Functional analysis

Lecture 5.

March 11. 2021

Review. Measure space

$(X, \mathcal{R}, \mu)$ is called *measure space*, if

- X is an arbitrary set,
- $\mathcal{R} \subset 2^X$ is a σ algebra.

  Elements of $\mathcal{R}$ are *measurable sets*.
- $\mu : \mathcal{R} \to \mathbb{R}^+ \cup \{+\infty\}$ is a measure,

  i.e. it is a σ-additive set function, nonegative.

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) \quad \text{if} \quad A_i \cap A_j = \emptyset,\ i \neq j.$$

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Measure spaces. Examples.

1. $X = \mathbb{N}$. $\mathcal{R} = 2^{\mathbb{N}}$, all subsets.

$$\mu(A) := \#(A) = \begin{cases} |A| & \text{if it is finite} \\ +\infty & \text{if it is not finite} \end{cases}.$$

2. $X = \{x_1, x_2, \dots, x_n, \dots\}$, (or finite). $\mathcal{R} = 2^X$.

   $p_1, p_2, \dots, p_n, \dots$ are nonnegative numbers s.t.

$$\sum_{k=1}^{\infty} p_k = 1.$$

   The measure is defined for $A \subset X$:

$$\mu(A) := \sum_{x_i \in A} p_i.$$

Lebesgue measure on $\mathbb{R}$.

1. $\mathcal{I} = \{$ finite intervals$\}$. $m(I) = b - a$.
2. $\mathcal{E} = \left\{ A \subset \mathbb{R} \ : \ A = \bigcup_{k=1}^{n} I_k, \ \ I_k \epsilon \mathcal{I} \text{ are disjoint} \right\}$, *simple sets*.
$$m(A) = \sum_{k=1}^{n} m(I_k).$$
3. $A \subset \mathbb{R}$: $\quad m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} m(I_k) : \ A \subset \bigcup_{k=1}^{\infty} I_k, \ I_k \epsilon \mathcal{I} \right\}$.
4. *"Heuristics"*: $\exists \mathcal{M}$ σ-algebra:
$$\mathcal{E} \subset \mathcal{M} \subset 2^{\mathrm{R}}, \qquad m^*|_{\mathcal{M}} \text{ is } \sigma \text{ -additive.}$$

$m \equiv$ *Lebesgue-measure*. $\mathcal{M} \equiv$ *Lebesgue-measurable sets*

Next step.

Extension of Lebesgue measure to $\mathbb{R}^n$.

Change 1. (*i.e. measure of finite intervals*) to

1^* Elements of $\mathcal{I}$ are $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$,

with measure $m(I) = \prod_{k=1}^{n} (b_k - a_k)$.

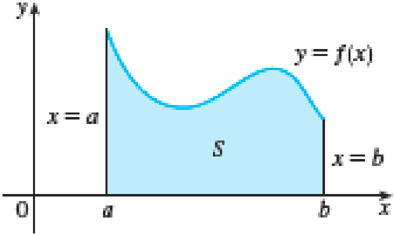
Then, "everything is the same..."

Purpose of measure: new concept of integral

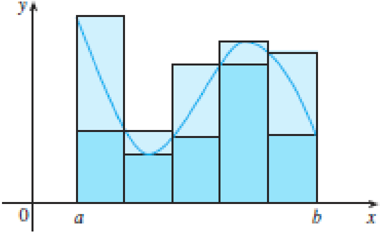
Review of the Riemann integral.

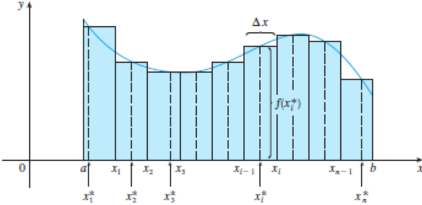
- Motivation: $f : [a, b] \to \mathbb{R}^+$ bounded function.

  How much is the area under the graph of f?



- Approximations of the area:





The **Riemann integral**:

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k,$$

(*The* lim *is independent of the choice of* $x_k^* \epsilon [x_{k-1}, x_k]$.)

One shortcoming of the Riemann integral.

Define $f_n : [0,1] \to \mathbb{R}$ by:

$$f_n(x) = \begin{cases} 1, & \text{if} \quad x = \dfrac{k}{m}, \quad k = 1, 2, \ldots m, \quad m \leq n \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then $\int_0^1 f_n(x)dx = 0$ for all $n = 1, 2, \ldots$.

But:

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1, & \text{if} \quad x \epsilon \mathbb{Q}, \\ \\ 0, & \text{if} \quad x \epsilon \mathbb{Q}^*, \end{cases} \qquad \nexists \int_0^1 f(x)dx.$$

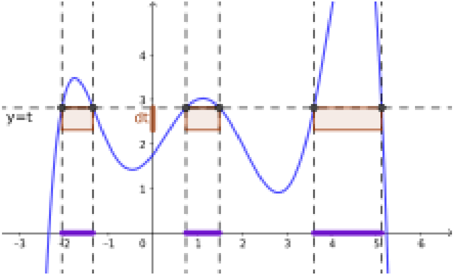
Remark. The limit function is called *Dirichlet function*.

Measurable functions

An $f : \mathbb{R} \to \mathbb{R}$ function is MEASURABLE, if:

$$\{x : f(x) < a\} \epsilon \mathcal{M} \qquad \forall a \epsilon \mathbb{R}$$

Equivalent statements are: write $\leq$ or $>$ or $\geq$ instead of $<$



If f is measurable, then the sets $\{x : f(x) = a\} \epsilon \mathcal{M}$ for all $a \epsilon \mathbb{R}$.

Measurable functions. Examples.

1. Continuous functions.

   The key: If f is cont., then $\{x : f(x) > a\}$ is open.

2. Continuous functions with finite number of discontinuities.

3. Dirichlet function.

4. Characteristic function χ_E, where $E \epsilon \mathcal{M}$:

$$\chi_E(x) = \begin{cases} 1 & x \epsilon E \\ 0 & x \not\epsilon E \end{cases}$$

Properties of measurable functions

Proposition. If f and g are measurable, then

- $f + g$ is measurable,
- $f \cdot g$ is measurable,
- $\min(f, g)$ is measurable, etc.

"Usual ways of combining functions preserves measurability"

Remark. The set of *measurable functions* is quite attractive...

Simple functions

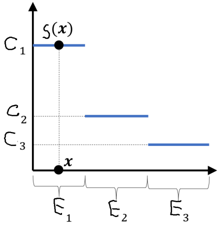
$s(x)$ is <u>SIMPLE function</u>, if the image is finite: $R_s = \{c_1, \ldots, c_n\}$.

Denote $E_k = \{x \ : \ s(x) = c_k\}$. Trivially $E_k \cap E_j = \emptyset$, $k \neq j$.

Thus $s(x)$ can be written as:

$$s(x) = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}(x).$$

The simple function s is measurable $\iff$ $E_k \epsilon \mathcal{M}$.



Also called:

STEP FUNCTION.

Lebesgue integral for simple functions

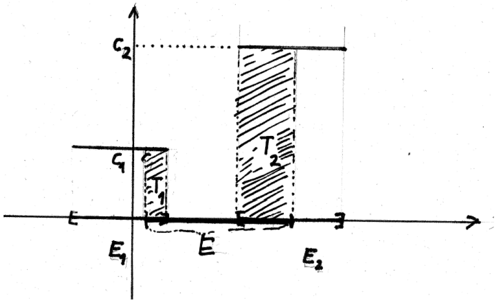
Instead of *"piecewise constant"*, consider *"simple"*.

Let $E_k \epsilon \mathcal{M}$, $k = 1, 2, \dots n$, $E_k \cap E_j = \emptyset$. $c_k \epsilon \mathbb{R}$.

$$s(x) = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}(x) = \begin{cases} c_k & \text{if} \quad x \epsilon E_k \\ 0 & \end{cases}$$

$E \epsilon \mathcal{M}$. The integral of s over E w.r.t measure m is:

$$\int_E s \, dm = \sum_{k=1}^{n} c_k m(E \cap E_k).$$



Example. $E = [0,1]$. $E_0 = [0,1] \cap \mathbb{Q}$.

$$s(x) := \chi_{E_0}(x), \quad m(E_0) = 0 \quad \Rightarrow \qquad \int_E s \, dm = 0 \ \surd$$

$s(x)$ is the *Dirichlet function*.

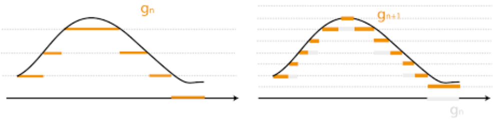
Preparation of the extension of the integral

Proposition. $f \geq 0$ measurable.

Then $\exists (s_n) \nearrow$ monotone increasing sequence of

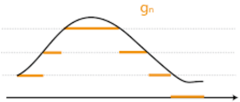
simple functions:

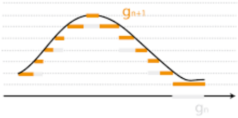
$$\lim_{n\to\infty} s_n(x) = f(x) \qquad \forall x.$$



Sketch of the proof.

We construct the sequence of simple functions.





$n \epsilon \mathbb{N}$. Let's define

$$E_k := \{x \ : \ \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}, \quad F_n := \{x \ : \ f(x) \geq n\}.$$

$$s_n(x) = \begin{cases} n & \text{if} \quad x \epsilon F_n \\ \\ \frac{k-1}{2^n} & \text{if} \quad x \epsilon E_k \end{cases} \qquad \Longrightarrow \qquad s_n(x) \to f(x).$$

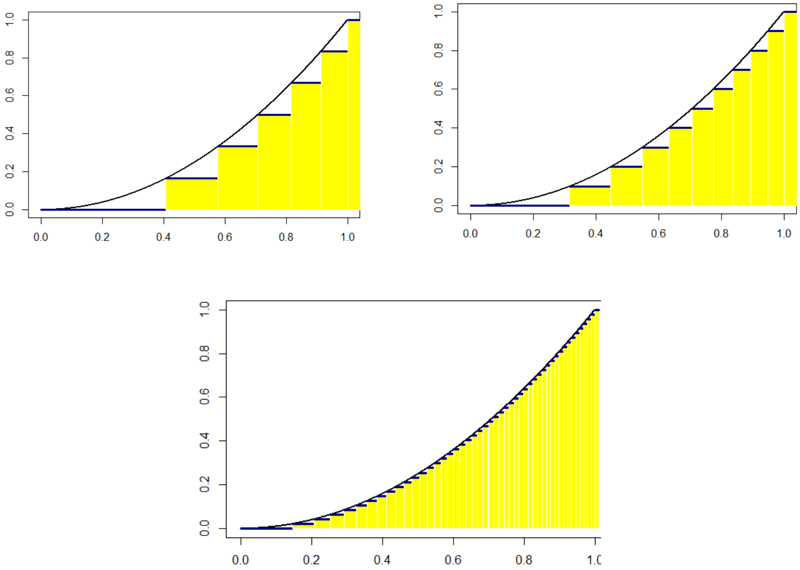
Lebesgue integral, extension

$f \geq 0$ is measurable. $E \epsilon \mathcal{M}$.

The Lebesgue integral of f over E w.r.t measure m is:

$$\int_E f\ dm = \sup \left\{ \int_E s\, dm\ :\ s \text{ simple},\ \ s(x) \leq f(x). \right\}$$

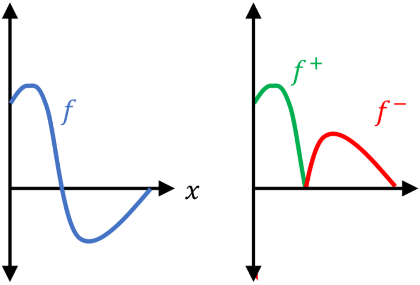
Remark. This value can be $+\infty$ (!)


0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0

Lebesgue integral extension, last step

f is an *arbitrary* measurable function. Write it as $f = f_+ - f_-$

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$



$f = f_+ - f_-$ where $f_+ \geq 0$ and $f_- \geq 0$ are measurable.

These integrals are well defined:

$$\int_E f_+ \, dm, \qquad \int_E f_- \, dm.$$

f is called LEBESGUE INTEGRABLE, if both integrals are finite.

The Lebesgue integral of f over E w.r.t measure m is:

$$\int_E f \, dm := \int_E f_+ \, dm - \int_E f_- \, dm.$$

$\mathcal{L}(E)$: the space of *Lebesgue-integrable* functions over set E.

Properties of the Lebesgue integral

- If $f, g \epsilon \mathcal{L}$, then

$$\int_E (f+g)\, dm = \int_E f\, dm + \int_E g\, dm, \qquad \int_E c \cdot f\, dm = c \cdot \int_E f\, dm$$

- If $m(E) < \infty$ and $a \leq f(x) \leq b$ measurable, then

$$a \cdot m(E) \leq \int_E f\, dm \leq b \cdot m(E).$$

- If $f, g \epsilon \mathcal{L}$ and $f(x) \leq g(x)$, then

$$\int_E f\, dm \leq \int_E g\, dm.$$

More properties of Lebesgue integral

- $f \epsilon \mathcal{L} \iff |f| \epsilon \mathcal{L}$, *(in the case of the Riemann integral $\not\Leftarrow$)*

$$\text{and} \quad \left| \int_E f\, dm \right| \leq \int_E |f|\, dm.$$

- If $m(E) = 0$, then for all measurable functions $\int_E f\, dm = 0$.
- If $E = E_1 \cup E_2$, E_1 and E_2 are disjoint, then

$$\int_{E_1 \cup E_2} f\, dm = \int_{E_1} f\, dm + \int_{E_2} f\, dm.$$

Almost everywhere

f, g are measurable functions.

They are called $f \sim g$ (equivalent), or

$$f = g \quad \text{almost everywhere (a.e.)}$$

if $m\Big(\{x \;:\; f(x) \neq g(x)\}\Big) = 0.$

Proposition. $f = g$ a.e. is an equivalence relation.

Proposition. If f and g are *continuous*, and $f = g$ a.e., then

$$f(x) = g(x) \quad \text{for all} \quad x.$$

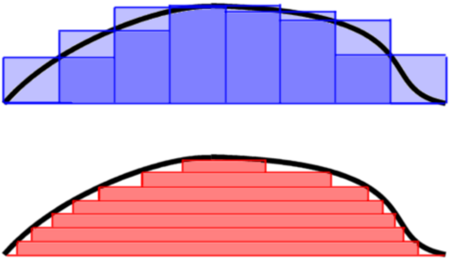
The properties of the $\mathcal{L}$-integral are valid for a.e. instead of $=$

Comparison of $\mathcal{R}$ and $\mathcal{L}$ integrals.

Proposition. Assume $f \epsilon \mathcal{R}[a, b]$ i.e. it is *Riemann integrable*.

Then $f \epsilon \mathcal{L}[a, b]$, i.e. it is *Lebesgue integrable* too, and

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm.$$



Corollary. If f is Riemann integrable, we can use that integral.

Review. Main properties of the Lebesgue integral

1. If E is measurable and f is measurable & bounded a.e.

$$\Rightarrow \exists \int_E f \, dm.$$

2. If $f \epsilon \mathcal{L}(E)$ and $f = g$ a.e. $\Rightarrow \displaystyle\int_E f \, dm = \int_E g \, dm.$

3. Assume $f \epsilon \mathcal{R}[a, b]$ i.e. Riemann integrable. Then $f \epsilon \mathcal{L}[a, b]$,

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, dm.$$