Functional analysis

Lecture 9.

April 22. 2021

A detour

Theorem. (*Classical Fourier theorem.*) Assume $f : [-\pi, \pi] \to \mathbb{R}$

satisfies the *Dirichlet conditions*. Do you remember??

Then $\forall x \epsilon [-\pi, \pi]$:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right), \quad \text{with}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx.$$

Corollary. The trig. system is complete in $\mathcal{L}^2[-\pi, \pi]$.

$\longrightarrow$ Moreover, the coefficients ARE KNOWN.

"Extension" to a Hilbert space

$(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space. (Can you recall the definition?)

Let $(\varphi_k) \subset H$ be an ON system:

$$\langle \varphi_k, \varphi_j \rangle = \delta_{k,j} = \begin{cases} 1 & \text{if} \quad k = j, \\ 0 & \text{if} \quad k \neq j. \end{cases}$$

Theorem. *Assume, that for an $f \epsilon H$ we have*

$$f = \sum_{k=1}^{\infty} c_k \varphi_k .$$

Then $c_k = \langle f, \varphi_k \rangle$. I.e. the coefficients can be recovered from f.

HW. Check it for the trigonometric system in $H = \mathcal{L}^2[-\pi, \pi]$

Proof of Theorem.

Let us define $s_n := \sum_{k=1}^{n} c_k \varphi_k$, the partial sum.

Then $$\lim_{n\to\infty} \|f - s_n\| = 0.$$

It follows, that for any $\varphi \epsilon H$

$$\lim_{n\to\infty} \langle f - s_n, \varphi \rangle = 0. \quad \text{(Why?)} \Longrightarrow \quad \langle f, \varphi \rangle = \lim_{n\to\infty} \langle s_n, \varphi \rangle$$

Let us choose $\varphi = \varphi_j$ for a fixed j. If $n \geq j$, then

$$\langle s_n, \varphi_j \rangle = \left\langle \sum_{k=1}^{n} c_k \varphi_k, \varphi_j \right\rangle = ??? = \sum_{k=1}^{n} c_k \langle \varphi_k, \varphi_j \rangle = c_j.$$

Thus $\lim_{n\to\infty} \langle s_n, \varphi_j \rangle = c_j$, and indeed: $c_j = \langle f, \varphi_j \rangle$.

Remark. If $(\varphi_n) \subset H$ is complete, then *every* $f \epsilon H$: $\exists (c_n)$

$$f = \sum_{n=1}^{\infty} c_n \varphi_n.$$

Corollary. $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space. (E.g. $H = \mathcal{L}^2[-\pi, \pi]$).

Let $(\varphi_k) \subset H$ be a complete ON system.

It means, that *every* $f \epsilon H$: $\exists (c_n)$

$$f = \sum_{n=1}^{\infty} c_n \varphi_n.$$

From the previous **Theorem.** it follows, that

$$c_n = \langle f, \varphi_n \rangle.$$

Fourier series expansion

Let $(\varphi_n) \subset H$ be a complete ON system. For any $f \epsilon H$ we define

- FOURIER COEFFICIENTS of f with respect to (φ_n) as

$$\boxed{\langle f, \varphi_n \rangle}, \; n = 1, 2, \dots$$

- FOURIER SERIES EXPANSION of f with respect to (φ_n) as

$$\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \cdot \varphi_n.$$

Notation. With $c_n = \langle f, \varphi_n \rangle$ we write $f \sim \sum_{n=1}^{\infty} c_n \, \varphi_n,$.

It is a *formal definition* yet. Why?

Sum of the Fourier series

Theorem. If (φ_n) is a *complete ON system*, then

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \, \varphi_n.$$

I.e. the sum of Fourier series gives back the original function.

Analogy. V is a *finite dim.* vector space. $v_1, \ldots, v_n \epsilon V$ is a BASIS, if

1. these vectors are *linearly independent*,
2. $\forall v \epsilon V$ can be written as $v = \sum_{k=1}^{n} c_k v_k$ (i.e. a generator system).

In infinite dimensional Hilbert space BASIS $\equiv$ *complete ON system.*

Parseval equality Try to recall "the original" one

Theorem. Let $f \epsilon H$.

1. $(\varphi_n) \subset H$ is an *ON system*. Then

$$\sum_{n=1}^{\infty} c_n^2 \leq \|f\|^2, \qquad c_n = \langle f, \varphi_n \rangle \; .$$

2. (φ_n) is ON *and complete* $\iff \sum_{n=1}^{\infty} c_n^2 = \|f\|^2$.

The latter identity is called PARSEVAL EQUALITY.

$$1.\ \sum_{n=1}^{\infty} c_n^2 \leq \|f\|^2 \quad \text{with} \quad c_n = \langle f, \varphi_n \rangle$$

Proof. Let us define $s_n := \sum_{k=1}^{n} c_k \varphi_k$. Geometrically it is *try to finish...*

the projection of f *onto* $\text{span}\{\varphi_1, ..., \varphi_n\}$. Thus $(f - s_n) \perp s_n$.

Then, by the Pythagorean theorem:

$$\|f\|^2 = \|f - s_n\|^2 + \|s_n\|^2 \implies \|s_n\|^2 \leq \|f\|^2 \quad \forall n.$$

For $k \neq j$ use orthogonality: $c_k \varphi_k \perp c_j \varphi_j$, thus $\|s_n\|^2 = \sum_{k=1}^{n} c_k^2$.

Finally, with $n \to \infty$ $\surd$

2. $\sum_{n=1}^{\infty} c_n^2 = \|f\|^2 \qquad \iff \qquad (\varphi_n)$ is complete,

Proof. To verify a proposition with $\iff$ inside has two parts.

Part A. $\Longleftarrow$ Assume (φ_n) is *complete*.

On the previous slide we have seen, that:

$$\|f\|^2 = \|f - s_n\|^2 + \|s_n\|^2. \tag{1}$$

From the completeness of (φ_n) follows, that $f = \sum_{n=1}^{\infty} c_n \varphi_n$,

thus $\lim_{n\to\infty} \|f - s_n\|^2 = 0$.

From (1) we get

$$\|f\|^2 = \lim_{n\to\infty} \|s_n\|^2 = \sum_{k=1}^{\infty} c_k^2 \surd$$

$$\sum_{n=1}^{\infty} c_n^2 = \|f\|^2 \qquad \Longrightarrow \qquad (\varphi_n) \quad \text{is complete}$$

Part B. Assuming $\|f\|^2 = \sum_{k=1}^{\infty} c_k^2 \; \forall f$, prove (φ_n) is COMPLETE.

Do it Yourself. HW.

Lemma.

$$(\varphi_n) \text{ is complete} \qquad \Longleftrightarrow \qquad \langle f, \varphi_n \rangle = 0 \quad \forall n \quad \Longrightarrow \quad f = 0.$$

Generalized Parseval equality

Theorem. Let (φ_n) be a complete ON system in H.

$f, g \epsilon H$ are arbitrary elements, with Fourier series expansions:

$$f = \sum_{n=1}^{\infty} c_n \varphi_n, \qquad g = \sum_{n=1}^{\infty} d_n \varphi_n$$

Then

$$\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k \quad \textit{equivalently} \quad \langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \langle g, \varphi_k \rangle,$$

where $c = (c_k)$ and $d = (d_k)$ are the Fourier coefficients of f and g.

A classical example

$H = \mathcal{L}^2[-\pi, \pi]$. An orthogonal system is:

$$(1,\ \cos(kx),\ \sin(kx)\ :\ k = 1, 2, ...)$$

After normalization we get:

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}}\ :\ k = 1, 2, \ldots\right)$$

Thus the Fourier coefficients of $f \epsilon \mathcal{L}^2[-\pi, \pi]$ are:

$$\int_{-\pi}^{\pi} \frac{\cos(kx)}{\sqrt{\pi}} f(x)dx = \alpha_k, \qquad \int_{-\pi}^{\pi} \frac{\sin(kx)}{\sqrt{\pi}} f(x)dx = \beta_k. \tag{2}$$

Thus the Fourier series of f is:

$$\alpha_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(\alpha_k \cdot \frac{\cos(kx)}{\sqrt{\pi}} + \beta_k \cdot \frac{\sin(kx)}{\sqrt{\pi}}\right)$$

Substituting (2) we get $a_k = \frac{\alpha_k}{\sqrt{\pi}}$, and $b_k = \frac{\beta_k}{\sqrt{\pi}}$.

Moreover, the trigonometric system is complete, Parseval equation$\surd$

Special case: $H = \mathcal{L}^2(R)$

Corollary. Consider a (φ_n) complete ON system in $\mathcal{L}^2(R)$.

For any $f \epsilon \mathcal{L}^2(R)$ it is possible to assign $(c_n) \epsilon \ell^2$, using (φ_n) such that

$$\|f\|_{\mathcal{L}^2} = \|(c_n)\|_{\ell^2} \quad \text{and} \quad \langle f, g \rangle_{\mathcal{L}^2} = \langle c, d \rangle_{\ell^2} \quad \forall g \epsilon \mathcal{L}^2(R).$$

The other direction is the following important Thm.

Theorem. (*Riesz-Fisher thm.*) Let $(d_k) \epsilon \ell^2$, i.e. $\sum_{k=1}^{\infty} d_k^2 < \infty$. Then

$\exists ! f \epsilon \mathcal{L}^2(R)$, such that $\|f\|^2 = \sum_{k=1}^{\infty} d_k^2$, and it's Fourier coefficients are d_k.

Proof. (*Hint*) A "candidate" is $f := \sum_{k=1}^{\infty} d_k \varphi_k$. It is OK. Finish the proof.

$\mathcal{L}^2$ and ℓ^2

Corollary. $\mathcal{L}^2(R)$ és ℓ^2 are isometrically isomorphic.

The linear isometry is based an any (φ_n) complete ON system,

using the Fourier coefficients: $f \quad \longleftrightarrow \quad (c_n)$.

This assignment is

1. *one-by one*,
2. *linear*,
3. *inner product reserving* (also norm-reserving)

Definition. (c_n) are the COORDINATES of f w.r.t (φ_n)

$\mathcal{L}^2$ and ℓ^2

PLEASE STOP FOR A WHILE, AND UNDERSTAND THIS POINT.

$\mathcal{L}^2(R)$ and ℓ^2 are the "same".

Here $\mathcal{L}^2(R) = \mathcal{L}^2(R, \mathcal{R}, \mu)$!

Example. $H = \mathcal{L}^2[-1,1]$

In $\mathcal{L}^2[-1,1]$ a complete ON system are the Legendre polynomials.

We have seen some elements of $(P_n(x))$:

$$P_0(x) = \frac{1}{\sqrt{2}}, \quad P_1(x) = \sqrt{\frac{3}{2}}x, \quad P_2(x) = \text{it was a HW} \quad \ldots..$$

Then every $f \epsilon \mathcal{L}^2[-1,1]$ can be written as

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \quad \text{with} \quad c_n = \int_{-1}^{1} f(x) P_n(x)\, dx.$$

Thus every $f \epsilon \mathcal{L}^2[-1,1]$ can be approximated by a polynomial of degree n with KNOWN coefficients. Can you recall sg. similar?

An example in $H = \mathcal{L}^2[0,1]$, Haar functions

This example for an ON system in $\mathcal{L}^2[0,1]$ are HAAR-FUNCTIONS.

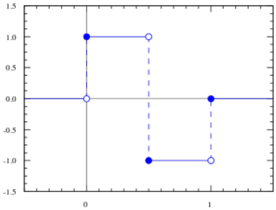
This is the simplest WAVELET FAMILY.

They are defined in blocks.

$$H_{n,k} : [0,1] \to \mathbb{R}, \quad \text{with} \quad n = 0, 1, 2, \dots k = 1, ..., 2^n.$$

The "zero element" is $H_{0,0}(x) = 1$. The *mother wavelet* is

$$H_{0,1}(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1/2 \\ -1 & \text{if} \quad 1/2 \le x \le 1 \end{cases}$$



Haar functions, n^{th} block.

For $n \geq 1$ divide $[0, 1]$ into 2^n equal parts with points $\dfrac{k}{2^n}$.

Let's define for $1 \leq k \leq 2^n$:

$$H_{n,k}(x) = \begin{cases} \sqrt{2^n} & \text{if} \quad \dfrac{k-1}{2^n} \leq x < \dfrac{k-1/2}{2^n} \\ -\sqrt{2^n} & \text{if} \quad \dfrac{k-1/2}{2^n} \leq x < \dfrac{k}{2^n} \\ 0 & \text{otherwise} \end{cases} .$$

The nonzero part is the "mother wavelet", *squished* and *stretched*.

Easy to check, that $\|H_{n,k}\| = 1$ and $H_{n,k} \perp H_{n,j}$ for $j \neq k$. DO IT.

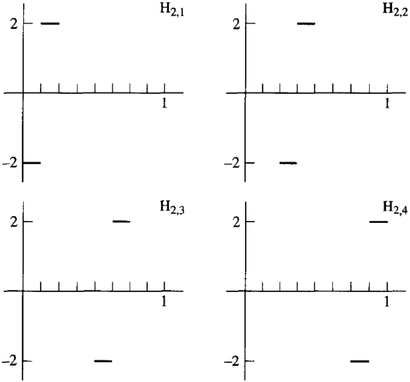
Theorem. This ON system is **complete**. (Not trivial to prove.)

E.g. Haar functions $H_{2,k}$

As an example, here are the graphs of the

$H_{2,k}$

Haar functions for $k = 1, 2, 3, 4$.



E.g. Haar function $H_{3,5}$

For example $H_{3,5}$ is the following:

$$H_{3,5}(x) = \begin{cases} 2^{3/2} & \text{if} \quad \frac{1}{2} \leq x < \frac{1}{2} + \frac{1}{2^4} \\ -2^{3/2} & \text{if} \quad \frac{1}{2} + \frac{1}{2^4} \leq x < \frac{5}{2^3}, \\ 0 & \text{otherwise} \end{cases}$$

Draw it!

Excercise. $\|H_{3,5}\| = ?$

Corollary

Let us consider $f \epsilon \mathcal{L}^2[0,1]$.

Then for any $\varepsilon > 0$ the function f can be approximated by

$$F_N = \sum_{n=0}^{N} \sum_{k=1}^{2^n} c_{k,n} H_{k,n}$$

such that *the error is less then* ε:

$$\|f - F_N\|_2 < \varepsilon.$$