# Functional analysis

Lecture 4.

March 4. 2021

# Complete metric spaces

# Review: Topological notions of $\mathbb{R}$ in a metric space

Definition. $(M, d)$ is a metric space. $(x_n) \subset M$ is a sequence.

$(x_n)$ is *convergent* and *the limit of $(x_n)$ is $x^* \epsilon M$*:

$$\lim_{n \to \infty} x_n = x^*$$

if $\forall \varepsilon > 0\ \exists N$, such that

$$d(x_n, x^*) < \varepsilon \qquad \text{for all} \quad n \geq N.$$

# Cauchy sequence

$(M, d)$ is a metric space.

**Definition.** $(x_n) \subset M$ is a *Cauchy sequence*, if $\forall \varepsilon > 0\ \exists N$ s.t.

$$d(x_n, x_m) < \varepsilon \qquad \forall n, m \geq N.$$

**Proposition.** If $(x_n)$ is convergent $\implies$ it is Cauchy seq.

*Proof*: Choose $\varepsilon > 0$. Then $\exists N$ such that

$$d(x_n, x_0) < \varepsilon/2, \quad \forall n \geq N.$$

Then for all $n, m \geq N$:

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

# An example

$(M, d) = (\mathbb{Q}, |\cdot|)$ is a metric space. Define a sequence as

$$x_1 = 3, \quad x_2 = 3.1, \quad x_3 = 3.14, \ldots \quad x_n = \text{"first } n \text{ digits in } \pi\text{"}.$$

Then $(x_n) \subset \mathbb{Q}$ is Cauchy sequence. (Why? $|x_n - x_m| <?$)

But $\lim_{n\to\infty} x_n = \pi \notin \mathbb{Q}$. There is a "hole" in the metric space .

$\longrightarrow$ If $(x_n)$ is Cauchy in $(\mathbb{Q}, |\cdot|) \not\Rightarrow$ it is convergent (...)

# Complete metric space

In a general $(M, d)$ a metric space

$$(x_n) \text{ is a Cauchy sequence} \not\Rightarrow \text{it is convergent (!)}$$

Definition. $(M, d)$ is COMPLETE METRIC SPACE, if

ALL Cauchy sequences are convergent.

If $(V, \|\cdot\|)$ is complete, it is called BANACH-SPACE.

If $(V, \langle\cdot,\cdot\rangle)$ is complete, it is called HILBERT-SPACE.

Next week: Bibliography of Banach or Hilbert?

# Examples

- $(\mathbb{R}, |\cdot|)$ is complete. In Calculus we learned, that

$$(x_n) \subset \mathbb{R} \text{ is convergent} \iff (x_n) \text{ is Cauchy}.$$

- $(\mathbb{Q}, |\cdot|)$ is NOT complete.

- $(\mathbb{R}^n, \|\cdot\|)$ is complete with the usual norms.

- $(M, d)$ is a metric space with the *discrete metric*.

  It is complete, because...

  ... any Cauchy sequence is constant after some $N$ index.

- $\ell^2$ is a HILBERT-SPACE.

# Example. Is $C[a, b]$ complete?

YES. Let $(f_n) \subset C[a, b]$ be a Cauchy sequence.

Then $\forall \varepsilon > 0$ $\exists N$ such that

$$\|f_n - f_m\| = \max_{x \epsilon [a,b]} |f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m \geq N$$

$$\implies \forall x : \ |f_n(x) - f_m(x)| < \varepsilon, \qquad \forall n, \ m > N.$$

Then for a fixed $x \epsilon [a, b]$ $(f_n(x)) \subset \mathbb{R}$ is Cauchy:

$$\exists \lim_{n \to \infty} f_n(x) =: f_0(x).$$

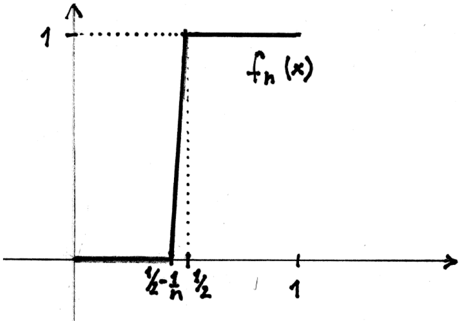
$\implies f_0 : [a, b] \to \mathbb{R}$ is well defined, $\lim_{n \to \infty} f_n = f_0$ *uniformly*.

Thus $f_0$ is continuous, i.e. $f_0 \epsilon C[a, b]$. $\quad f_0 = \lim_{n \to \infty} f_n \surd$

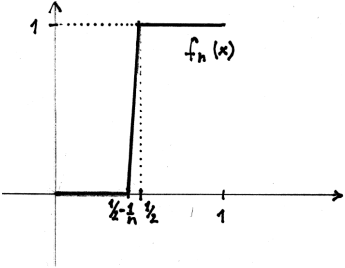
# Example. Is $C_2[0, 1]$ complete?

**NOT.** We'll show $(f_n) \subset C_2[a, b]$ Cauchy sequence, that is not convergent. For $n \geq 3$ let us define

$$f_n(x) := \begin{cases} 0 & \text{if} \quad x < 1/2 - 1/n \\ 1 & \text{if} \quad x > 1/2 \\ \text{linear} & \text{if} \quad 1/2 - 1/n \leq x \leq 1/2 \end{cases}$$




$f_n \in C_2[0, 1]$.




$$\int_0^1 (f_n(x) - f_m(x))^2\,dx = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{m}\right)^2,$$

$$\implies \lim_{n,m\to\infty} \|f_n - f_m\|_2 \to 0,$$

$(f_n)$ is a *Cauchy sequence* in $C_2[0,1]$.

For a fixed $x \epsilon [0,1]$:

$$\lim_{n\to\infty} f_n(x) = f_0(x) = \begin{cases} 1 \text{ if } x \geq 1/2 \\ 0 \text{ if } x < 1/2 \end{cases}$$

$f_0$ is not continuous. $f_0 \not\epsilon C_2[0,1]$!

The Cauchy sequence has no limit in $C_2[0,1]$.

There is a hole in it.

# Example. $C_2[a, b]$ is not complete!

"Completition" of $(\mathbb{Q}, |\cdot|)$ was $(\mathbb{R}, |\cdot|)$, by adding the *irrational* numbers.

It would be desirable to extend the set of

continuous functions in such a way,

that any Cauchy sequence in squared distance

has limit.

# Motivation. A bit of history.

We have seen, that $C_2[0, 1]$ is NOT complete!

Moreover, $\mathcal{R}[0, 1]$ is not complete with the quadratic norm.

$\longrightarrow$ Need for a *new concept of integral*...

Around 1900 it was *Henri Lebesgue*, who developed

a *new concept of integral* that is based on *measure theory*.

*"One of the greatest mathematical achievements of the twentieth century."*

# Measure

”measure” $\quad \approxeq$

- length of an interval
- area of a domain in the plain
- volume
- weight
- …

Definition. $X$ is an arbitrary set.

Define the *set of all subsets* of $X$ by

$$2^X = \{A \subset X\}$$

# Measurable space

Definition. $\mathcal{R} \subset 2^X$ is an ALGEBRA ($\sigma$-algebra), if:

1. $X \epsilon \mathcal{R}$,
2. $A, B \epsilon \mathcal{R} \Longrightarrow A \cup B \epsilon \mathcal{R}$,

2.+ $A_k \epsilon \mathcal{R}, \quad k = 1, 2, \ldots \Longrightarrow \bigcup_{k=1}^{\infty} A_k \epsilon \mathcal{R}$

3. $A, B \epsilon \mathcal{R} \Longrightarrow A \setminus B \epsilon \mathcal{R}$.

Definition. $(X, \mathcal{R})$ is a MEASURABLE SPACE, if $\mathcal{R} \subset 2^X$ is $\sigma$-algebra. The elements of $\mathcal{R}$ are MEASURABLE SETS.

*Example.* $X$ is metric space, $\mathcal{B}$ is the *Borel sets*:

$$\mathcal{B} = \sigma\left(\{ \text{open sets}, \quad \text{closed sets} \}\right)$$

# Measure space

Definition. Let $\mathcal{R}$ be a $\sigma$-algebra.

A function $\mu : \mathcal{R} \to \mathbb{R}^+ \cup \{+\infty\}$ is a MEASURE, if

- $\sigma$*-additive*: i.e.

  if $A_1, \ldots, A_n, \ldots \epsilon \mathcal{R}$ are disjoints, and $\bigcup_{k=1}^{\infty} A_k \epsilon \mathcal{R}$,

  $$\implies \quad \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

- $\mu(\emptyset) = 0$.

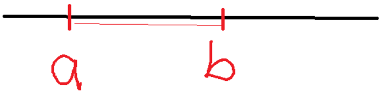
$(X, \mathcal{R}, \mu)$ is called MEASURE SPACE.

# Lebesgue measure on $\mathbb{R}$

$X = \mathbb{R}$. We introduce *measurable sets* and *measure* gradually.

*Goal*: $(\mathbb{R}, \mathcal{M}, m)$.

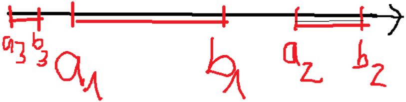
*Step 1.* $\mathcal{I}$ is the set of *finite intervals*. E.g. $I = \{x : a \leq x \leq b\}$




The measure in $\mathcal{I}$ is ("*=length*"): $m(I) = b - a$.

# Step 2.

Simple sets: $\mathcal{E} = \left\{ A \subset \mathbb{R} \; : \; A = \bigcup_{k=1}^{n} I_k, \quad I_k \epsilon \mathcal{I} \text{ are disjoint} \right\}$




If $A \epsilon \mathcal{E}$, let $m(A) = \sum_{k=1}^{n} m(I_k)$.

$m$ is $\sigma$-additive on $\mathcal{E}$. BUT $\mathcal{E}$ is not $\sigma$-algebra.

$(\mathbb{R}, \mathcal{E}, m)$ is *not a measure space* yet.

*Step 3.* Define an outer measure on $2^{\mathbb{R}}$.

$$A \subset \mathbb{R}: \quad m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} m(I_k) : \; A \subset \bigcup_{k=1}^{\infty} I_k, \; I_k \epsilon \mathcal{I} \right\}.$$

$m^* : 2^{\mathbb{R}} \to \mathbb{R}^+ \cup \{+\infty\}$.

If $A \epsilon \mathcal{E}$, then $m^*(A) = m(A)$. $\surd$ (But!) $m^*$ is <u>not $\sigma$-additive</u>.

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Up to this point we have:

| $\mathcal{E}$ | $\subset$ | ? | $\subset$ | $2^{\mathbb{R}}$ |
|---|---|---|---|---|
| not a $\sigma$-algebra | | | | $\sigma$-algebra |
| $m$ is $\sigma$-additive | | | | $m^*$ is not $\sigma$-additive |

*Step 4.* There's a middle ground.

$\exists \mathcal{M}$ $\sigma$-algebra:

$$\mathcal{E} \subset \mathcal{M} \subset 2^{\mathbb{R}},$$

s.t. $m^*|_{\mathcal{M}}$ is $\sigma$-additive.

Thus the restriction of $m^*$ onto $\mathcal{M}$ is <u>measure</u>.

It is called LEBESGUE MEASURE.

Elements of $\mathcal{M}$ are *Lebesgue-measurable sets* of $\mathbb{R}$.

The restriction of $m^*$ onto $\mathcal{M}$ is the *Lebesgue measure*.

It is denoted by $m$.

# Lebesgue measurable sets

**Q.** What are the elements of $\mathcal{M}$, *Lebesgue measurable sets*?

*Answer*, partly.

- All open and closed sets.
- Countable union and intersections of open & closed sets.

$$\mathcal{B} \subset \mathcal{M}, \quad \text{but} \quad \mathcal{B} \neq \mathcal{M}$$

- The Cantor set.
- Some *very strange* sets. Etc.

It is a good question,*whether there are non-measurable sets*.

YES. It is far from trivial to find such an example!