Functional analysis

Lecture 7.

March 25. 2021

$\mathcal{L}^p$ space. *Review.*

$1 \leq p < \infty$. $R \subset \mathbb{R}$. The $\mathcal{L}^p(R)$, "BIG ELL p" space, is defined as:

$$\mathcal{L}^p(R) = \{f : R \to \mathbb{R}, \quad \int_R |f|^p dm < \infty\},$$

where a.e. identical functions are IDENTIFIED.

Short notation is $\mathcal{L}^p$, with general set R.

The norm is

$$\|f\|_p = \left(\int_R |f|^p dm\right)^{1/p}.$$

$\mathcal{L}^\infty$ space. Review.

"BIG ELL ∞" space: $\mathcal{L}^\infty(R) = \{f : R \to \mathbb{C}$, essentially bounded$\}$.

Again, the a.e. equal functions are considered identical.

It is a normed space with norm:

$$\|f\|_\infty := \mathrm{ess\ sup}\, f.$$

$f : R \to \mathbb{C}$ is *essentially bounded*, if $\exists M \epsilon \mathbf{R}$, s.t $|f(x)| \leq M$ a.e.

$$\mathrm{ess\ sup}\, f := \inf\{M \mid \exists E,\ m(E) = 0 : |f(x)| \leq M, \forall x \not\epsilon E\}$$

Integral in a *general* measure space.

$(X, \mathcal{R}, \mu)$ is a measure space, i.e.:

- X is an arbitrary set.
- $\mathcal{R} \subset 2^X$ is a σ-algebra.
- $\mu : \mathcal{R} \to \mathbb{R}^+ \cup \{\infty\}$ is a measure.

Integral w.r.t the meas. μ: similarly to the def. of the Lebesgue-int.

1. Start for simple functions, i.e $s(x) = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}(x)$, $E_k \epsilon \mathcal{R}$.

$$\int_E s \, d\mu := \sum_{k=1}^{n} c_k \mu(E_k \cap E), \quad E \epsilon \mathcal{R}.$$

2. Extend it for non negative measurable functions. Review it.
3. Extend it for measurable functions. $\surd$ (Can you recall it?)

A special case.

$X = \mathbb{N}$. The σ-algebra is $\mathcal{R} = 2^{\mathbb{N}}$. μ is the counting measure:

For $A \subset \mathbb{N}$ define : $\mu(A) :=$ number of elements in A.

A function defined on X is $f : \mathbb{N} \to \mathbb{R}$ is the same as

$\longrightarrow$ a *sequence of numbers.* $f \equiv (x_n)$.

$\Longrightarrow$ f is always measurable w.r.to μ. Why?

Let $E \subset \mathbf{N}$ and $f : \mathbb{N} \to \mathbb{C}$. The integral is: (Try to write it!)

$$\int_E f\, d\mu = \sum_{n \epsilon E} f(n),$$

if the right hand side is finite.

General $\mathcal{L}^p(X, \mathcal{R}, \mu)$ space

Let $1 \le p < \infty$. Then

$$\mathcal{L}^p(X, \mathcal{R}, \mu) := \{f : X \to \mathbb{C}, \int_X |f|^p \, d\mu < \infty\}$$

The norm in the space is: guess?

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Let $p = \infty$. The definition of $\mathcal{L}^\infty(X, \mathcal{R}, \mu)$ is: ...Do it yourself...

Completeness.

Theorem. (Riesz theorem, in general)

$\mathcal{L}^p(X, \mathcal{R}, \mu)$ is complete normed space, i.e. it is Banach space.

Proposition. $\mathcal{L}^p$ is an inner product space $\iff$ $p = 2$.

In $\mathcal{L}^2 = \mathcal{L}^2(X, \mathcal{R}, \mu)$ the inner product is

$$\langle f, g \rangle = \int_X f \cdot \bar{g} \, d\mu$$

The most important Lebesgue space is $\mathcal{L}^2(X)$.

It is a HILBERT space.

Focus on $\boxed{p = 2}$

$$\mathcal{L}^2[a,b] = \{f : [a,b] \to \mathbf{R}^n,\ \text{meas.},\ \int_{[a,b]} f^2\, dm < \infty\},\ \text{+ a.e. equality.}$$

Comparison of $\mathcal{L}^2$ and $\mathbf{R}^n$

An element : $(f(t),\ t \epsilon [a,b])$ $\qquad (x_k,\ k = 1, 2, \ldots n)$

The norm: $\|f\|_2 = \left(\int_{[a,b]} f^2 dm \right)^{1/2}$ $\qquad \|x\|_2 = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2}$

The inner product: $\langle f, g \rangle = \int_{[a,b]} f\, g\, dm$ $\qquad \langle x,\ y \rangle = \sum_{k=1}^{n} x_k y_k$

$\longrightarrow$ *The infinite dimensional companion of finite dimensional* $\mathbf{R}^n$.

$\mathcal{L}^2$ *is the infinite dimensional companion of* $\mathbb{R}^n$.

Please stop for a while, and understand up to this point.

Extension of some basic definitions from $\mathbb{R}^n$ to $\mathcal{L}^2$

The functions $f,\ g \epsilon \mathcal{L}^2(X)$ are ORTHOGONAL, if $\langle f, g\rangle = 0$, i.e.

$$\langle f, g\rangle = \int_X f\bar{g}\, d\mu = 0. \qquad \text{Notation:} \quad f \perp g$$

E.g. $X = [-\pi, \pi]$, $f(x) = sin(x)$ and $g(x) = \cos(x)$. Check it.

The *system of functions* $(f_k, k = 1, ..., n)$ is ORTHOGONAL, if

$$\langle f_k, f_j\rangle = 0, \quad \forall k \neq j, \qquad f_j \perp f_k.$$

E.g. $f_k(x) = sin(kx)$, $X = [-\pi, \pi]$. Check it.

Definition. The functions $(f_k,\ k = 1, 2, \dots n) \subset \mathcal{L}^2(X)$ are LINEARLY INDEPENDENT, if

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + ... + \alpha_n f_n(x) = 0 \quad \text{a.e. } x \epsilon X \ \Rightarrow \ \alpha_1 = ... = \alpha_n = 0.$$

The functions $(f_n,\ n \epsilon \mathbb{N}) \subset \mathcal{L}^2(X))$ are LINEARLY INDEPENDENT, if $\forall n\ (f_1,\ f_2, \dots f_n)$ are linearly independent.

E.g. $(f_n(x) = x^n,\ n \epsilon \mathbb{N})$ are linearly independent in $\mathcal{L}^2[0, 1]$. Check.

Proposition. If $f_1,\ f_2,\ \dots f_n \epsilon \mathcal{L}^2$ are *pairwise orthogonal*, then they are *independent*. Check it.

Definition. $f \epsilon \mathcal{L}^2(X)$ is NORMALIZED, if $\|f\|_2 = 1$.

Definition. $(\varphi_k,\ k\epsilon\mathbb{N})$ is ORTHONORMAL (ON), if

$$\langle\varphi_k,\varphi_j\rangle=\delta_{k,j}=\begin{cases}1 & \text{if} \quad k=j,\\ 0 & \text{if} \quad k\neq j.\end{cases}$$

(This $\delta_{k,j}$ is the KRONECKER DELTA.)

Example. The following functions are ON in $\mathcal{L}^2[-\pi,\pi]$

$$\left(\frac{1}{\sqrt{2\pi}},\frac{\cos(kx)}{\sqrt{\pi}},\frac{\sin(kx)}{\sqrt{\pi}}\ :\ k=1,2,...\right)$$

E.g. the normality of $\varphi_0=\dfrac{1}{\sqrt{2\pi}}$ can be seen as: (try first)

$$\|\varphi_0\|_2=\left(\int_{-\pi}^{\pi}(\frac{1}{\sqrt{2\pi}})^2dx\right)^{1/2}=\left(\int_{-\pi}^{\pi}\frac{1}{2\pi}\right)^{1/2}=1.$$

To check the other parts is left as an Exercise.

Completeness of functions.

Definition. The (f_k) is *linearly independent system* is COMPLETE, if

$$\forall f \epsilon \mathcal{L}^2(X) \quad \exists (c_n) \subset \mathbb{R} : \quad f = \sum_{k=1}^{\infty} c_k f_k .$$

This infinite equality means, that the *convergence is in mean*, i.e.

$$\lim_{n \to \infty} \| f - \sum_{k=1}^{n} c_k f_k \|_2 = 0,$$

Proposition. (f_k) is complete $\iff$ $\forall f \epsilon \mathcal{L}^2$ and $\forall \varepsilon > 0$ $\exists c_1, \dots c_n$ s.t.

$$\| f - \sum_{k=1}^{n} c_k f_k \|_2 < \varepsilon .$$

An example of a complete system of functions.

Let us consider the system of functions $\{1, x, x^2, ...x^n, ...\}$. *Properties*:

- $\{1, x, x^2, ...x^n, ...\} \subset \mathcal{L}^2[-1, 1]$, since $\int_{[-1,1]} (x^k)^2 dm < \infty$.
- They are linearly independent. Indeed, suppose

$$\sum_{k=0}^{n} \alpha_k x^k = 0, \quad a.e. \ x \epsilon [-1, 1].$$

  Then $\alpha_k = 0$ for all k, since it is a polinom.

- It is a *complete system*. See the following thm:

Theorem. *(Weierstrass Approximation Theorem)*

$\forall f \epsilon \mathcal{L}^2$ and $\forall \varepsilon > 0$ <u>there is</u> a polynomial p, such that $\|f - p\|_2 < \varepsilon$.

Completeness.

ATTENTION! Two kinds of *completeness* was defined up to this point.

- A *metric space* is complete, if :

  $\longrightarrow$ every Cauchy sequence is convergent.

- A system of *linearly independent functions* is complete, if:

  $\longrightarrow$ these functions are *dense* in the space.

Do not mix them up.

Review of the classical Fourier theorem.

Theorem. Assume $f : [-\pi, \pi] \to \mathbb{R}$ satisfies the *Dirichlet conditions*:

- it is piecewise continuously differentiable,
- it can have discontinuity only of first kind,
- if at x_0 there is a discontinuity, then $f(x_0) = \dfrac{f(x_0+0)+f(x_0-0)}{2}$.

Then $\forall x \epsilon [-\pi, \pi]$:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right), \quad \text{with}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx.$$

Review it, please, if you don't remember!

Apply the previous Thm.

Corollary.

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} \;\; : \;\; k = 1, 2, ...\right)$$

is complete in $\mathcal{L}^2[-\pi, \pi]$.

Exercise. Verify that

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \quad k = 1, 2, \dots\right)$$

is a complete system in $\mathcal{L}^2[0, \pi]$

Two complete systems.

We have seen two complete systems in $\mathcal{L}^2([-\pi, \pi])$.

Then $\forall f \epsilon \mathcal{L}^2([-\pi, \pi])$: $f = \sum_k \alpha_k f_k$, where (f_n) is the complete system.

Question.

$\longrightarrow$ Is it possible to get the α_k coefficients?

Answer.

- In the case of the trigonometric system $\Rightarrow$ *formula* $\surd$
- In the case of the polynomial system $\Rightarrow$ *existence theorem*

  NO FORMULA.

Why is this difference????

$\longrightarrow$ If you have a *guess*, please write to me.