Functional analysis

Lecture 3.

February 25, 2021

Review of function space
$C[a, b]$

The *max*-norm in $C[a,b]$

$[a,b] \subset \mathbb{R}$ is a bounded interval.

$$C[a,b] = \{f : [a,b] \to \mathbb{R}, \quad \text{continuous}\}, \quad \|f\|_\infty = \max_{x\epsilon[a,b]} |f(x)|$$

Proposition. $(f_n) \subset C[a,b]$ and $f \epsilon C[a,b]$. Then

the following two statements are equivalent:

1. $f_n \to f$ in *max*-norm $\|\cdot\|_\infty$.
2. f_n tends to f uniformly on $[a,b]$.

Proof. 1. $\Rightarrow$ 2. If $f_n \to f$ in max-norm $\Longrightarrow \forall \varepsilon > 0$: $\exists N$ s.t.

$$\max_{x\epsilon[a,b]} |f_n(x) - f(x)| < \varepsilon \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon \quad \forall x, \quad \forall n \geq N.$$

Uniform convergence $\surd$

2. $\Rightarrow$ 1. HW

Two norms in $C[a, b]$

In $C[a, b]$ the *sup-norm* *doesn't come* from an inner product.

$$\|f\|_\infty = \max_{x \epsilon [a,b]} |f(x)|$$

However THERE IS an inner product in this space, defined as

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx. \qquad \Longrightarrow \quad \|f\|_2 = ...$$

In general $\|f\|_\infty \neq \|f\|_2$.

Which of the two norms is more desirable? It depends...

We do not have a preference yet.

Finite dimension

V is a *linear space* (with or without a norm on it).

$v_1, \; v_2, \ldots, \; v_n \epsilon V$ are linearly independent, if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0 \qquad \Longleftrightarrow \; \alpha_1 = \alpha_2 = \ldots \alpha_n = 0.$$

The dimension of V is n, if

1. $\exists v_1, v_2, \ldots, v_n \epsilon V$ linearly independent,
2. s.t. $\forall w \epsilon V$: $w = \sum_{k=1}^{n} a_k v_k$.

Then $v_1, v_2, \ldots, v_n$ is a *basis* in V

In this case V is FINITE DIMENSIONAL.

Infinite dimension

Example. $\mathbb{R}^n$ is finite dimensional. A possible basis:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots 0) \\ e_2 &= (0, 1, 0, \ldots 0) \\ &\vdots \\ e_n &= (0, 0, 0, \ldots 1) \end{aligned}$$

Definition. The DIMENSION OF V IS ∞, if

$$\forall n \epsilon \mathbb{N} \qquad \exists v_1,\ v_2, \ldots,\ v_n \epsilon V \quad \text{linearly independent.}$$

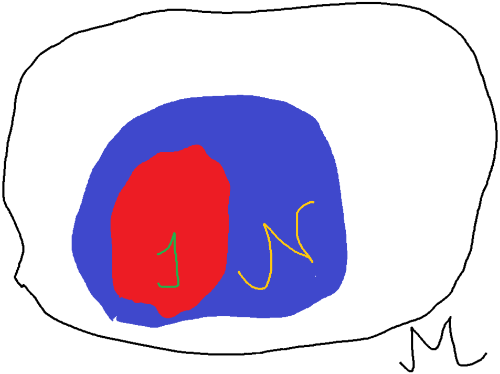
$\Longrightarrow$ There is no finite subset, that spans the entire space.

Example of infinite dimensional space?

The topology of metric spaces

Abstract spaces in FA

Metric space $\ll$ Normed space $\ll$ Inner product space



We introduce some basic concepts s.t. we can use it in more complex structures. It is advisable to consider *the simplest structure* now.

Open sets

(M, d) is a metric space.

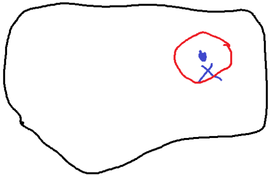
Definition. An *open ball* centered at $x \epsilon M$ with radius $r > 0$ is:

$$B_r(x) = \{y \epsilon M \ : \ d(x, y) < r\}. \quad (\textit{open neighborhood})$$

$E \subset M$ is a subset.

$x \epsilon E$ is INTERIOR POINT OF E,

if $\exists r > 0$, s.t. $B_r(x) \subset E$.



Definition. $E \subset M$ is an OPEN SET, if $\forall x \epsilon E$ is interior point.

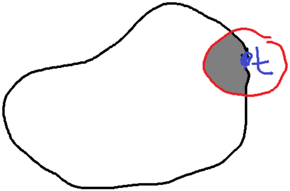
Proposition. Union of *any number* of open sets is open.

Closed sets

$F \subset M$ is a subset.

$t \epsilon M$ is a LIMIT POINT of F, if

$$\forall \varepsilon > 0 : \quad \left(B_\varepsilon(t) \backslash \{t\} \right) \cap F \neq \emptyset.$$



Definition. $F \subset M$ is CLOSED SET, if it contains $\forall t$ limit points.

Proposition. Intersection of *any number* of closed sets is closed.

Open and closed sets

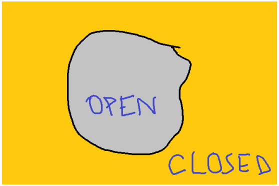
Proposition. The following properties are equivalent

1. $F \subset M$ is closed.
2. $\forall (x_n) \subset F$ convergent sequences: $\lim_{n\to\infty} x_n \epsilon F$.

Proposition. $E \subset M$ is *open*

if and only if

$M \setminus E = F$ is *closed*.



Remark. $E \subset M$ is a subset. It is open ~~OR~~ closed.

+ neither
+ both (?)

Examples.

1. $M = \mathbb{R}$ with Eucledian distance. $a < b$.

   Then $[a, b]$ is closed, (a, b) is open.

2. $M = C[a, b]$. $k > 0$ is a fixed number.

$$E = \{f : |f(x)| < k \quad \forall x, \quad f \text{ is continuous}\}$$

$$F = \{f : |f(x)| \leq k \quad \forall x, \quad f \text{ is continuous}\}$$

$$\implies \quad E \subset C[a, b] \textit{ is open}, \qquad F \subset C[a, b] \textit{ is closed}.$$

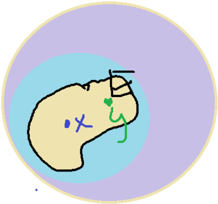
3. $(V, \|\cdot\|)$ is a normed space.

$$B_1 = \{v \ : \ \|v\| = 1\} \quad \implies \quad \textit{is closed}.$$

$$B_2 = \{v \ : \ \|v\| \leq 1\} \quad \implies \quad \textit{is closed}.$$

Bounded and compact sets

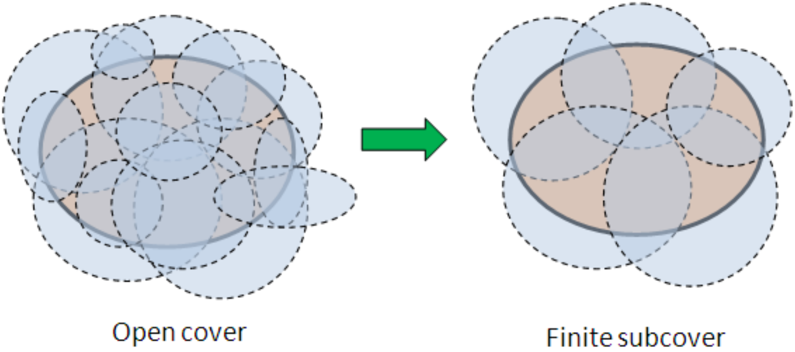
- $E \subset M$ is BOUNDED, if for an $x \epsilon E$
  $\exists r > 0$, s.t. $E \subset B_r(x)$.



- $E \subset M$ is a subset.
  A collection of sets $\{U_\alpha,\ \alpha \epsilon I\}$ is a COVER, if $E \subset \bigcup_{\alpha \epsilon I} U_\alpha$.
  - $\{U_\alpha,\ \alpha \epsilon I\}$ is an *open cover*, if U_α is open $\forall \alpha$.
  - It is a *finite cover*, if the number of sets is finite.
- $E \subset M$ is COMPACT , if any *open cover* contains a *finite subcover*.

Compactness

Again. $E \subset M$ is COMPACT, if any *open cover* contains a *finite subcover*:



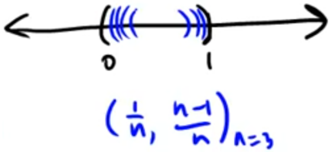
Compact set – not compact set

Example. In $(\mathbb{R}, |\cdot|)$ metric space $E_1 = [0, 1]$ is compact and $E_2 = (0, 1)$ is not compact. Which one is easier to prove?

Let's show, that E_2 is not compact.

Let us consider the open sets

$U_n = \left(\frac{1}{n}, \frac{n-1}{n}\right)$ for $n \geq 3$



Then $(0, 1) = \bigcup_{n=3}^{\infty} U_n$, and there is NO finite subcover! WHY?

Next: Let's prove that E_1 is compact. Seems more difficult...

An alternative definition.

$E \subset M$ is SEQUENTIALLY COMPACT, if

$\forall (x_n) \subset E$ there is a *convergent* (x_{n_k}) subsequence s.t.

$$\lim_{n_k \to \infty} x_{n_k} = x_0 \epsilon E.$$

Theorem. E is compact $\iff$ it is sequentially compact. ~~Proof~~

Let's prove that E_1 is compact. Not very difficult...

If $(x_n) \subset [0,1] \Longrightarrow$ it is bounded. $\overset{\text{B.W}}{\Longrightarrow}$ $\exists (x_{n_k})$ convergent,

$$\lim_{n_k \to \infty} x_{n_k} = x_0 \epsilon [0,1].$$

Thus E_1 is sequentially compact.

Properties

1. If $E \subset M$ is compact, then it is *bounded*. (Exc.)
2. If $E \subset M$ is *compact*, then it is *closed.* (Exc.)

Converse?

$$\text{"}E \text{ is closed and bounded"} \quad \overset{?}{\Longrightarrow} \quad \text{"compact"}$$

Guess? *(Partly true $\surd$)*

Theorem. (Heine-Borel thm.)

$E \subset \mathbb{R}^n$ is compact $\iff$ E is *closed* and *bounded*.

(With any norm defined on $\mathbb{R}^n$)

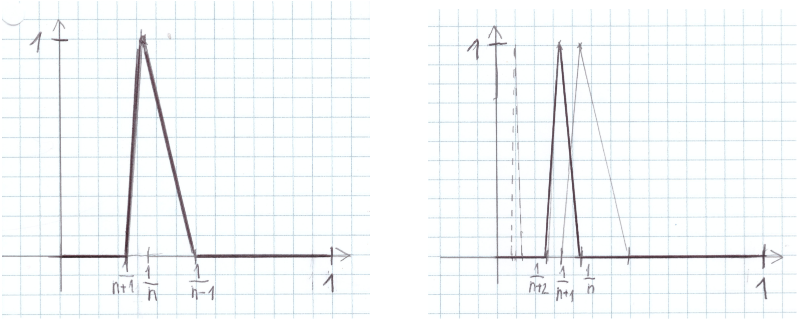
In infinite dimension?

In $C[0,1]$ the closed unit ball is bounded and closed.

$$
\begin{aligned}
B_1(0) &= \{f : [0,1] \to \mathbb{R}, \text{cont. } \max |f(x)| \leq 1\} = \\
&= \{f \epsilon C[0,1] \ : \ \|f\|_\infty \leq 1\}. \quad \text{compact?}
\end{aligned}
$$

Let us define the functions for $n \geq 2$ Draw!:

$$
f_n(x) = \begin{cases}
1, & \text{if} \quad x = \frac{1}{n} \\
0, & \text{if} \quad x \leq \frac{1}{n+1} \ \text{ or } \ x \geq \frac{1}{n-1} \\
\text{linear}, & \text{if} \quad x \epsilon \left(\frac{1}{n+1}, \frac{1}{n}\right) \\
\text{linear}, & \text{if} \quad x \epsilon \left(\frac{1}{n}, \frac{1}{n-1}\right)
\end{cases}
$$



Then $(f_n) \subset C[0,1]$, and $\|f_n\| = 1 \ \forall n$.

But (f_n) has no convergent subsequence. $B_1(0)$ is not compact.

Thus in infinite dimension "closed +bounded" $\not\Longrightarrow$ "compact".

Application of compact sets

(M, d) is a metric space. $E \subset M$ is a compact set.

Assume $f : E \to \mathbb{R}$ is continuous.

Theorem. 1. of Weierstrass

Then f is bounded.

Theorem. 2. of Weierstrass

Then f attains its minimum and maximum on E.