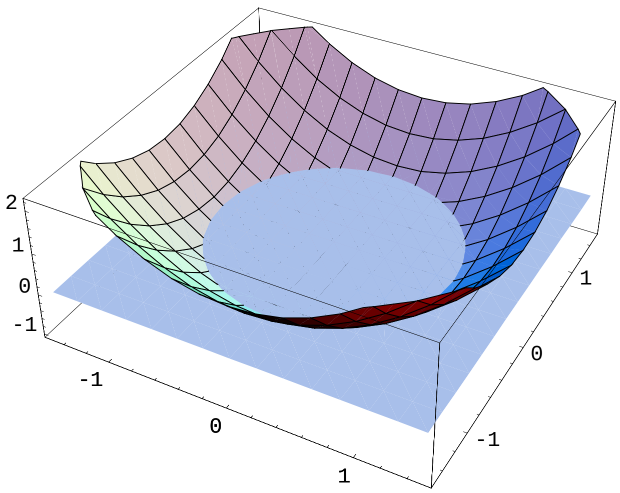
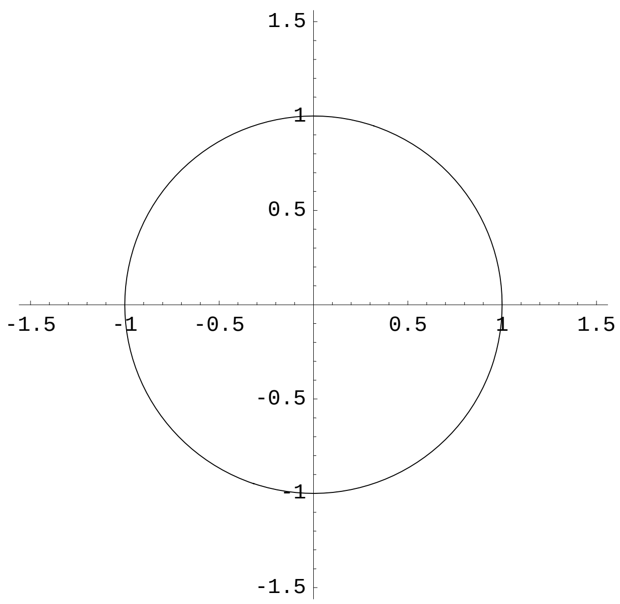
```
(**********************************************)
(********** Implicit függvény tétel **********)
(**********************************************)
<< Graphics`ImplicitPlot`
pr = 1.5;
F[x_, y_] = x^2 + y^2 - 1;

p1 = Plot3D[F[x, y], {x, -pr, pr}, {y, -pr, pr}, DisplayFunction → Identity];
p2 = Plot3D[0, {x, -pr, pr}, {y, -pr, pr}, Mesh → False, DisplayFunction → Identity];
Show[p1, p2, DisplayFunction → $DisplayFunction]

Solve[F[x, y] == 0, y]
ImplicitPlot[F[x, y] == 0, {x, -pr, pr}, {y, -pr, pr}, AxesOrigin → {0, 0}]
```



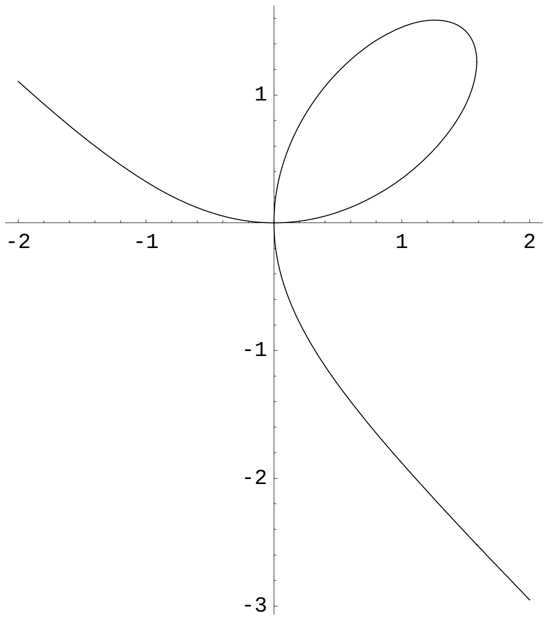
- Graphics3D -

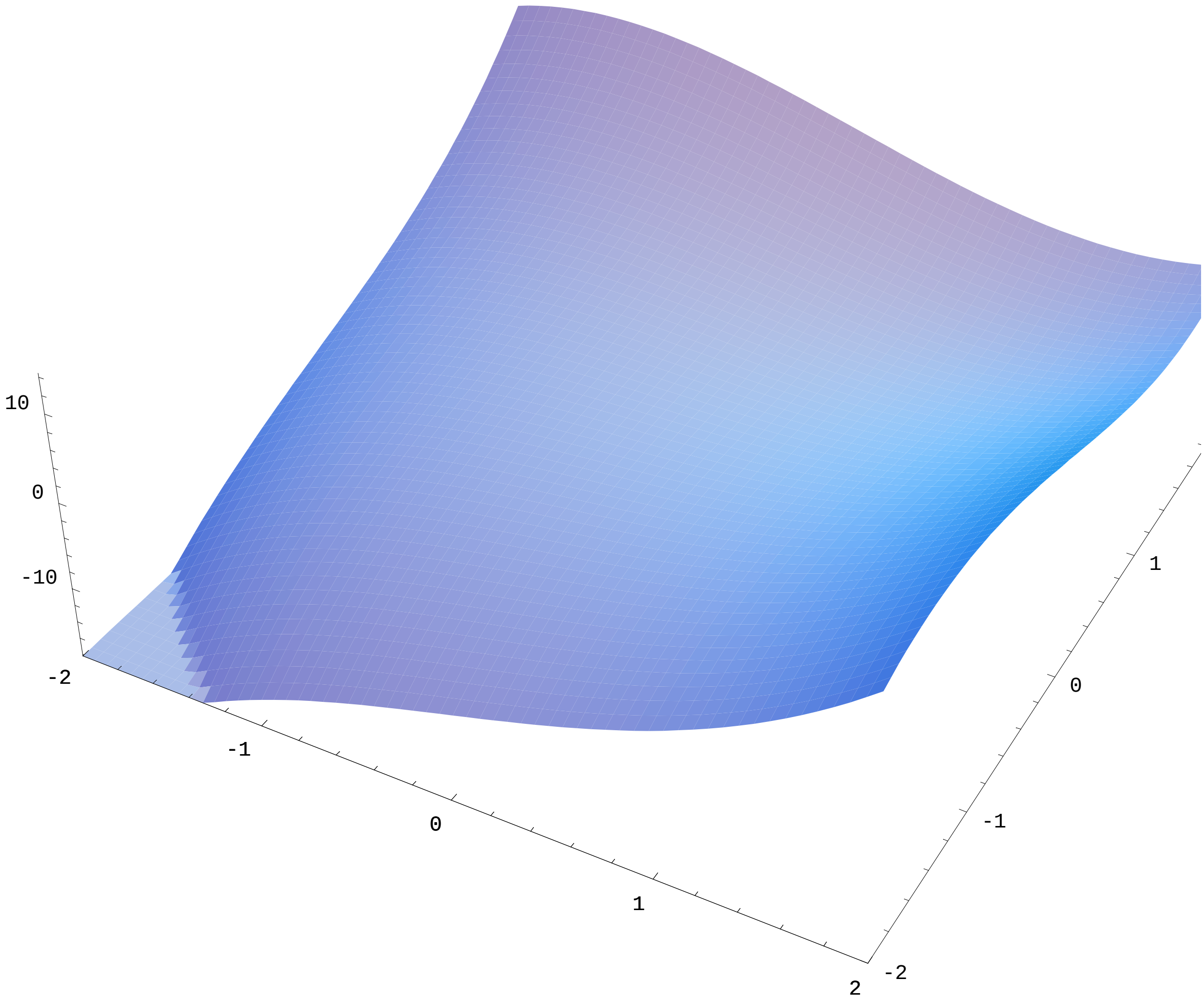
$\{\{y \to -\sqrt{1-x^2}\}, \{y \to \sqrt{1-x^2}\}\}$

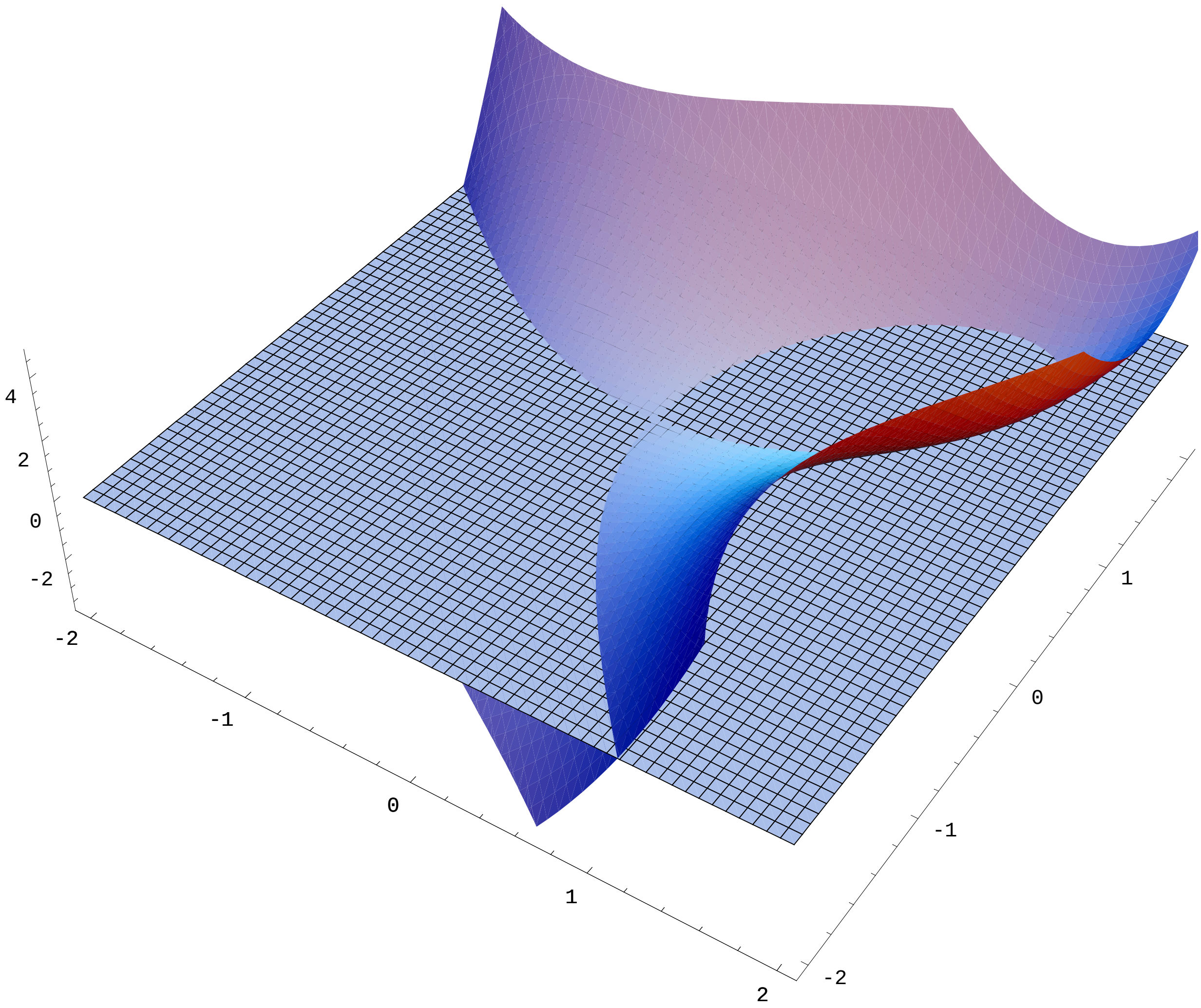


- ContourGraphics -

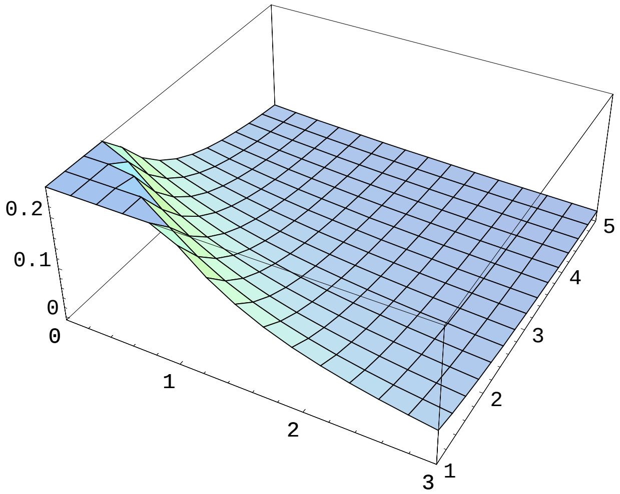
```
ClearAll[f]; ClearAll[x]; ClearAll[y]; ClearAll[x0];
ClearAll[y0]; ClearAll[Dx]; ClearAll[Dy]; ClearAll[Dα];
ClearAll[nv]; ClearAll[dx]; ClearAll[pr];
F[x_, y_] := x^3 + y^3 - 3 * x * y

<< Graphics`ImplicitPlot`
ImplicitPlot[F[x, y] == 0, {x, -2, 2}];

a = Plot3D[F[x, y], {x, -2, 2}, {y, -2, 2},
   Mesh → False, PlotPoints → 60, Boxed → False, ImageSize → 600];
b = Plot3D[0, {x, -2, 2}, {y, -2, 2}, PlotPoints → 60, Boxed → False,
   Axes → False, DisplayFunction → Identity];
(* a felület és a z=0 sík metszete lesz az implicit függvény görbéje *)
Show[a, b, ViewPoint -> {1.3, -2, 2}];
```



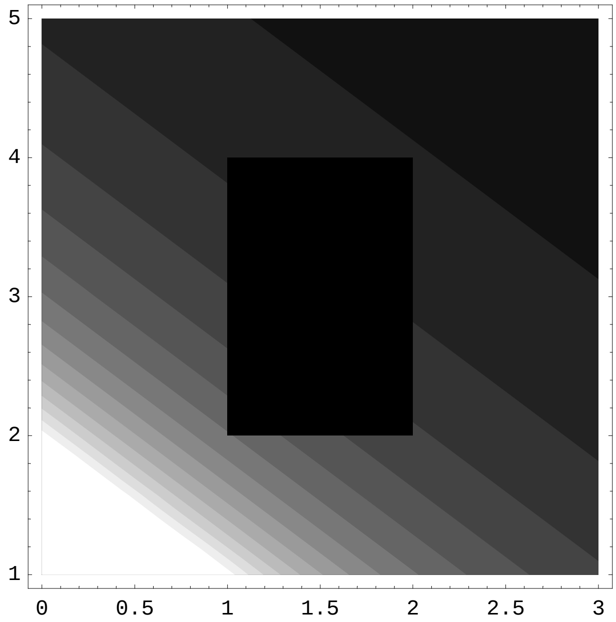

10
0
-10
-2
-1
0
1
2
-2
-1
0
1




```
(*********************************************)
(************* Kettősintegrálok **************)
(*********************************************)

(************* téglalaptartomány ***********)
f[x_, y_] = 1/(x + y)^2;
Plot3D[f[x, y], {x, 0, 3}, {y, 1, 5}]
p1 = ContourPlot[f[x, y], {x, 0, 3}, {y, 1, 5},
   DisplayFunction → Identity, ContourLines → False, Contours → 15];
(* ábrázoljuk az x-y síkon az integrálási (téglalap alakú) tartományt is *)
Show[p1, Graphics[{Rectangle[{1, 2}, {2, 4}]}], DisplayFunction → $DisplayFunction]
(* a kettős integrál eredménye een tartomány fölött *)
Integrate[f[x, y], {x, 1, 2}, {y, 2, 4}]
```



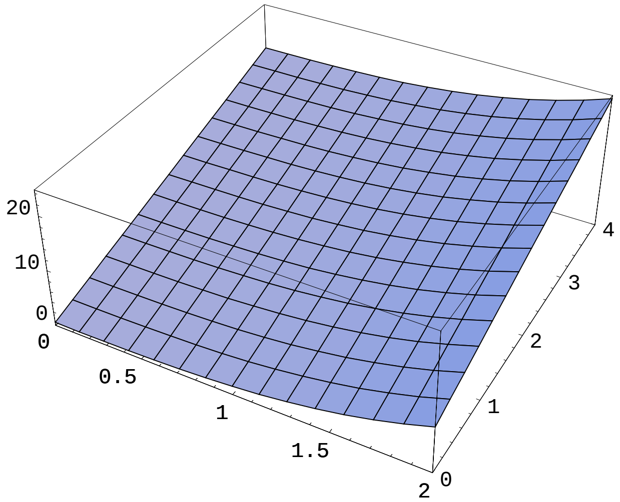
- SurfaceGraphics -



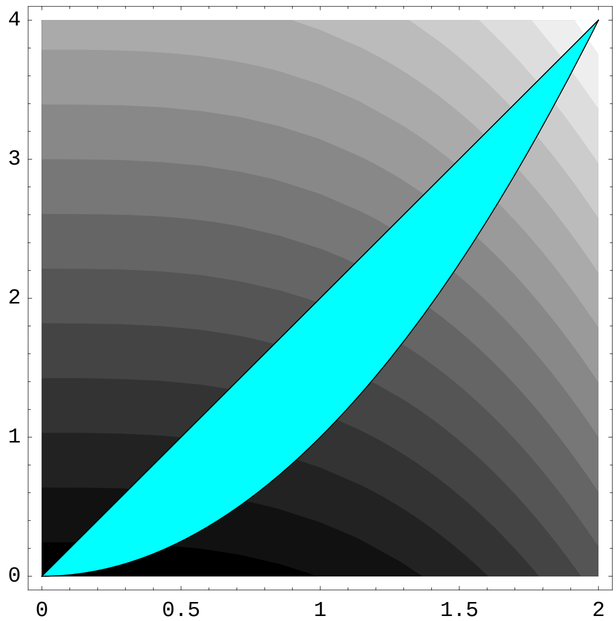
- Graphics -

-Log[3] + Log[4] + Log[5] - Log[6]

```
(************* normáltartomány esetén ***********)
<< Graphics`FilledPlot`
f[x_, y_] = x^3 + 4 y;
Plot3D[f[x, y], {x, 0, 2}, {y, 0, 4}]
p1 = ContourPlot[f[x, y], {x, 0, 2}, {y, 0, 4},
   DisplayFunction → Identity, ContourLines → False, Contours → 15];
p2 = FilledPlot[{2 x, x^2}, {x, 0, 2}, DisplayFunction → Identity];
(* maga a tartomány, ami fölött integrálunk *)
Show[p1, p2, DisplayFunction → $DisplayFunction]
(* a kettős integrálás eredménye is egy számérték pl. 32/3 köbméter (pl.),
 mivel felület alatti térfogatról van szó *)
Integrate[f[x, y], {x, 0, 2}, {y, x^2, 2 x}]
```



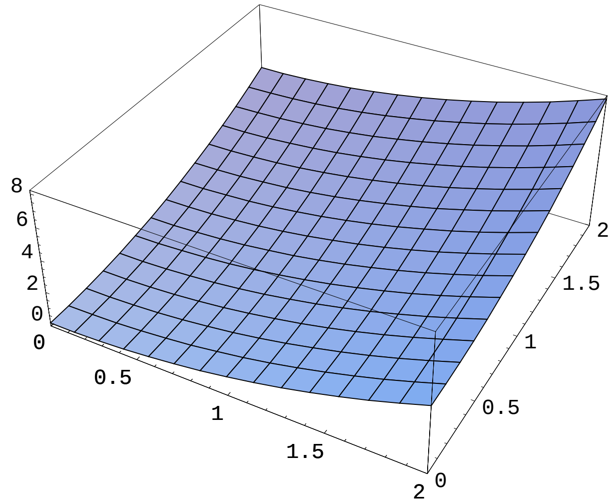
- SurfaceGraphics -



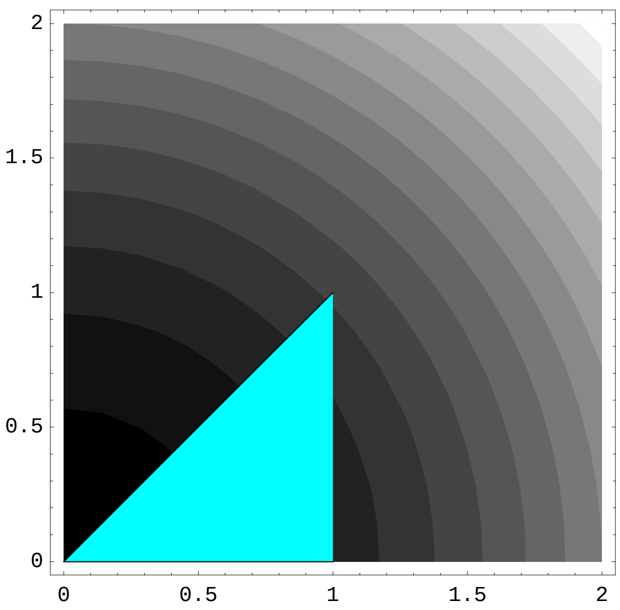
- Graphics -

$$\frac{32}{3}$$

```
(************* feladat ******************)
(* Számold ki az x²+y² függvény integrálját a [0,1] x [y,1]
   normáltartományon és jelenítsd meg az integrálási tartományt! *)

f[x_, y_] = x^2 + y^2;
Plot3D[f[x, y], {x, 0, 2}, {y, 0, 2}]
p1 = ContourPlot[f[x, y], {x, 0, 2}, {y, 0, 2},
   DisplayFunction → Identity, ContourLines → False, Contours → 15];
p2 = FilledPlot[{0, x}, {x, 0, 1}, DisplayFunction → Identity];
Show[p1, p2, DisplayFunction → $DisplayFunction]
Integrate[f[x, y], {y, 0, 1}, {x, y, 1}]
```



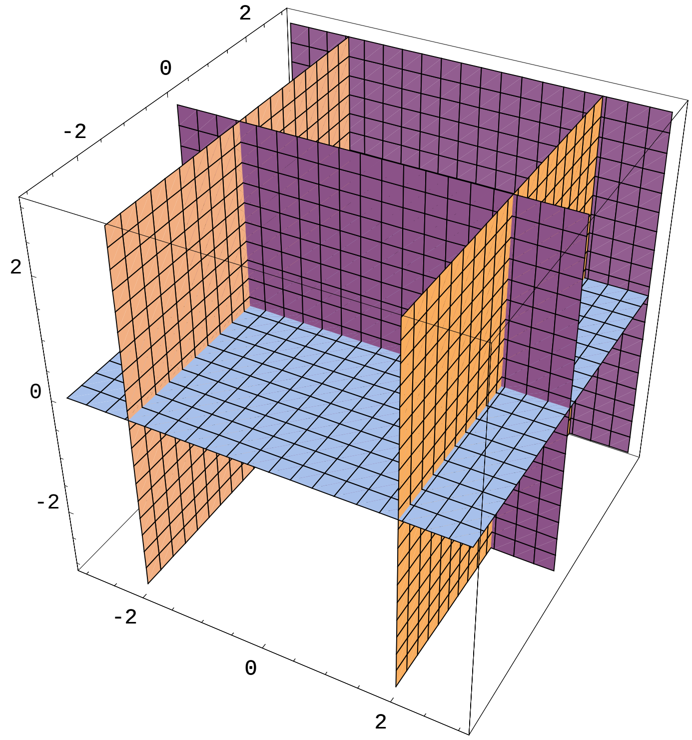
- SurfaceGraphics -

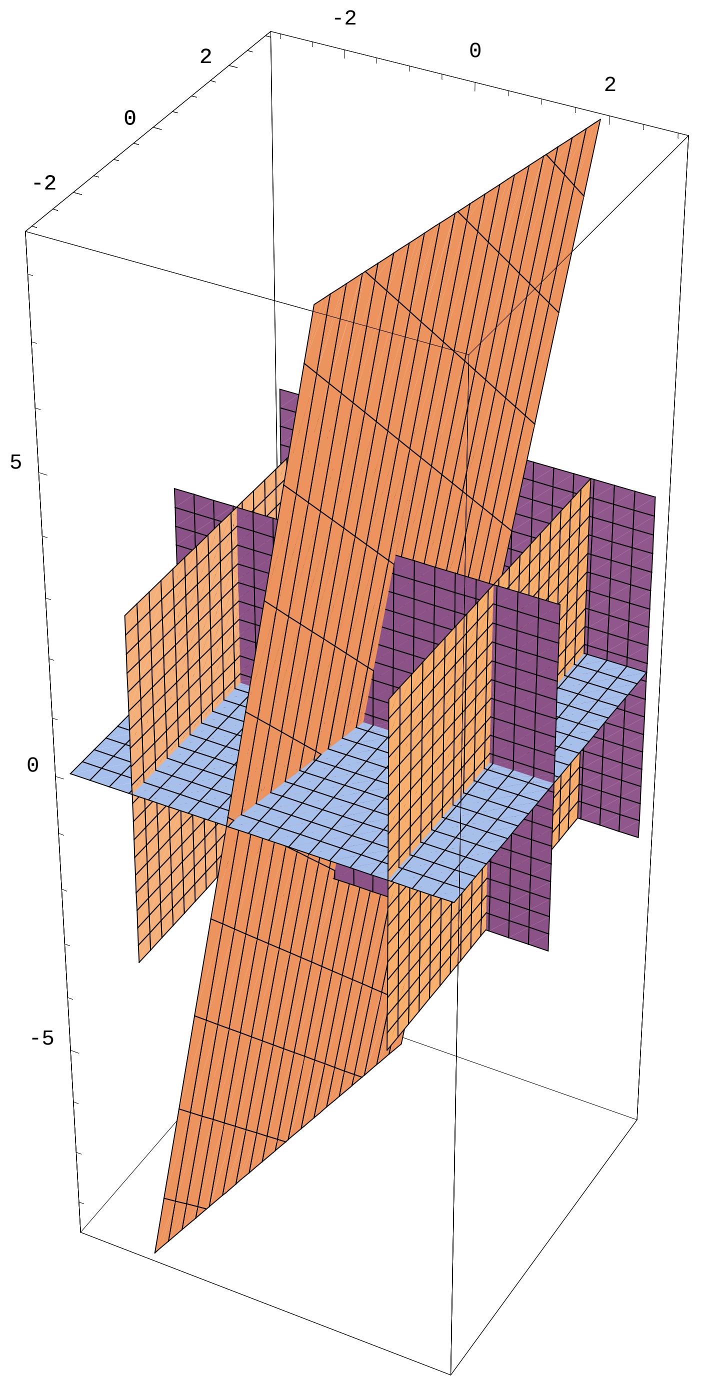


- Graphics -

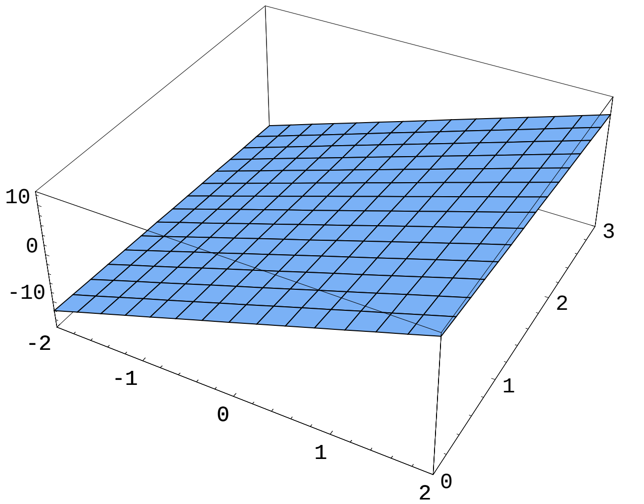
$\frac{1}{3}$

```
(************ Térfogatszámítás *************)
ClearAll[x0, x1, y0, y1, z0, z1, n, r, x, y, z, f, F, vectors];
x0 = -3; y0 = -3; z0 = -3;  (* ábrák széleibek definiálása *)
x1 = 3; y1 = 3; z1 = 3; (* ábrák széleibek definiálása *)
(* adott a z=6x-y függvény, és a határoló síkok,
 ami egy térfogatot zár körül (-2<x<2 és 0<y<3)*)

F = {{z == 6 x - y}, {x == 2}, {x == -2}, {y == 3}, {y == 0}, {z == 0}};
<< Graphics`ImplicitPlot`
<< Graphics`ParametricPlot3D`
p1 = ParametricPlot3D[{x, y, 6 x - y},
   {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
p2 = ParametricPlot3D[{2, y, z}, {y, y0, y1}, {z, z0, z1}, DisplayFunction → Identity];
p3 = ParametricPlot3D[{-2, y, z}, {y, y0, y1}, {z, z0, z1}, DisplayFunction → Identity];
p4 = ParametricPlot3D[{x, 3, z}, {x, x0, x1}, {z, z0, z1}, DisplayFunction → Identity];
p5 = ParametricPlot3D[{x, 0, z}, {x, x0, x1}, {z, z0, z1}, DisplayFunction → Identity];
p6 = ParametricPlot3D[{x, y, 0}, {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
Show[p2, p3, p4, p5, p6, DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
(* a függvénnyel együtt ábrázolva *)
Show[p1, p2, p3, p4, p5, p6,
  DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
```



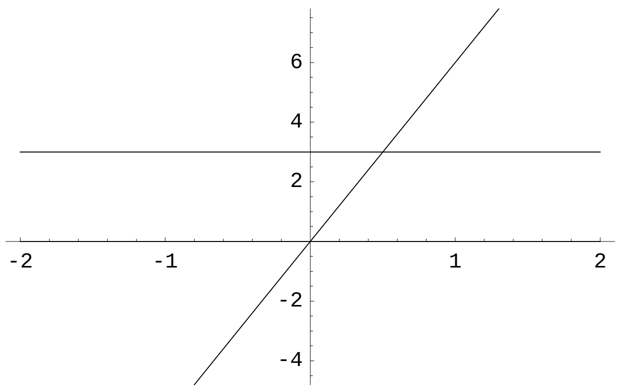

-2
0
2
2
0
-2
5
0
-5

```
Plot3D[6 x - y, {x, -2, 2}, {y, 0, 3}, AspectRatio → Automatic, PlotRange → All];
(* Ha kiszámoljuk az integrált így ahogy van: *)
∫_{-2}^{2} ∫_{0}^{3} (6 x - y) ⅆy ⅆx (*Ez nem térfogatot számol,
 mert a z tengely alatti rész térfogatát (-1)-gyel szorozza!, ELŐJELES TÉRFOGAT*)
```



−18

```
(*nézzük részletesebben a problémát: *)
p7 = Plot[6 x, {x, -2, 2}, DisplayFunction → Identity]; (*a z=0 síkmetszet*)
p8 = Plot[0, {x, -2, 2}, DisplayFunction → Identity]; (*a z=0 síkmetszet*)
p9 = Plot[3, {x, -2, 2}, DisplayFunction → Identity]; (*a z=0 síkmetszet*)
Show[p7, p8, p9, DisplayFunction → $DisplayFunction];
(*a z=0 síkmetszet egy ábrában*)
Solve[{y == 3, 0 == 6 x - y}, {x, y}]
(* ezek alapján felbontva a tartományt 2 részre: *)
Integrate[6 x - y, {y, 0, 3}, {x, y / 6, 2}]
(*Annak a résznek a térfogata, ahol 6 x-y>0*)
Integrate[- (6 x - y), {y, 0, 3}, {x, -2, y / 6}]
(*Annak a résznek a térfogata, ahol 6 x-y<0*)
```

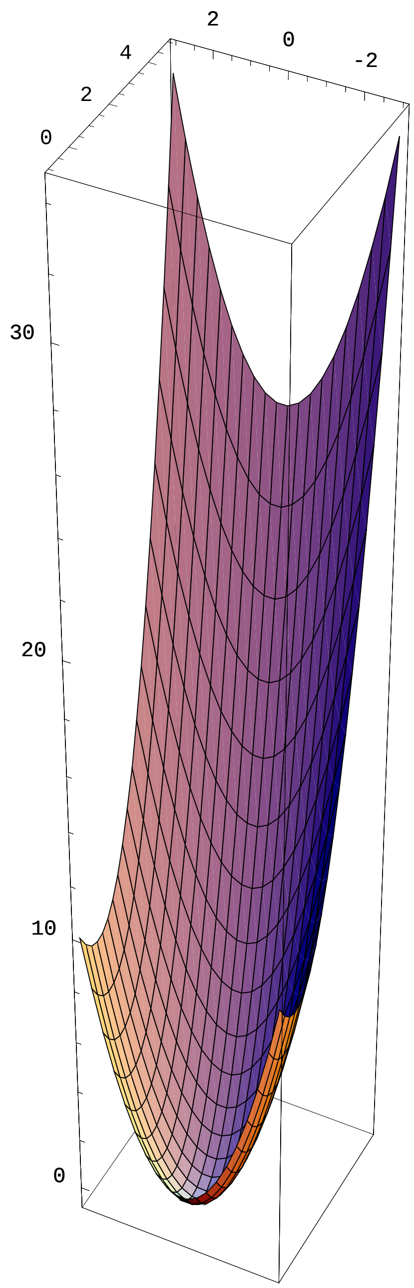


$\{\{x \to \frac{1}{2}, y \to 3\}\}$

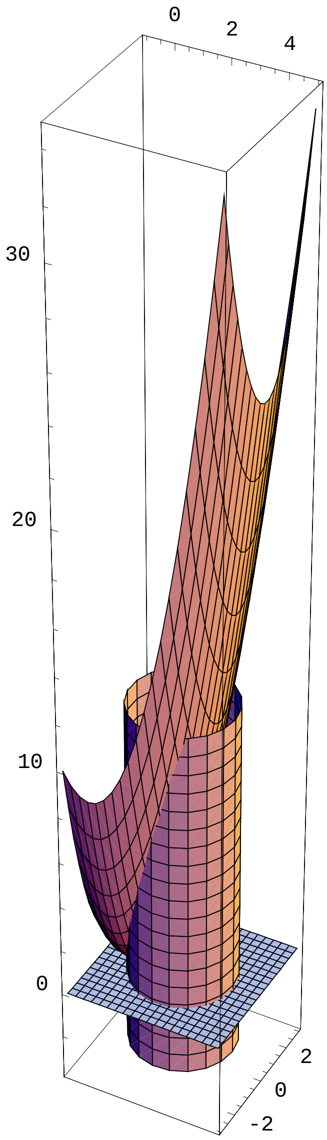
$\frac{111}{4}$

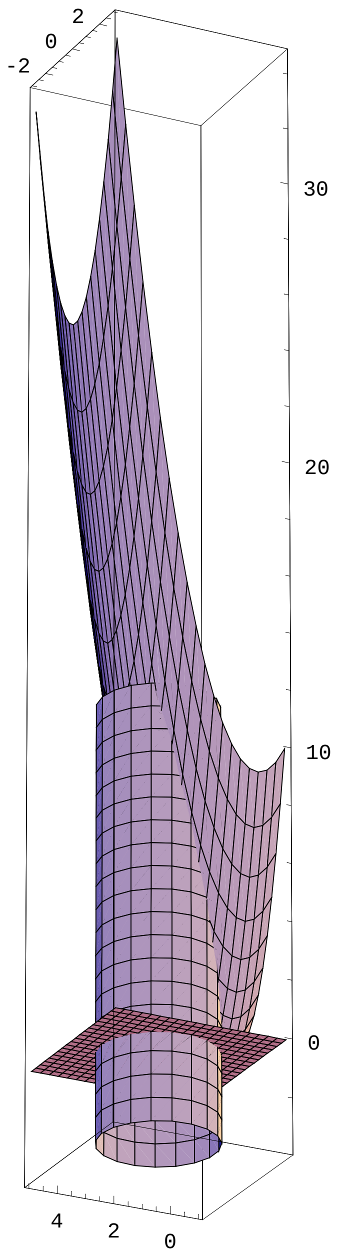
$\frac{183}{4}$

```
(**************** Az egyik gyakorlatpélda **************)
ClearAll[x0, x1, y0, y1, z0, z1, n, r, x, y, z, f, F, vectors];
x0 = -1; y0 = -3; z0 = -3;  (* ábrahatárok *)
x1 = 5; y1 = 3; z1 = 12;
F = {{z == x^2 + y^2}, {(x - 2)^2 + y^2 == 4}, {z == 0}};
<< Graphics`ImplicitPlot`
<< Graphics`ParametricPlot3D`
p1 = ParametricPlot3D[{x, y, x^2 + y^2},
   {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
Show[p1, DisplayFunction → $DisplayFunction, ViewPoint → {-2, -1, 2},
  AspectRatio → Automatic];(*Nézzük a belsejét*)
```



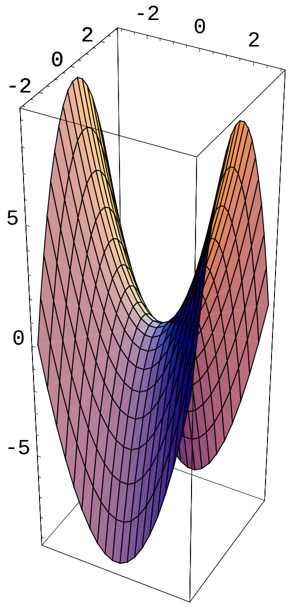
```
p2 = ParametricPlot3D[{2 + 2 Cos[t], 2 Sin[t], z},
   {t, 0, 2 Pi}, {z, z0, z1}, DisplayFunction → Identity];
p3 = ParametricPlot3D[{x, y, 0}, {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
Show[p1, p2, p3, DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
(* Elforgatva is nézzünk bele! *)
Show[p1, p2, p3, DisplayFunction → $DisplayFunction,
  ViewPoint → {4, 8, -4}, AspectRatio → Automatic];
(* áttérünk polárkoordináta rendszerbe *)
x[r_, φ_] := r * Cos[φ];
y[r_, φ_] := r * Sin[φ];
(* integrálunk, és itt is a végeredmény egy szám lesz *)
Integrate[(x[r, φ]^2 + y[r, φ]^2) r, {φ, -Pi / 2, Pi / 2}, {r, 0, 4 Cos[φ]}]
```

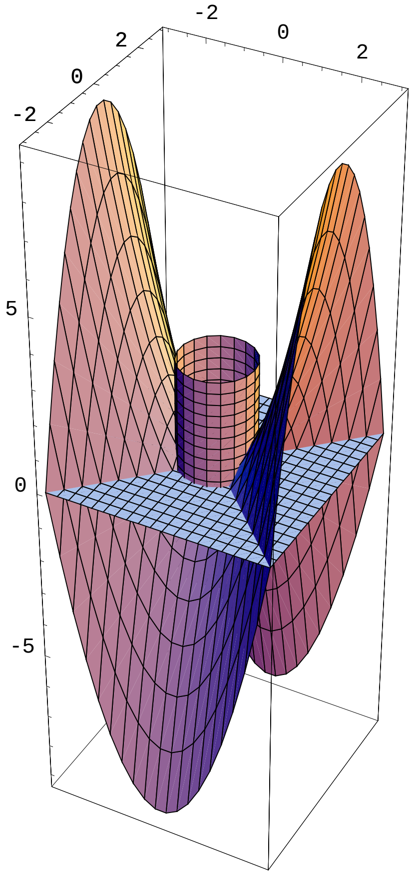




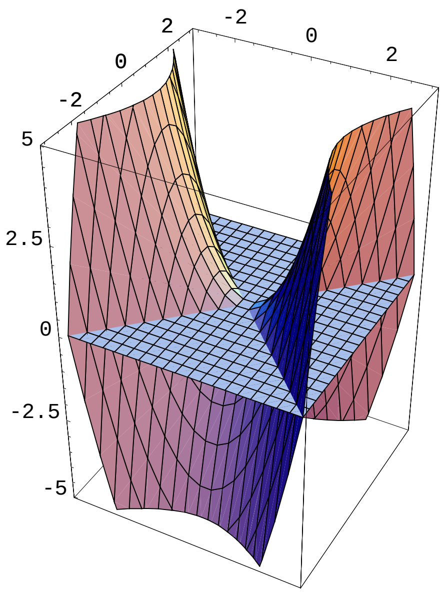
24 π

```
(**************** Egy másik gyakorlatpélda **************)
ClearAll[x0, x1, y0, y1, z0, z1, n, r, x, y, z, f, F, vectors];
x0 = -3; y0 = -3; z0 = -3;
x1 = 3; y1 = 3; z1 = 3;
F = {{z == x^2 - y^2}, {x^2 + y^2 == 1}, {z == 0}};
<< Graphics`ImplicitPlot`
<< Graphics`ParametricPlot3D`
p1 = ParametricPlot3D[{x, y, x^2 - y^2},
   {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
p2 = ParametricPlot3D[{Cos[t], Sin[t], z}, {t, 0, 2 Pi},
   {z, z0, z1}, DisplayFunction → Identity];
p3 = ParametricPlot3D[{x, y, 0}, {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
Show[p1, DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
Show[p1, p2, p3, DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
```



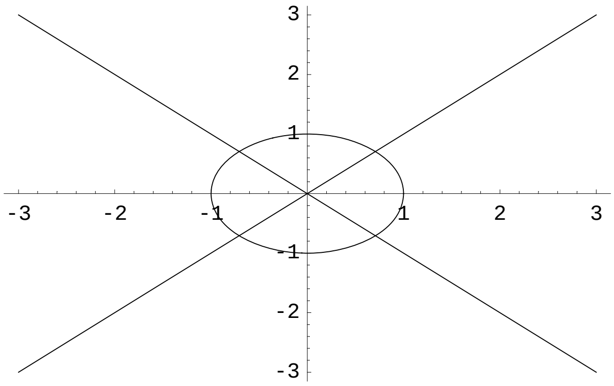



```
Show[p1, p3, DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
x[r_, φ_] := r * Cos[φ];
y[r_, φ_] := r * Sin[φ];
Integrate[(x[r, φ]^2 - y[r, φ]^2) r, {φ, 0, 2 Pi}, {r, 0, 2}]
(*Ez nem térfogatot számol,
 mert a z tengely alatti rész térfogatát (-1)-gyel szorozza!*)
(* érdekes is az eredmény! *)
```



0

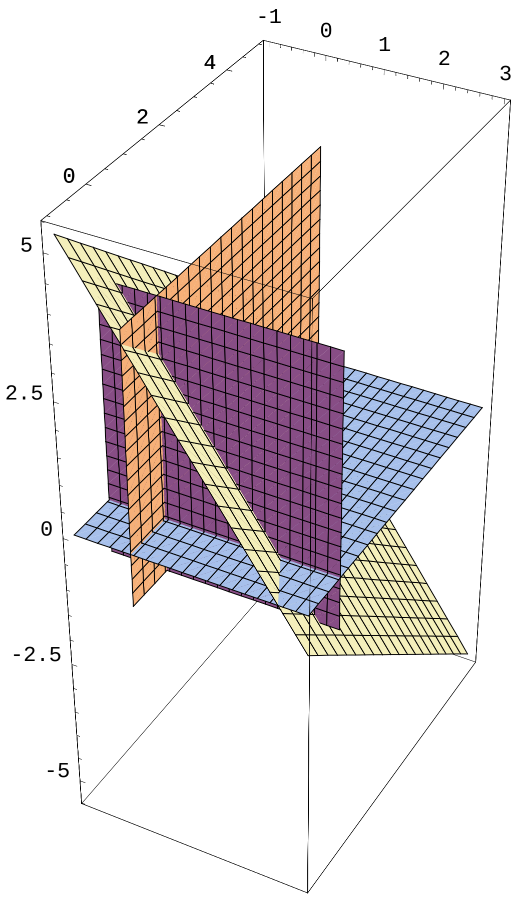
```
p7 = Plot[x, {x, x0, x1}, DisplayFunction → Identity]; (*a z=0 síkmetszet*)
p8 = Plot[-x, {x, x0, x1}, DisplayFunction → Identity];
p9 = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 Pi}, DisplayFunction → Identity];
(* Felülnézetből *)
Show[p7, p8, p9, DisplayFunction → $DisplayFunction];
Integrate[(x[r, φ]^2 - y[r, φ]^2) r, {φ, -Pi / 4, Pi / 4}, {r, 0, 1}]
(*Az egyik negyedrész térfogata, ebből számítható a teljes térfogat*)
```



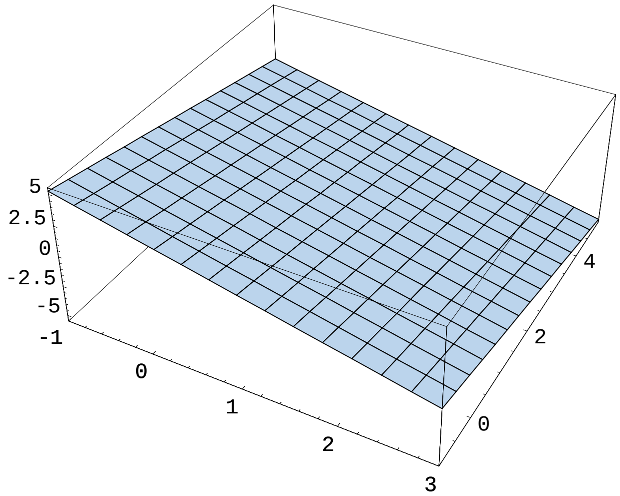
$\frac{1}{4}$

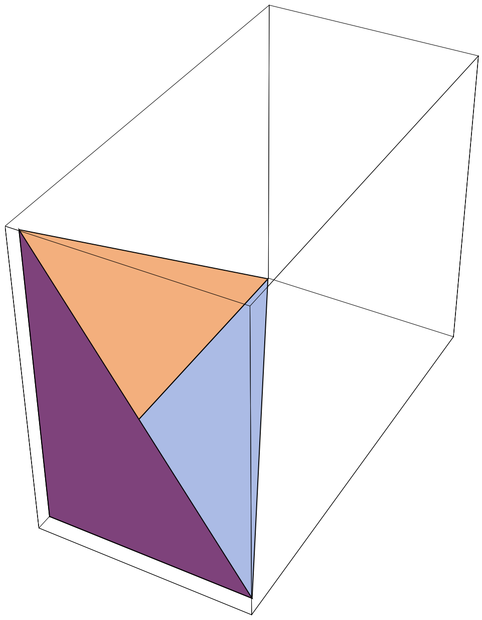
```
(*********************************************)
(************* Hármasintegrálok **************)
(*********************************************)
```

```
(**************** Tetraéder tartomány **************)
ClearAll[a, b, c, d, x0, x1, y0, y1, z0, z1, n, r, x, y, z, f, F, vectors];
x0 = -1; y0 = -1; z0 = -1;
a = 6; b = 3; c = 4; d = 12;
x1 = d / a + 1; y1 = d / b + 1; z1 = d / c + 1;
F = {{ax + by + cz == d}, {x == 1}, {y == 0}, {z == 0}};
<< Graphics`ImplicitPlot`
<< Graphics`ParametricPlot3D`
p1 = ParametricPlot3D[{x, y, 0}, {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
p2 = ParametricPlot3D[{x, 0, z}, {x, x0, x1}, {z, z0, z1}, DisplayFunction → Identity];
p3 = ParametricPlot3D[{0, y, z}, {y, y0, y1}, {z, z0, z1}, DisplayFunction → Identity];
p4 = ParametricPlot3D[{x, y, (d - a x - b y) / c},
   {x, x0, x1}, {y, y0, y1}, DisplayFunction → Identity];
Show[p1, p2, p3, p4, DisplayFunction → $DisplayFunction, AspectRatio → Automatic];
```




```
Plot3D[(d - a x - b y) / c, {x, x0, x1}, {y, y0, y1}, AspectRatio → Automatic];
p5 = Graphics3D[Polygon[{{0, 0, 0}, {0, d / b, 0}, {0, 0, d / c}}]];
(*az x=0 síkmetszet*)
p6 = Graphics3D[Polygon[{{0, 0, 0}, {d / a, 0, 0}, {0, 0, d / c}}]];
(*az y=0 síkmetszet*)
p7 = Graphics3D[Polygon[{{0, 0, 0}, {d / a, 0, 0}, {0, d / b, 0}}]];(*a z=0 síkmetszet*)
Show[p5, p6, p7, DisplayFunction → $DisplayFunction];
f[x_, y_] = 2 x y;(*Integráljuk ezt a függvényt a tetraéderen*)
Integrate[f[x, y], {x, 0, d / a}, {y, 0, (d - a x) / b}, {z, 0, (d - a x - b y) / c}]
Integrate[f[x, y], {x, 0, d / a}, {z, 0, (d - a x) / c},
 {y, 0, (d - a x - c z) / b}](*A másik négy lehetőség HF!*)
```



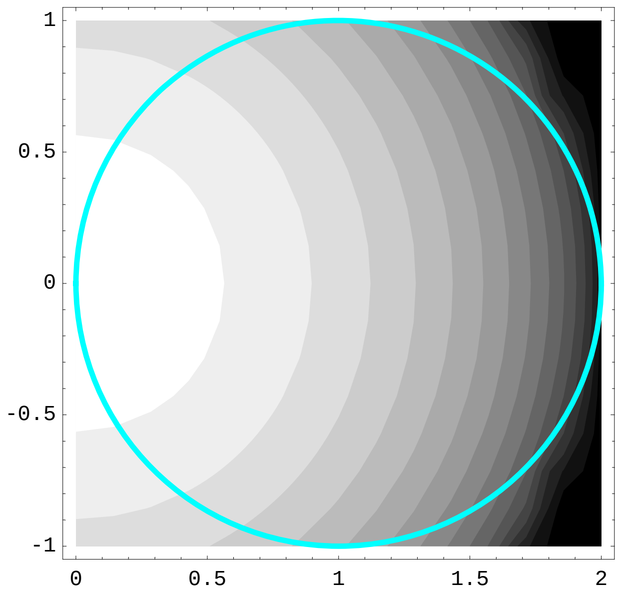


$$\frac{16}{5}$$

$$\frac{16}{5}$$

```
(************* Viviani-test ************)
<< Graphics`ImplicitPlot`
R = 1;
G = {x^2 + y^2 + z^2 == 4 R^2};
H = {(x - R)^2 + y^2 == R^2};
f[x_, y_] = z /. Solve[G, z][[2]]
p1 = ContourPlot[Re[f[x, y]], {x, 0, 2}, {y, -1, 1},
   DisplayFunction → Identity, ContourLines → False, Contours → 15];
p2 = ImplicitPlot[H, {x, 0, 2}, DisplayFunction → Identity,
   PlotStyle → {{Thickness[0.01], Hue[0.5]}}];
Show[p1, p2, DisplayFunction → $DisplayFunction]
```

$\sqrt{4 - x^2 - y^2}$

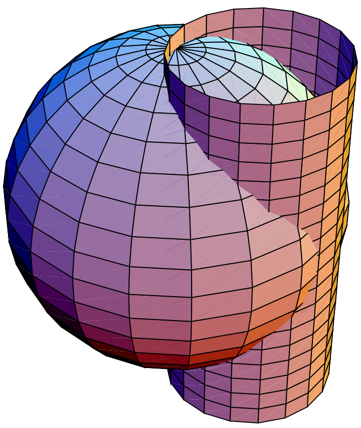


- Graphics -

```
p3 = ParametricPlot3D[{2 Cos[φ] Cos[θ], 2 Sin[φ] Cos[θ], 2 Sin[θ]},
  {φ, 0, 2 π}, {θ, -π/2, π/2}, DisplayFunction → Identity]
p4 = ParametricPlot3D[{1 + Cos[φ] , Sin[φ] , z}, {φ, 0, 2 π},
  {z, -2.2, 2.2}, DisplayFunction → Identity]
Show[p3, p4, DisplayFunction → $DisplayFunction, Boxed → False, Axes → False]
Integrate[f[x, y], {x, 0, 2}, {y, -Sqrt[R^2 - (x - R)^2], Sqrt[R^2 - (x - R)^2]}]
Integrate[1, {x, 0, 2}, {y, -Sqrt[R^2 - (x - R)^2], Sqrt[R^2 - (x - R)^2]}, {z, 0, Sqrt[4 R^2 - x^2 - y^2]}]
```

- Graphics3D -

- Graphics3D -

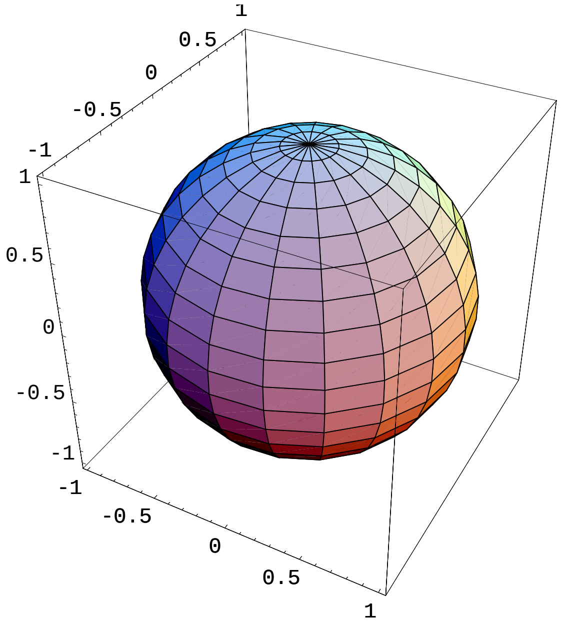


- Graphics3D -

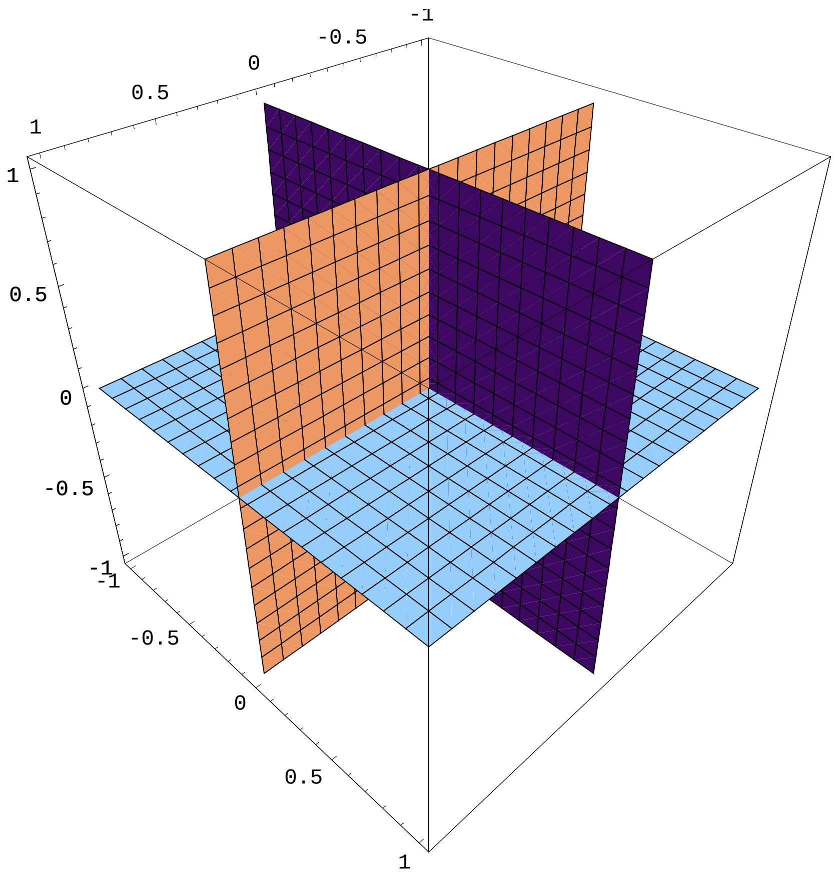
$\frac{8}{9}(-4+3\pi)$

$\frac{8}{9}(-4+3\pi)$

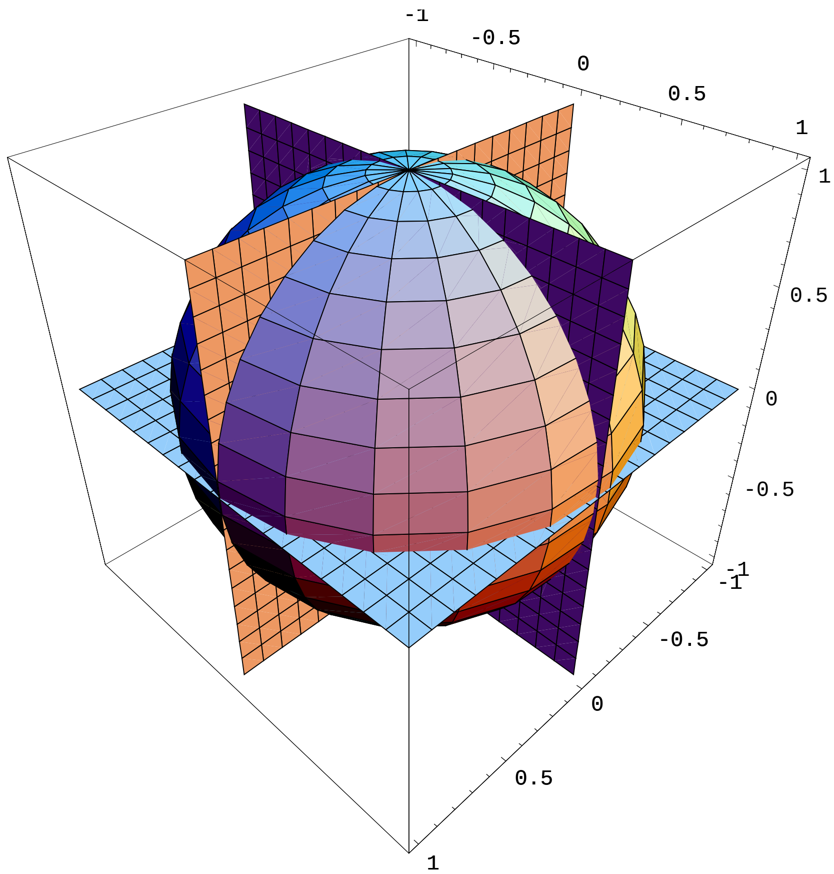
```
(************* feladat ************)
(* z=x^2-y^2 hiperbolikus paraboloidot ábrázold 3D-ben,
 és számold ki azt a területet,
 amit az xy tengelysíkkal és az x^2+y^2=1 hengerpalásttal bezár *)
sik1 = ParametricPlot3D[{0, x, y}, {x, -1, 1}, {y, -1, 1}, DisplayFunction → Identity];
sik2 = ParametricPlot3D[{x, 0, y}, {x, -1, 1}, {y, -1, 1}, DisplayFunction → Identity];
sik3 = ParametricPlot3D[{x, y, 0}, {x, -1, 1}, {y, -1, 1}, DisplayFunction → Identity];
sik4 = ParametricPlot3D[
  {Cos[t] Cos[u], Sin[t] Cos[u], Sin[u]}, {t, 0, 2 Pi}, {u, -Pi / 2, Pi / 2}]
Show[sik1, sik2, sik3, ViewPoint -> {1.2, 1.2, 1.2},
 DisplayFunction → $DisplayFunction, ImageSize → 400]
Show[sik1, sik2, sik3, sik4, ViewPoint -> {1.2, 1.2, 1.2},
 ImageSize → 400, DisplayFunction → $DisplayFunction]
```



- Graphics3D -



- Graphics3D -



- Graphics3D -

(**************** Egység sugarú gömb
térfogata Descartes koordinátarendszerben **************)

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dy\, dx$$

$$\frac{4\pi}{3}$$

(**************** Órai példa Descartes koordinátarendszerben **************)
(********** ugye, hogy érdemes áttérni gömbi koordinátarendszerre? *********)

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \frac{x*y*z}{x^2+y^2+z^2}\, dz\, dy\, dx$$

$$\frac{1}{32}$$

Implicit függvények ábrázolása

www.wolframalpha.com, mivel ez Mathematica 6-os és 7-es parancsokat elfogad

ContourPlot[x^2/9+y^2/(1/4)-1==0, {x, -4, 4}, {y, -4, 4}]

ContourPlot[x^3 + y^3 - 3*x*y==0, {x, -4, 4}, {y, -4, 4}]

Ehhez a $P_0(1.5, 1.5)$ pontba érintőt húzva:
ContourPlot[{x^3 + y^3 - 3*x*y==0,y==-x+3}, {x, -4, 4}, {y, -4, 4}]

Integráláshoz segédlet (vizualizáció és végeredmény)

ContourPlot3D[{z = x^2 - y^2, x^2 + y^2 =1, z = 0}, {x, -2, 2}, {y, -2, 2},{z,-3,3}]
(tessék változtatgatni a határokat)

Vagy akár egy többes integrált is beírhatunk:

Integrate[(x*y*z)/(x^2+y^2+z^2),{x, 0, 1},{y,0,Sqrt[1-x^2]},{z,0,Sqrt[1- x^2-y^2]}]

Definite integral: More digits

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{x\,y\,z}{x^2+y^2+z^2}\,dz\,dy\,dx = \frac{1}{32} \approx 0.03125\ldots$$

Integrate[(x*y*z)/(x^2 + y^2 + z^2), {x, 0, 1}, {y, 0, Sqrt[1 – x^2... WolframAlpha