Functional analysis

Lecture 8.

April 15. 2021

Complete systems in $\mathcal{L}^2(X)$

Review

Review. Completeness of functions.

$(f_k) \subset \mathcal{L}^2(X)$. It is a COMPLETE SYSTEM, if

1. they are *linearly independent*,
2. $\forall f \epsilon \mathcal{L}^2(X)$: $\quad f = \sum_{k=1}^{\infty} c_k f_k$ with some $(c_n) \subset \mathbb{R}$.

Analogy. Let V be a finite dimensional vector space. Assume

1. $v_1, \ldots, v_n \epsilon V$ are *linearly independent*,
2. $\forall v \epsilon V$ can be written as $v = \sum_{k=1}^{n} c_k v_k$ (i.e. a generator system).

Remember? It was called: BASIS OF THE VECTOR SPACE.

In infinite dimension we need also *convergence* in the infinite sum.

Two complete systems in $\mathcal{L}^2[-\pi, \pi]$. Review.

1. Trigonometric system: $\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} \ : \ k \epsilon \mathbf{N}\right)$
2. Polynomial system: $\left(1, x, x^2, ...x^k, ... \ : \ k = 1, 2, ...\right)$.

Then $\forall f \epsilon \mathcal{L}^2(X)$: $f = \sum_k \alpha_k f_k$, where (f_n) is the complete system.

$\longrightarrow$ Is it possible to get the α_k coefficients for a *given* f?

1. $\Rightarrow$ *formula* $\surd$
2. $\Rightarrow$ *existence theorem* ↯ NO FORMULA.

Difference: guess???

1. The functions in the trigonometric system are *orthogonal*.

2. The functions in the polinomial systems are *not orthogonal* , e.g.

$$\langle x^4, x^2 \rangle = \int_{-1}^{1} x^4 x^2 dx = \left[\frac{x^7}{7} \right]_{-1}^{1} = \frac{2}{7} \neq 0$$

Next step: Let's orthogonalize the polynomials!

For simplicity let $X = [a, b] \subset \mathbb{R}$ and consider $\mathcal{L}^2 = \mathcal{L}^2[a, b]$.

Theorem. Assume $(f_n) \subset \mathcal{L}^2$ are *linearly independent*.

Then there exists another $(\varphi_n) \subset \mathcal{L}^2$ system of functions s.t:

1. (φ_n) is ON ($\Rightarrow$ linearly independent too.)
2. $\forall n$: $f_n = \sum_{k=1}^{n} \alpha_{kn} \varphi_k$, such that $\alpha_{nn} \neq 0$.
3. $\forall n$: $\varphi_n = \sum_{k=1}^{n} \beta_{kn} f_k$, such that $\beta_{nn} \neq 0$.

Moreover (φ_n) is *unique* up to the sign.

Remark. Properties 2. and 3. imply that *linear subspace* spanned by

$$\{\varphi_1, ..., \varphi_n\} \quad \text{and} \quad \{f_1, ..., f_n\}$$

are the same. Why?

Proof. (Sketch) We use the *Gram-Schmidt orthogonalization* (G-S)

(Visualization: $f \epsilon \mathcal{L}^2$ is a *"vector"*)

STEP 1. Define

$$\varphi_1 := \frac{f_1}{\|f_1\|}.$$

Why this way?

STEP 2. Our dual purpose is:

1. $\{\varphi_1, \varphi_2\}$ should be ON,
2. f_2 can be written as $f_2 = \alpha_{12}\varphi_1 + \alpha_{22}\varphi_2$ with some α_{12}, α_{22}.

The method is the following: try to follow by "drawing"

1. Project f_2 onto φ_1: $f_2|_{\varphi_1} = \langle f_2, \varphi_1 \rangle \cdot \varphi_1$.
2. Subtract it form f_2: $\widehat{\varphi}_2 := f_2 - \langle f_2, \varphi_1 \rangle \cdot \varphi_1$, $\Rightarrow$ $\widehat{\varphi}_2 \perp \varphi_1$.

   (What is still missing?)

3. Normalize:

$$\varphi_2 = \frac{\widehat{\varphi}_2}{\|\widehat{\varphi}_2\|} = \frac{f_2 - \langle f_2, \varphi_1 \rangle \varphi_1}{\|f_2 - \langle f_2, \varphi_1 \rangle \varphi_1\|} \quad \Longrightarrow \quad \|\varphi_2\| = 1, \quad \varphi_1 \perp \varphi_2. \checkmark$$

GENERAL STEP. Assume $\varphi_1, ..., \varphi_{n-1}$: as requested. $\surd$.

f_n is the "new" element. $\varphi_n = ?$

1. Project f_n onto $\{\varphi_1, ..., \varphi_{n-1}\}$. I.e.

$$\min_{c_1,\ldots,c_{n-1}} \left\| f_n - \sum_{k=1}^{n-1} c_k \varphi_k \right\| = ? \quad \Longrightarrow \quad c_k = \langle f_n, \varphi_k \rangle$$

2. Subtract it form f_n. Finish it yourself. $\widehat{\varphi}_n := \ldots$

3. Normalize:

$$\varphi_n = \frac{\widehat{\varphi}_n}{\|\widehat{\varphi}_n\|} = \frac{f_n - \sum_{k=1}^{n-1} \langle f_n, \varphi_k \rangle \varphi_k}{\left\| f_n - \sum_{k=1}^{n-1} \langle f_n, \varphi_k \rangle \varphi_k \right\|} \quad \Rightarrow \quad \|\varphi_n\| = 1,\ \varphi_n \perp \{\varphi_1, ..., \varphi_{n-1}\}.$$

Summary: starting from a linearly independent system $(f_n) \subset \mathcal{L}^2$

the construction of the ON system $(\varphi_n) \subset \mathcal{L}^2$ is done. $\surd$

Corollary. (f_n) is complete $\iff$ (φ_n) is complete.

Definition. A (φ_n) COMPLETE ON system is called ON BASIS.

ON system of polynomials

EXAMPLE. Apply the Thm. for $X = [-1, 1]$.

In $\mathcal{L}^2([-1, 1])$ the system $(1, x, x^2, ...x^n, ...)$, is *lin. independent*.

G-S orthogonalization $\Longrightarrow$ a system of functions $(P_0, P_1, ...P_n, ...)$ s.t.

1. $P_n(x) = \sum_{k=0}^{n} \beta_{kn} x^k$, where $\beta_{nn} \neq 0$.

   I.e. P_n is a *polynomial of degree exactly n* .

2. $\langle P_n, P_m \rangle = 0$ for $n \neq m$. I.e.

$$\int_{-1}^{1} P_n(x) \cdot P_m(x)\, dx = 0 \text{ for } n \neq m, \qquad \int_{-1}^{1} P_n^2(x)\, dx = 1.$$

Definition. These are the LEGENDRE POLYNOMIALS.

Corollary. As $(1, x, x^2, ...x^k, ... \ : \ k = 1, 2, ...)$ is complete,

the Legendre polynomial system is *complete ON system.*

I.e. the Legendre polynomials are an ON BASIS in $\mathcal{L}^2([-1, 1])$

Try to write down the first 2 (or 3) Legendre polynomial.

Theorem. The Legendre polynomials can be written as:

$$P_n(x) = c_n \cdot \frac{d^n}{dx^n}(x^2 - 1)^n = c_n \cdot \left((x^2 - 1)^n\right)^{(n)},$$

where c_n is the normalizing constant, $c_n = \sqrt{\frac{2n+1}{2}} \cdot \frac{1}{2^n \cdot n!}$.

$$P_n(x) = c_n \cdot \frac{d^n}{dx^n}(x^2-1)^n$$

To prove the formula we have to see two main properties of (P_n):

1. P_n is a polynomial of degree n.
2. $P_n \perp P_m$ for $n \neq m$.

Proof. (Sketch)

1. $(x^2-1)^n$ is a polynomial of degree $2n$. Finish it.
2. *Trick*: See that for all $k \leq m < n$

$$\left\langle P_n, x^k \right\rangle = \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \cdot x^k dx = 0.$$

Why is it enough to verify orthogonality of P_n, P_m?

Remark. You may find as "*Legendre polynomial system*"

(p_n) polynomials with different c_n main coefficients.

The reason is, that *normalization* might be different to $\mathcal{L}^2$-norm.

The two most important property is, that

1. p_n IS POLYNOMIAL OF DEGREE n,
2. for $n \neq m$: p_n and p_m ORTHOGONAL, i.e. $\langle p_n, p_m \rangle = 0$.

Up to this point we considered $\mathcal{L}^2(X, \mathcal{M}, m)$, m is the Lebesgue meas.

We extend the results to $\mathcal{L}^2$ with *general measure*.

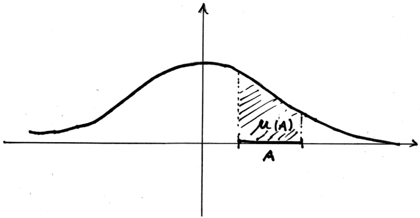
$R \subset \mathbb{R}$. The measure will be given by a "weight function" $\varrho : R \to \mathbb{R}^+$,

that is Lebesgue integrable.

$A \subset R$ is a subset.

The measure of A is defined:

$$m_\varrho(A) := \int_A \varrho \, dm.$$



A possible "meaning": Imagine a wire with variable thickness.

The measure of $A \subset R$ is the *weight of that part* of the wire.

Weighted $\mathcal{L}^2$ space

The integral w.r.t the measure m_ϱ over an E measurable set is:

$$\int_E f \, dm_\varrho = \int_E f \varrho \, dm.$$

Formally we can write "$dm_\varrho = \varrho \, dm$".

The WEIGHTED $\mathcal{L}^2$ SPACE is $\mathcal{L}^2_\varrho(R)$:

$$\mathcal{L}^2_\varrho(R) = \{f : R \to \mathbf{R} \ : \ \int_R f^2 \, dm_\varrho = \int_R f^2 \varrho \, dm < \infty\},$$

also considering m_ϱ-a.e. functions identical.

$\mathcal{L}^2_\varrho(R)$ space

$\mathcal{L}^2_\varrho(R)$ is a *Hilbert space*, with inner product:

$$\langle f, g \rangle_\varrho := \int_R f\, g\, \varrho\, dm,$$

In $\mathcal{L}^2_\varrho(R)$ the *norm* is defined as:

$$\|f\|_{\varrho,2} = \left(\int_R |f|^2\, \varrho\, dm \right)^{1/2}.$$

Thus in this space *orthogonality* means:

$$f \perp g \qquad \equiv \qquad \int_R f\, g\, \varrho\, dm = 0.$$

In $\mathcal{L}^2_\varrho(R)$ let us consider the linearly independent system

$$\{1, x, x^2, \dots\} \subset \mathcal{L}^2_\varrho(R),$$

(assuming these functions are part of the space.)

Let us *orthogonalize* and *normalize* this system.

$\longrightarrow$ We get $\underline{\text{ON polynomials in } \mathcal{L}^2_\varrho(R)}$.

Example 1. Let us consider $R = (-1, 1)$.

The CHEBYSHEV POLYNOMIALS of the *first kind*,

and of the *second kind* are defined by the weight functions:

$$\varrho_1(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \varrho_2(x) = \sqrt{1-x^2}.$$

Proposition. These polynomials are (without normalization):

$$T_n(x) = \cos\left(n \cdot \arccos(x)\right), \qquad U_n(x) = \frac{\sin\left((n+1)\arccos(x)\right)}{\sin\left(\arccos(x)\right)}.$$

These polynomials can be written using $x = \cos(\theta)$:

$$T_n(x) = \cos(n\theta), \qquad U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

Verify, that $T_n(x)$ and $U_n(x)$ are *polynomial of degree exactly n.*

(Left as an exercise.)

Let's verify the orthogonality. I.e. for $m \neq n$:

$$\langle T_n, T_m \rangle_\rho = \int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} \, dx = 0,$$

$$\langle U_n, U_m \rangle_\rho = \int_{-1}^{1} U_n(x) U_m(x) \sqrt{1-x^2} \, dx = 0.$$

(Left as an exercise.)

Example 2. Let us consider $R = \mathbb{R}$.

THE HERMITE POLYNOMIALS are defined with the weight function:

$$\varrho(x) = e^{-x^2}.$$

As $\int_R (x^k)^2 dm_\varrho = \int_{-\infty}^{\infty} (x^k)^2 e^{-x^2}\, dx < \infty$, we get $x^k \epsilon \mathcal{L}^2_\varrho(\mathbf{R})$.

After orthogonalization we get:

$$H_n(x) = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

Verify, that $H_n(x)$ is *polynomial of degree exactly n.*

Questions.

- Why are these systems of orthogonal polynomials important?
- What can we use the ON polynomials for?

+1 example in $\mathcal{L}^2[0, 1]$

This example gives an *ON system* in $\mathcal{L}^2[0, 1]$.

They are called *Haar-functions*.

The functions are not polynomials, but this is the

simplest WAVELET FAMILY.

(*More details on that can be found in the in the book.*)

The functions are defined in blocks.

$$H_{n,k} \quad \text{with} \quad n = 0, 1, 2, \dots \text{ and } k = 1, ..., 2^n.$$

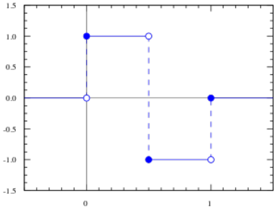
For all indices $H_{n,k} : [0, 1] \to \mathbb{R}$.

Haar functions

For $n = 0$ there are: $H_{0,0}$ and $H_{0,1}$.

$H_{0,0}(x) \equiv 1$. This is an "extra" element

$$H_{0,1}(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2 \\ -1 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$



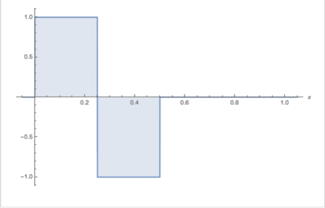
$H_{0,1}(x)$ is the so called *mother wavelet*.

Easy to check, that $\|H_{0,0}\| = \|H_{0,1}\| = 1$ and $H_{0,0} \perp H_{0,1}$. DO IT.

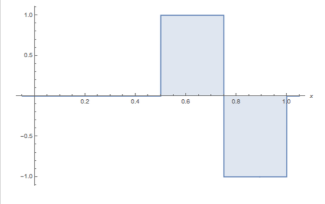
Haar functions, 1^{st} block.

For $n = 1$ there are 2 functions: $H_{1,1}$ and $H_{1,2}$.

$$H_{1,1}(x) = \begin{cases} \sqrt{2} & \text{if } 0 \le x < 1/4 \\ -\sqrt{2} & \text{if } 1/4 \le x < 1/2 \\ 0 & \text{if } 1/2 \le x < 1 \end{cases}$$



$$H_{1,2}(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/2 \\ \sqrt{2} & \text{if } 1/2 \le x < 3/4 \\ -\sqrt{2} & \text{if } 3/4 \le x < 1 \end{cases}$$



Easy to check, that $\|H_{1,k}\| = 1$ and $H_{1,1} \perp H_{1,2}$. DO IT.

Haar functions, n^{th} block.

For $n \geq 1$ divide $[0, 1]$ into 2^n equal parts with points $\frac{k}{2^n}$. Let's define:

$$H_{n,k}(x) = \begin{cases} \sqrt{2^n} & \text{if} \quad \frac{k-1}{2^n} \leq x < \frac{k-1/2}{2^n} \\ -\sqrt{2^n} & \text{if} \quad \frac{k-1/2}{2^n} \leq x < \frac{k}{2^n} \\ 0 & \text{otherwise} \end{cases} , \quad n \geq 1,\ 1 \leq k \leq 2^n.$$

The nonzero part is the "*mother wavelet*", *squished* and *stretched*.

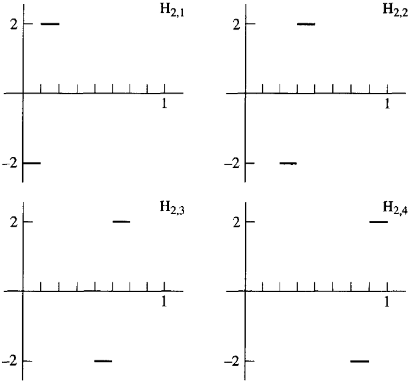
Easy to check, that $\|H_{n,k}\| = 1$ and $H_{n,k} \perp H_{n,j}$ for $j \neq k$. DO IT.

E.g. Haar functions $H_{2,k}$

As an example, here are the graphs of the

$H_{2,k}$

Haar functions for $k = 1, 2, 3, 4$.



Proposition.

1. $(H_{n,k})$ is ON. (*We have 'almost' seen it.*)
2. This ON system is **complete**. (Not trivial to prove.)