Functional analysis

Lecture 6.

March 18. 2021

Review. Lebesgue integral

The *basic step* was: define the integral for simple functions.

$$s(x) = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}(x), \qquad E_k \cap E_j = \emptyset, \quad c_k \epsilon \mathbb{R}.$$

Let $E \epsilon \mathcal{M}$. Then the Lebesgue integral of s over E is:

$$\int_E s\, dm = \sum_{k=1}^{n} c_k m(E \cap E_k). \quad \text{Geometric meaning?}$$

We extended it to measurable functions: $\int_E f\, dm \surd$. How?

Notation. $\mathcal{L}(E)$ is the set of $\mathcal{L}$-integrable functions over E.

Review. Main properties of the Lebesgue integral

1. If E is measurable, f is measurable & bounded a.e. $\Rightarrow \exists \int\limits_E f\, dm.$

2. If $f \epsilon \mathcal{L}(E)$ and $f = g$ a.e. $\Rightarrow \int\limits_E f\, dm = \int\limits_E g\, dm.$

3. Assume $f \epsilon \mathcal{R}[a, b]$ i.e. Riemann integrable. Then $f \epsilon \mathcal{L}[a, b]$,

$$\int\limits_a^b f(x)\, dx = \int\limits_{[a,b]} f\, dm.$$

What is the meaning of the two integrals?

Advantages of $\mathcal{L}$ integral

Advantage 1. More functions are Lebesgue integrable.

E.g. the Dirichlet function is not Riemann integrable. $f : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if } x \epsilon [0,1], \text{ rational} \\ 0 & \text{if } x \epsilon [0,1], \text{ irrational} \end{cases} \qquad \not\int_0^1 f(x)dx \quad \textit{Why?}$$

Nevertheless, as $f = 0$ a.e. it is Lebesgue integrable, and

$$\int_{[0,1]} f\, dm = \int_{[0,1]} 0\, dm = 0.$$

Advantages of $\mathcal{L}$ integrals, cont.

Advantage 2. Easy to interchange with the *limit*.

The fact, that the pointwise limit and the Lebesgue integral

are interchangeable

is a key property.

This property makes $\mathcal{L}$ intergal **more useful** than $\mathcal{R}$ integral.

In $\mathcal{R}$ integrals the uniform limit and the integral are interchangeable.

We state two basic results.

Lebesgue's Monotone Convergence thm.

Theorem. $E \epsilon \mathcal{M}$ is a measurable set.

Assume $0 \leq (f_n)$ is a measurable function sequence, that is $\nearrow$ a.e..

Define the pointwise limit: $f(x) := \lim\limits_{n\to\infty} f_n(x)$.

Then

$$\int_E f\,dm = \lim_{n\to\infty} \int_E f_n\,dm. \tag{1}$$

Remark. In the case of the *Riemann integral* (1) is true IF ONLY

the convergence is uniform .

Lebesgue's Dominated Convergence thm.

Theorem. $E \epsilon \mathcal{M}$ is a measurable set.

(f_n) are measurable, $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. (*Pointwise limit.*)

Assume $\exists g \epsilon \mathcal{L}(E)$:

$$|f_n(x)| \leq g(x), \qquad \text{for a.e. } x \epsilon E, \ \forall n.$$

Then

$$\int_E f\, dm = \lim_{n\to\infty} \int_E f_n\, dm$$

No need for **uniform** convergence!

Lebesgue's function spaces

$\mathcal{L}^p(R)$ function space

Let $p \geq 1$ be a real number, and $R = [a, b]$.

Definition. The $\mathcal{L}^p(R)$ *function set* is defined as:

$$\mathcal{L}^p(R) = \{f : R \to \mathbb{R} \text{ measurable}, \int_R |f|^p dm < \infty\}.$$

These are the "BIG ELL p" spaces. Recall the "little ell p spaces"

(Short notation is $\mathcal{L}^p$, with general set R.)

Proposition. $\mathcal{L}^p$ is a vector space.

Proof. *Think: What do we have to prove?*

1. If $f \epsilon \mathcal{L}^p$, $c \epsilon \mathbb{R}$ $\Rightarrow \int_R |c \cdot f|^p dm = |c|^p \int_R |f|^p dm < \infty$. $cf \epsilon \mathcal{L}^p \surd$
2. $f, g \epsilon \mathcal{L}^p \quad \overset{?}{\Longrightarrow} \quad f + g \epsilon \mathcal{L}^p$

Trick: $|f(x) + g(x)| \leq 2\max(|f(x)|, |g(x)|)$. Use it:

$$|f(x) + g(x)|^p \leq 2^p \max(|f(x)|, |g(x)|)^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right)$$

Integrate: $\int_R |f + g|^p dm \leq 2^p \left(\int_R |f|^p dm + \int_R |g|^p dm \right) < \infty$. $\surd$

The proof is finished.

Please stop for a while, and understand up to this point:

$$\mathcal{L}^p(R) = \{f : R \to \mathbf{R} \quad \text{is measurable,} \quad \int_R |f|^p dm < \infty\}$$

is a vector space

Norm in $\mathcal{L}^p(R)$?

Review. $\ell^p = \{(x_n) \ : \ \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, the norm $\|x\|_p = \left(\sum |x_n|^p\right)^{1/p}$.

Similarly, let's try

$$\|f\|_p = \left(\int_R |f|^p dm \right)^{1/p}.$$

Is it really a norm?

- Nonnegative? answer? $\surd$
- Non degenerative? answer?

$$\|f\|_p = 0 \quad \iff \quad f = 0 \qquad ?$$

NOT. Why?

Norm in $\mathcal{L}^p(R)$?

For the Dirichlet function f we have $\|f\|_p = 0$, but $f \not\equiv 0$!

In $\mathcal{L}^p$ we'll IDENTIFY functions that are identical a.e..

I.e. new definition of $\mathcal{L}^p$: equivalence classes:

$$\text{If } f = g \text{ a.e.} \epsilon \mathcal{L}^p \implies "f = g"$$

Then $\|f\|_p = \left(\int_R |f|^p dm \right)^{1/p}$ is a norm, indeed.

Verify, that $\int_R |f| dm = 0 \implies f = 0$ a.e.

Triangle inequality in $\mathcal{L}^p$ spaces

Theorem. (Minkovskii-inequality) For all $1 \leq p < +\infty$:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. If $p = 1$, use the *original triangle inequality*:

$$\forall x : \qquad |f(x)+g(x)| \leq |f(x)| + |g(x)|.$$

Then integrate

$$\|f+g\|_1 \leq \int_R |f| dm + \int_R |g| dm = \|f\|_1 + \|g\|_1 \surd$$

For $p > 1$ the proof is very hard...

$\mathcal{L}^\infty$ space

How to define the $\mathcal{L}^p(R)$ Lebesgue space for $p = \infty$?

Do you have an idea, how to do it?

Think of ℓ^∞

Let $p = +\infty$. We will define the function space

$$\mathcal{L}^\infty(R)$$

Essentially bounded function

$f : R \to \mathbb{C}$ is called ESSENTIALLY BOUNDED, if

- $\exists M \epsilon \mathbf{R}$ constant, and
- $\exists E \epsilon \mathcal{M}$, $m(E) = 0$, such that

$$|f(x)| \leq M, \qquad \forall x \not\epsilon E.$$

If f is essentially bouded, then the *essential supremum* is

$$\mathrm{ess\ sup} f := \inf\{M \mid \exists E,\ m(E) = 0 : |f(x)| \leq M,\ \forall x \not\epsilon E\}$$

Essential supremum. Example.

Let's see the Dirichlet function. $f : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if } x \epsilon [0,1], \text{ rational} \\ \\ 0 & \text{if } x \epsilon [0,1], \text{ irrational} \end{cases}$$

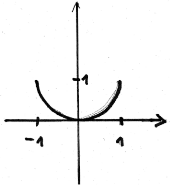
Try to "draw" this function. Obviously $\sup f = \max f = 1$.

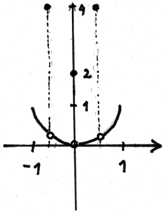
Then $\text{ess } \sup f = 0$.

Difference between sup and ess sup?

Example. Let us consider two function over $R = [-1, 1]$.

$$f(x) = x^2, \qquad g(x) = \begin{cases} x^2 & \text{if} \quad x \neq 0,\ x \neq \pm\frac{1}{2} \\ 2 & \text{if} \quad x = 0 \\ 4 & \text{if} \quad x = \pm\frac{1}{2} \end{cases}$$





$\sup f = 1,$

$\sup g = 4.$

BUT $f = g$ *a.e.* and thus: $\operatorname{ess\ sup} f = \operatorname{ess\ sup} g = 1.$

Definition of $\mathcal{L}^\infty$

Definition. $\mathcal{L}^\infty(R)$ FUNCTION SPACE is the set of functions defined over R, that are *essentially bounded*.

Again, we'll consider a.e. equal functions identical.

$$\mathcal{L}^\infty(R) = \{f : R \to \mathbb{C} \text{ is measurable, essentially bounded}\}.$$

$\mathcal{L}^\infty(R)$ is a vector space. Check it!

It is a normed space with norm: (Guess??)

$$\|f\|_\infty := \text{ess}\ \sup f.$$

The relation of $\mathcal{L}^p(R)$ spaces.

Assume $m(R) < \infty$ (e.g. $R = [a, b]$).

Then

$$\forall f \epsilon \mathcal{L}^\infty(R) \implies f \epsilon \mathcal{L}^p(R) \quad \forall p \geq 1.$$

In general, we also have in the case of $R = [a, b]$:

$$1 \leq p < q : \qquad \forall f \epsilon \mathcal{L}^q(R) \implies f \epsilon \mathcal{L}^p(R).$$

Thus $\mathcal{L}^\infty(R) \subset \mathcal{L}^p(R) \subset \mathcal{L}^1(R)$ for all $p > 1$.

Remark. The background of the notation $\mathcal{L}^\infty(R)$ is:

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty .$$

Completeness

Theorem. (Riesz)

For any $1 \le p \le +\infty$ $\mathcal{L}^p(R)$ is COMPLETE.

Rewiew. Completeness of $\mathcal{L}^p(R)$ means: All $(f_n) \subset \mathcal{L}^p(R)$

Cauchy sequences are convergent, i.e.

$$\exists \lim f_n = f \epsilon \mathcal{L}^p(X).$$

Thus $\mathcal{L}^p(R)$ is always a Banach space.

The proof is very HARD.

$C^2[a, b]$ again

Recall, that $C^2[a, b] = \{f : [a, b] \to \mathbb{R}, \quad \text{continuous}\}$, with norm

$$\|f\|_2 = \left(\int_a^b f^2(x)dx \right)^{1/2} \qquad \textit{(same as Lebesgue int.)}$$

We have seen, that "unfortunately"

$C^2[a, b]$ is not complete.

But, every $f[a, b] \to \mathbb{R}$ continuous function: $f \epsilon \mathcal{L}^2[a, b]$. (Why?).

Thus $C^2[a, b] \subset \mathcal{L}^2[a, b]$. And $\mathcal{L}^2[a, b]$ is COMPLETE.

"Completing $C^2[a,b]$"

We have "completed" $C^2[a,b]$ with the limits of it's Cauchy sequences.

The most important Lebesue space is $\mathcal{L}^2[a,b]$.

It is a HILBERT space. The infinite dimensional companion of $\mathbf{R}^n$.

Focus on $\boxed{p=2}$

$$\mathcal{L}^2[a,b] = \{f : [a,b] \to \mathbf{R},\ \text{meas.},\ \int_{[a,b]} f^2\, dm < \infty\},\ \textit{+ a.e. equality.}$$