-2a, 0)$, $(0, 2a]$, u is 1. Obviously, $u = 0$ at all other points.

At the time $t_3 = 3a/2c$, the support of the forward (backward) wave is $[a/2, 5a/2]$ ($[-5a/2, -a/2]$, respectively), and there is no interaction between the waves. Therefore, the solution at these intervals equals 1, and it equals zero otherwise.

To conclude, the first step of the graphical method is to compute and to draw the graphs of the forward and backward waves. Then, for a given time t, we shift these two shapes to the right and, respectively, to the left by ct units. Finally, we add the two graphs.

In the next example we use the graphical method to investigate the influence of the initial velocity on the solution.

Example 4.7 Find the graphs of the solution $u(x, t_i)$, $t_i = i$, $i = 1, 4$ for the problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0 && -\infty < x < \infty,\ t > 0,\\ &u(x,0) = f(x) = 0, && -\infty < x < \infty,\\ &u_t(x,0) = g(x) = \begin{cases} 0 & x < 0,\\ 1 & x \geq 0. \end{cases} \end{aligned}$$

We apply d'Alembert's formula to write the solution as the sum of forward and backward waves:

$$u(x,t) = \frac{1}{2}\int_{x-t}^{0} g(s)\,\mathrm{d}s + \frac{1}{2}\int_{0}^{x+t} g(s)\,\mathrm{d}s = -\frac{\max\{0, x-t\}}{2} + \frac{\max\{0, x+t\}}{2}.$$

Since both the forward and backward waves are piecewise linear functions, the solution $u(\cdot, t)$ for all times t is a piecewise linear function of x.

We draw in the plane (x, t) the characteristics emanating from the points where the initial condition is nonsmooth. In our case this happens at just one point, namely $x = 0$. We also depict the lines $t = t_i$ that form the abscissas for the graph of $u(x, t_i)$ (see Figure 4.4).

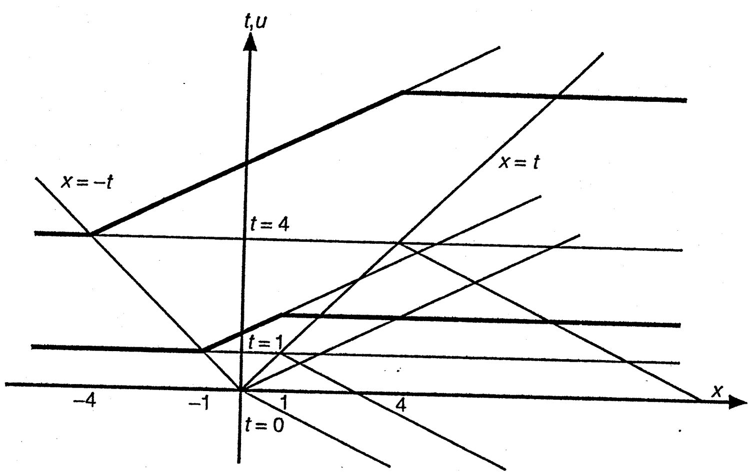

Figure 4.4 The graphical solution for Example 4.7.

At time t_1, the forward wave has moved one unit to the right, and the backward wave has moved one unit to the left. The forward wave is supported on the interval $[1, \infty)$, while the backward wave is supported on $[-1, \infty)$. Therefore the solution is supported on $[-1, \infty)$. It is clear that on the interval $[-1, 1]$ the solution forms a linear function thanks to the backward wave. Specifically, $u = (x + 1)/2$ there. In the interval $[1, \infty)$ the forward wave is a linear function with slope $-1/2$, while the backward wave is a linear function with slope $1/2$. Therefore the solution itself, being a superposition of these two waves, is constant $u = 1$ there.

A similar consideration can be used to draw the graph for $t = 4$. Actually, the solution can be written explicitly as

$$u(x,t) = \begin{cases} 0 & x < -t,\\ \dfrac{x+t}{2} & -t \leq x \leq t,\\ t & x > t. \end{cases}$$

Notice that for each fixed $x_0 \in \mathbb{R}$, the solution $u(x_0, t)$ (considered as a function of t) is not bounded.

4.5 The Cauchy problem for the nonhomogeneous wave equation

Consider the following Cauchy problem

$$u_{tt} - c^2 u_{xx} = F(x,t) \qquad -\infty < x < \infty,\ t > 0, \tag{4.13}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \qquad -\infty < x < \infty. \tag{4.14}$$

This problem models, for example, the vibration of a very long string in the presence of an external force F. As in the homogeneous case f, g are given functions that represent the shape and the vertical velocity of the string at time $t = 0$.

As in every linear problem, the uniqueness for the homogeneous problem implies the uniqueness for the nonhomogeneous problem.

Proposition 4.8 *The Cauchy problem (4.13)–(4.14) admits at most one solution.*

Proof Assume that u_1, u_2 are solutions of problem (4.13)–(4.14). We should prove that $u_1 = u_2$. The function $u = u_1 - u_2$ is a solution of the homogeneous problem

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty, < x < \infty,\ t > 0, \tag{4.15}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0 \qquad -\infty < x < \infty. \tag{4.16}$$

On the other hand, $v(x, t) = 0$ is also a solution of the same (homogeneous) problem. By Theorem 4.4, $u = v = 0$, hence, $u_1 = u_2$. □

Next, using an explicit formula, we prove, as in the homogeneous case, the existence of a solution of the Cauchy problem (4.13)–(4.14). For this purpose, recall Green's formula for a pair of functions P, Q in a planar domain Ω with a piecewise smooth boundary Γ:

$$\iint_\Omega [Q(x,t)_x - P(x,t)_t]\,\mathrm{d}x\,\mathrm{d}t = \oint_\Gamma [P(x,t)\,dx + Q(x,t)\,\mathrm{d}t].$$

Let $u(x,t)$ be a solution of problem (4.13)–(4.14). Integrate the two sides of the PDE (4.13) over a characteristic triangle Δ with a fixed upper vertex (x_0, t_0). The three edges of this triangle (base, right and left edges) will be denoted by B, R, L, respectively (see Figure 4.1). We have

$$-\iint_\Delta F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint_\Delta (c^2u_{xx} - u_{tt})\,\mathrm{d}x\,\mathrm{d}t.$$

Using Green's formula with $Q = c^2u_x$ and $P = u_t$, we obtain

$$-\iint_\Delta F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \oint_\Gamma (u_t\,\mathrm{d}x + c^2u_x\,\mathrm{d}t) = \int_B + \int_R + \int_L (u_t\,\mathrm{d}x + c^2u_x\,\mathrm{d}t).$$

On the base B we have $\mathrm{d}t = 0$; therefore, using the initial conditions, we get

$$\int_B (u_t\,\mathrm{d}x + c^2u_x\,\mathrm{d}t) = \int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,\mathrm{d}x = \int_{x_0-ct_0}^{x_0+ct_0} g(x)\,\mathrm{d}x.$$

On the right edge R, $x + ct = x_0 + ct_0$, hence $\mathrm{d}x = -c\mathrm{d}t$. Consequently,

$$\int_R (u_t\,\mathrm{d}x + c^2u_x\,\mathrm{d}t) = -c\int_R (u_t\,\mathrm{d}t + u_x\,\mathrm{d}x) = -c\int_R \mathrm{d}u$$
$$= -c[u(x_0,t_0) - u(x_0+ct_0,0)] = -c[u(x_0,t_0) - f(x_0+ct_0)].$$

Similarly, on the left edge L, $x - ct = x_0 - ct_0$, implying $\mathrm{d}x = c\mathrm{d}t$, and

$$\int_L (u_t\,\mathrm{d}x + c^2u_x\,\mathrm{d}t) = c\int_L (u_t\,\mathrm{d}t + u_x\,\mathrm{d}x) = c\int_L \mathrm{d}u$$
$$= c[u(x_0-ct_0,0) - u(x_0,t_0)] = c[f(x_0-ct_0) - u(x_0,t_0)].$$

Therefore,

$$-\iint_\Delta F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{x_0-ct_0}^{x_0+ct_0} g(x)\,\mathrm{d}x + c[f(x_0-ct_0) + f(x_0+ct_0) - 2u(x_0,t_0)].$$

Solving for u gives

$$u(x_0,t_0) = \frac{f(x_0+ct_0) + f(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,\mathrm{d}x + \frac{1}{2c}\iint_\Delta F(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

We finally obtain an explicit formula for the solution at an arbitrary point (x,t):

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s + \frac{1}{2c}\iint_\Delta F(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{4.17}$$

This formula is also called d'Alembert's formula.

Remark 4.9 (1) Note that for $F = 0$ the two d'Alembert's formulas coincide, and actually, we have obtained another proof of the d'Alembert formula (4.11).
(2) The value of u at a point (x_0, t_0) is determined by the values of the given data on the *whole* characteristic triangle whose upper vertex is the point (x_0, t_0). This is the domain of dependence for the nonhomogeneous Cauchy problem.

It remains to prove that the function u in (4.17) is indeed a solution of the Cauchy problem. From the superposition principle it follows that u in (4.17) is the desired solution, if and only if the function

$$v(x,t) = \frac{1}{2c}\iint_\Delta F(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

is a solution of the Cauchy problem

$$u_{tt} - c^2u_{xx} = F(x,t) \qquad -\infty < x < \infty,\ t > 0, \tag{4.18}$$
$$u(x,0) = 0, \quad u_t(x,0) = 0 \qquad -\infty < x < \infty. \tag{4.19}$$

We shall prove that v is a solution of the initial value problem (4.18)–(4.19) under the assumption that F and F_x are continuous. Clearly,

$$v(x,0) = 0.$$

In order to take derivatives, we shall use the formula

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} G(\xi,t)\,\mathrm{d}\xi = G(b(t),t)b'(t) - G(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}G(\xi,t)\,\mathrm{d}\xi.$$

Hence,

$$v_t(x,t) = \frac{1}{2c}\int_x^x F(\xi,t)\,\mathrm{d}\xi + \frac{1}{2}\int_0^t [F(x+c(t-\tau),\tau) + F(x-c(t-\tau),\tau)]\,\mathrm{d}\tau$$
$$= \frac{1}{2}\int_0^t [F(x+c(t-\tau),\tau) + F(x-c(t-\tau),\tau)]\,\mathrm{d}\tau.$$

In particular,

$$v_t(x,0) = 0.$$

By taking the second derivative with respect to t, we have

$$v_{tt}(x,t) = F(x,t) + \frac{c}{2}\int_0^t [F_x(x + c(t-\tau), \tau) - F_x(x - c(t-\tau), \tau)]\, d\tau.$$

Similarly,

$$v_x(x,t) = \frac{1}{2c}\int_0^t [F(x + c(t-\tau), \tau) - F(x - c(t-\tau), \tau)]\, d\tau,$$

$$v_{xx}(x,t) = \frac{1}{2c}\int_0^t [F_x(x + c(t-\tau), \tau) - F_x(x - c(t-\tau), \tau)]\, d\tau.$$

Therefore, $v(x,t)$ is a solution of the nonhomogeneous wave equation (4.18), and the homogeneous initial conditions (4.19) are satisfied. Note that all the above differentiations are justified provided that $F, F_x \in C(\mathbb{R}^2)$.

Theorem 4.10 *Fix $T > 0$. The Cauchy problem (4.13)–(4.14) in the domain $-\infty < x < \infty$, $0 \le t \le T$ is well-posed for $F, F_x \in C(\mathbb{R}^2)$, $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$.*

Proof Recall that the uniqueness has already been proved, and the existence follows from d'Alembert's formula. It remains to prove stability, i.e. we need to show that small changes in the initial conditions and the external force give rise to a small change in the solution. For $i = 1, 2$, let u_i be the solution of the Cauchy problem with the corresponding function F_i, and the initial conditions f_i, g_i. Now, if

$$|F_1(x,t) - F_2(x,t)| < \delta, \quad |f_1(x) - f_2(x)| < \delta, \quad |g_1(x) - g_2(x)| < \delta,$$

for all $x \in \mathbb{R}$, $0 \le t \le T$, then for all $x \in \mathbb{R}$ and $0 \le t \le T$ we have

$$|u_1(x,t) - u_2(x,t)| \le \frac{|f_1(x+ct) - f_2(x+ct)|}{2} + \frac{|f_1(x-ct) - f_2(x-ct)|}{2}$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} |g_1(s) - g_2(s)|\, ds + \frac{1}{2c}\iint_\Delta |F_1(\xi,\tau) - F_2(\xi,\tau)|\, d\xi\, d\tau$$
$$< \frac{1}{2}(\delta + \delta) + \frac{1}{2c}2ct\delta + \frac{1}{2c}ct^2\delta \le (1 + T + T^2/2)\delta.$$

Therefore, for a given $\varepsilon > 0$, we choose $\delta < \varepsilon/(1 + T + T^2/2)$. Thus, for all $x \in \mathbb{R}$ and $0 \le t \le T$, we have

$$|u_1(x,t) - u_2(x,t)| < \varepsilon.$$

Note that δ does not depend on the wave speed c. □

Corollary 4.11 *Suppose that f, g are even functions, and for every $t \ge 0$ the function $F(\cdot, t)$ is even too. Then for every $t \ge 0$ the solution $u(\cdot, t)$ of the Cauchy problem (4.13)–(4.14) is also even. Similarly, the solution is an odd function or a periodic function with a period L (as a function of x) if the data are odd functions or periodic functions with a period L.*

Proof We prove the first part of the corollary. The other parts can be shown similarly. Let u be the solution of the problem and define the function $v(x,t) = u(-x,t)$. Clearly,

$$v_x(x,t) = -u_x(-x,t), \quad v_t(x,t) = u_t(-x,t)$$

and

$$v_{xx}(x,t) = u_{xx}(-x,t), \quad v_{tt}(x,t) = u_{tt}(-x,t).$$

Therefore,

$$\begin{aligned} v_{tt}(x,t) - c^2 v_{xx}(x,t) &= u_{tt}(-x,t) - c^2 u_{xx}(-x,t) \\ &= F(-x,t) = F(x,t) \quad -\infty < x < \infty,\ t > 0. \end{aligned}$$

Thus, v is a solution of the nonhomogeneous wave equation (4.13). Furthermore,

$$v(x,0) = u(-x,0) = f(-x) = f(x), \quad v_t(x,0) = u_t(-x,0) = g(-x) = g(x).$$

It means that v is also a solution of the initial value problem (4.13)–(4.14). Since the solution of this problem is unique, we have $v(x,t) = u(x,t)$, which implies $u(-x,t) = u(x,t)$. □

Example 4.12 Solve the following Cauchy problem

$$\begin{aligned} &u_{tt} - 9u_{xx} = e^x - e^{-x} && -\infty < x < \infty,\ t > 0, \\ &u(x,0) = x && -\infty < x < \infty, \\ &u_t(x,0) = \sin x && -\infty < x < \infty. \end{aligned}$$

Using the d'Alembert formula, we have

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$
$$+ \frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-c(t-\tau)}^{\xi=x+c(t-\tau)} F(\xi,\tau)\, d\xi\, d\tau.$$

Hence,

$$u(x,t) = \frac{1}{2}[x + 3t + x - 3t] + \frac{1}{6}\int_{x-3t}^{x+3t} \sin s\, ds$$
$$+ \frac{1}{6}\int_{\tau=0}^{\tau=t}\int_{\xi=x-3(t-\tau)}^{\xi=x+3(t-\tau)} (e^\xi - e^{-\xi})\, d\xi\, d\tau$$
$$= x + \frac{1}{3}\sin x \sin 3t - \frac{2}{9}\sinh x + \frac{2}{9}\sinh x \cosh 3t.$$

As expected, for all $t \ge 0$, the solution u is an odd function of x.

Remark 4.13 In many cases it is possible to reduce a nonhomogeneous problem to a homogeneous problem if we can find a particular solution v of the given nonhomogeneous equation. This will eliminate the need to perform the double integration which appears in the d'Alembert formula (4.17). The technique is particularly useful when F has a simple form, for example, when $F = F(x)$ or $F = F(t)$. Suppose that such a particular solution v is found, and consider the function $w = u - v$. By the superposition principle, w should solve the following homogeneous Cauchy problem:

$$\begin{aligned} &w_{tt} - w_{xx} = 0 && -\infty < x < \infty,\ t > 0,\\ &w(x,0) = f(x) - v(x,0) && -\infty < x < \infty,\\ &w_t(x,0) = g(x) - v_t(x,0) && -\infty < x < \infty. \end{aligned}$$

Hence, w can be found using the d'Alembert formula for the homogeneous equation. Then $u = v + w$ is the solution of our original problem.

We illustrate this idea through the following example.

Example 4.14 Solve the problem

$$\begin{aligned} &u_{tt} - u_{xx} = t^7 && -\infty < x < \infty,\ t > 0,\\ &u(x,0) = 2x + \sin x && -\infty < x < \infty,\\ &u_t(x,0) = 0 && -\infty < x < \infty. \end{aligned}$$

Because of the special form of the nonhomogeneous equation, we look for a particular solution of the form $v = v(t)$. Indeed it can be easily verified that $v(x,t) = \frac{1}{72}t^9$ is such a solution. Consequently, we need to solve the homogeneous problem

$$\begin{aligned} &w_{tt} - w_{xx} = 0 && -\infty < x < \infty,\ t > 0,\\ &w(x,0) = f(x) - v(x,0) = 2x + \sin x && -\infty < x < \infty,\\ &w_t(x,0) = g(x) - v_t(x,0) = 0 && -\infty < x < \infty. \end{aligned}$$

Using d'Alembert's formula for the homogeneous equation, we have

$$w(x,t) = 2x + \tfrac{1}{2}\sin(x+t) + \tfrac{1}{2}\sin(x-t),$$

and the solution of the original problem is given by

$$u(x,t) = 2x + \sin x\cos(t) + \frac{t^9}{72}.$$

4.6 Exercises

4.1 Complete the proof of Corollary 4.11.

4.2 Solve the problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0 && 0 < x < \infty,\ t > 0,\\ &u(0,t) = t^2 && t > 0,\\ &u(x,0) = x^2 && 0 \le x < \infty,\\ &u_t(x,0) = 6x && 0 \le x < \infty, \end{aligned}$$

and evaluate $u(4,1)$ and $u(1,4)$.

4.3 Consider the problem

$$u_{tt} - 4u_{xx} = 0 \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 1 - x^2 & |x| \le 1,\\ 0 & \text{otherwise,} \end{cases}$$

$$u_t(x,0) = \begin{cases} 4 & 1 \le x \le 2,\\ 0 & \text{otherwise.} \end{cases}$$

(a) Using the graphical method, find $u(x,1)$.
(b) Find $\lim_{t\to\infty} u(5,t)$.
(c) Find the set of all points where the solution is singular (nonclassical).
(d) Find the set of all points where the solution is not continuous.

4.4 (a) Solve the following initial boundary value problem for a vibrating semi-infinite string which is fixed at $x = 0$:

$$\begin{aligned} &u_{tt} - u_{xx} = 0 && 0 < x < \infty,\ t > 0,\\ &u(0,t) = 0 && t > 0,\\ &u(x,0) = f(x) && 0 \le x < \infty,\\ &u_t(x,0) = g(x) && 0 \le x < \infty, \end{aligned}$$

where $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$ satisfy the compatibility conditions $f(0) = f''(0) = g(0) = 0$.
Hint Extend the functions f and g as odd functions $\tilde{f}$ and $\tilde{g}$ over the real line. Solve the Cauchy problem with initial data $\tilde{f}$ and $\tilde{g}$, and show that the restriction of this solution to the half-plane $x \ge 0$ is a solution of the problem. Recall that the solution of the Cauchy problem with odd data is odd. In particular, the solution with odd data is zero for $x = 0$ and all $t \ge 0$.
(b) Solve the problem with $f(x) = x^3 + x^6$, and $g(x) = \sin^2 x$, and evaluate $u(1,i)$ for $i = 1, 2, 3$. Is the solution classical?

4.5 Consider the problem

$$u_{tt} - u_{xx} = 0 \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 8x - 2x^2 & 0 \le x \le 4,\\ 0 & \text{otherwise,} \end{cases}$$

$$u_t(x,0) = \begin{cases} 16 & 0 \le x \le 4,\\ 0 & \text{otherwise.} \end{cases}$$

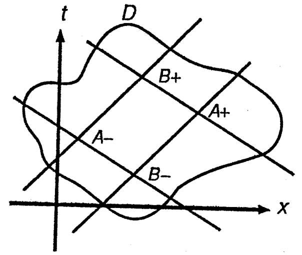

Figure 4.5 A drawing for the parallelogram identity.

(a) Find a formula for the forward and backward waves.
(b) Using the graphical method, draw the graph of $u(x,i)$ for $i = 4, 8, 12$.
(c) Find $u(\pm 5, 2)$, $u(\pm 3, 4)$.
(d) Find $\lim_{t\to\infty} u(5,t)$.

4.6 (a) Solve the following initial boundary value problem for a vibrating semi-infinite string with a free boundary condition:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && 0 < x < \infty,\ t > 0,\\ u_x(0,t) &= 0 && t > 0,\\ u(x,0) &= f(x) && 0 \le x < \infty,\\ u_t(x,0) &= g(x) && 0 \le x < \infty, \end{aligned}$$

where $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$ satisfy the compatibility conditions $f'_+(0) = g'_+(0) = 0$.
Hint Extend the functions f and g as even functions $\tilde f$ and $\tilde g$ on the line. Solve the Cauchy problem with initial data $\tilde f$ and $\tilde g$, and show that the restriction of this solution to the half-plane $x \ge 0$ is a solution of the problem.
(b) Solve the problem with $f(x) = x^3 + x^6$, $g(x) = \sin^3 x$, and evaluate $u(1,i)$ for $i = 1, 2, 3$. Is the solution classical?

4.7 (a) Let $u(x,t)$ be a solution of the wave equation $u_{tt} - u_{xx} = 0$ in a domain $D \subset \mathbb{R}^2$. Let a, b be real numbers such that the parallelogram with vertices $A_\pm = (x_0 \pm a, t_0 \pm b)$, $B_\pm = (x_0 \pm b, t_0 \pm a)$ is contained in D (see Figure 4.5). Prove the *parallelogram identity*:

$$u(x_0 - a, t_0 - b) + u(x_0 + a, t_0 + b) = u(x_0 - b, t_0 - a) + u(x_0 + b, t_0 + a).$$

(b) Derive the corresponding identity when the wave speed $c \ne 1$.
(c) Using the parallelogram identity, solve the following initial boundary value problem for a vibrating semi-infinite string with a nonhomogeneous boundary condition:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && 0 < x < \infty,\ t > 0,\\ u(0,t) &= h(t) && t > 0,\\ u(x,0) &= f(x) && 0 \le x < \infty,\\ u_t(x,0) &= g(x) && 0 \le x < \infty, \end{aligned}$$

where $f, g, h \in C^2([0,\infty))$.

Hint Distinguish between the cases $x - t > 0$ and $x - t \le 0$.
(d) From the explicit formula that was obtained in part (c), derive the corresponding compatibility conditions, and prove that the problem is well-posed.
(e) Derive an explicit formula for the solution and deduce the corresponding compatibility conditions for the case $c \ne 1$.

4.8 Solve the following initial boundary value problem using the parallelogram identity

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && 0 < x < \infty,\ 0 < t < 2x,\\ u(x,0) &= f(x) && 0 \le x < \infty,\\ u_t(x,0) &= g(x) && 0 \le x < \infty,\\ u(x,2x) &= h(x) && x \ge 0, \end{aligned}$$

where $f, g, h \in C^2([0,\infty))$.

4.9 Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 1 && -\infty < x < \infty,\ t > 0,\\ u(x,0) &= x^2 && -\infty < x < \infty,\\ u_t(x,0) &= 1 && -\infty < x < \infty. \end{aligned}$$

4.10 (a) Solve the *Darboux problem*:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad t > \max\{-x, x\},\ t \ge 0,\\ u(x,t) &= \begin{cases} \phi(t) & x = t,\ t \ge 0,\\ \psi(t) & x = -t,\ t \ge 0, \end{cases} \end{aligned}$$

where $\phi, \psi \in C^2([0,\infty)$ satisfies $\phi(0) = \psi(0)$.
(b) Prove that the problem is well posed.

4.11 A pressure wave generated as a result of an explosion satisfies the equation

$$P_{tt} - 16P_{xx} = 0$$

in the domain $\{(x,t) \mid -\infty < x < \infty,\ t > 0\}$, where $P(x,t)$ is the pressure at the point x and time t. The initial conditions at the explosion time $t = 0$ are

$$P(x,0) = \begin{cases} 10 & |x| \le 1,\\ 0 & |x| > 1, \end{cases}$$

$$P_t(x,0) = \begin{cases} 1 & |x| \le 1,\\ 0 & |x| > 1. \end{cases}$$

A building is located at the point $x_0 = 10$. The engineer who designed the building determined that it will sustain a pressure up to $P = 6$. Find the time t_0 when the pressure at the building is maximal. Will the building collapse?

4.12 (a) Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && 0 < x < \infty,\ 0 < t,\\ u(0,t) &= \frac{t}{1+t} && 0 \le t,\\ u(x,0) = u_t(x,0) &= 0 && 0 \le x < \infty. \end{aligned}$$

(b) Show that the limit

$$\lim_{x\to\infty} u(cx, x) := \phi(c)$$

exists for all $c > 0$. What is the limit?

4.13 Consider the Cauchy problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= F(x,t) && -\infty < x < \infty,\ t > 0,\\ u(x,0) = f(x),\quad u_t(x,0) &= g(x) && -\infty < x < \infty, \end{aligned}$$

where

$$f(x) = \begin{cases} x & 0 < x < 1,\\ 1 & 1 < x < 2,\\ 3 - x & 2 < x < 3,\\ 0 & x > 3, x < 0, \end{cases}$$

$$g(x) = \begin{cases} 1 - x^2 & |x| < 1,\\ 0 & |x| > 1, \end{cases}$$

and $F(x,t) = -4e^x$ on $t > 0$, $-\infty < x < \infty$.

(a) Is the d'Alembert solution of the problem a classical solution? If your answer is negative, find all the points where the solution is singular.

(b) Evaluate the solution at $(1, 1)$.

4.14 Solve the problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= e^x + \sin t && -\infty < x < \infty,\ t > 0,\\ u(x,0) &= 0 && -\infty < x < \infty,\\ u_t(x,0) &= \frac{1}{1+x^2} && -\infty < x < \infty. \end{aligned}$$

4.15 Find the general solution of the problem

$$u_{ttx} - u_{xxx} = 0,\quad u_x(x,0) = 0,\quad u_{xt}(x,0) = \sin x,$$

in the domain $\{(x,t) \mid -\infty < x < \infty,\ t > 0\}$.

4.16 Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= xt && -\infty < x < \infty,\ t > 0,\\ u(x,0) &= 0 && -\infty < x < \infty,\\ u_t(x,0) &= e^x && -\infty < x < \infty. \end{aligned}$$

4.17 (a) Without using the d'Alembert formula find a solution $u(x,t)$ of the problem

$$\begin{aligned} u_{tt} - u_{xx} &= \cos(x+t) && -\infty < x < \infty,\ t > 0,\\ u(x,0) = x,\quad u_t(x,0) &= \sin x && -\infty < x < \infty. \end{aligned}$$

(b) Without using the d'Alembert formula find $v(x,t)$ that is a solution of the problem

$$\begin{aligned} v_{tt} - v_{xx} &= \cos(x+t) && -\infty < x < \infty,\ t > 0,\\ v(x,0) = 0,\quad v_t(x,0) &= 0 && -\infty < x < \infty. \end{aligned}$$

(c) Find the PDE and initial conditions that are satisfied by the function $w := u - v$.

(d) Which of the functions u, v, w (as a function of x) is even? Odd? Periodic?

(e) Evaluate $v(2\pi, \pi)$, $w(0, \pi)$.

4.18 Solve the problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= 6t && -\infty < x < \infty,\ t > 0,\\ u(x,0) &= x && -\infty < x < \infty,\\ u_t(x,0) &= 0 && -\infty < x < \infty, \end{aligned}$$

without using the d'Alembert formula.

4.19 Let $u(x,t)$ be a solution of the equation $u_{tt} - u_{xx} = 0$ in the whole plane. Suppose that $u_x(x,t)$ is constant on the line $x = 1 + t$. Assume also that $u(x,0) = 1$ and $u(1,1) = 3$. Find such a solution $u(x,t)$. Is this solution uniquely determined?