Functional analysis

Lecture 2.

February 18, 2021

Inner product space over $\mathbb{K}$ (real or complex)

$(V, \langle\cdot,\cdot\rangle)$ is an INNER PRODUCT SPACE, if

1. V is a linear space over $\mathbb{K}$.
2. $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ is an operation with properties:
   - $\langle v, v\rangle \geq 0$, and $\langle v, v\rangle = 0 \iff v = 0$
   - $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$, $\lambda \epsilon \mathbb{K}$.
   - $\langle v, w\rangle = \langle w, v\rangle$, or $\langle v, w\rangle = \overline{\langle w, v\rangle}$
   - $\langle v, w + u\rangle = \langle v, w\rangle + \langle v, u\rangle$, distributive property.

Definition. v and w are ORTHOGONAL, if $\langle v, w\rangle = 0$.

Remark. Obviously $v = 0$ is orthogonal to every vetcor.

Inner product space $\longrightarrow$ Normed space

Theorem. *Inner products* always give rise to *Norms*.

Specifically, assume $(V, \langle\cdot,\cdot\rangle)$ is an inner product space.

$$\Downarrow$$

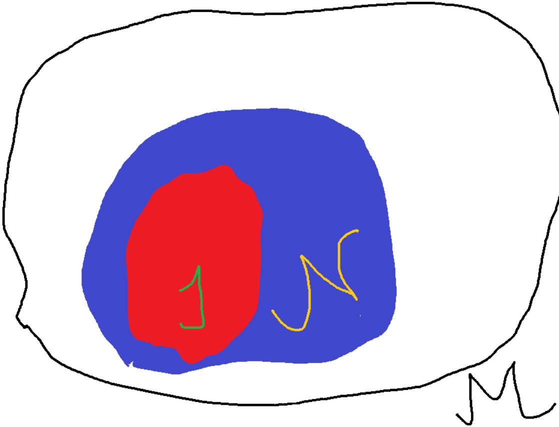
Then it is also a normed space with the following norm:

$$\|v\| = \langle v, v\rangle^{1/2}.$$

(HW: check it!)

Inner product space $\longrightarrow$ Normed space

The other direction is not true!



Inner product in $\mathbb{R}^n$?

In $\mathbb{R}^n$ we have defined norms:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad \|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

The extension of these norms is possible:

For $1 \leq p < \infty$ define $\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$.

Proposition.

$\left(\mathbb{R}^n, \|\cdot\|_p\right)$, $1 \leq p \leq \infty$ is an inner product space $\iff$ $p = 2$.

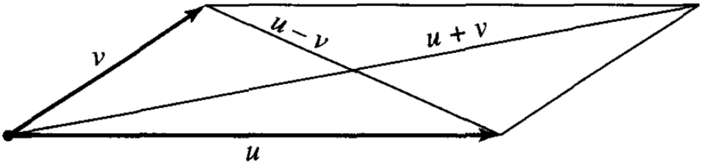
$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k.$$

The paralelogram equality

Theorem. $(V, \|\cdot\|)$ is a normed space. Then

the norm can be derived from an inner product $\Longleftrightarrow$

$$\|u+v\|^2 + \|u-v\|^2 = 2\big(\,\|u\|^2 + \|v\|^2\,\big) \qquad \forall u, v \epsilon V.$$



Details on Practical Class.

A corollary of the Paralelogram rule

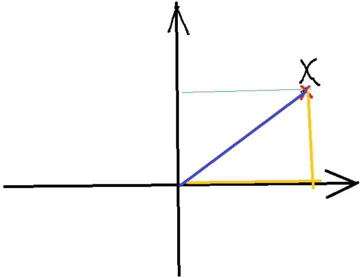
In $\mathbb{R}^2$ we have seen three norms up to this point.

For any $x = (x_1, x_2)$ we have defined

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|_1 = |x_1| + |x_2|,$$

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}$$



You will see, that only $\|x\|_2$ comes form an inner product, the others *do not.*

An important inequality: (Cauchy-Schwarz-Bunyakovskii)

Theorem. If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then

$$|\langle v, w\rangle| \leq \sqrt{\langle v, v\rangle} \cdot \sqrt{\langle w, w\rangle}$$

Outline of proof: Assume first $\|w\| = 1$.

Define $u := v - \langle v, w\rangle\, w$.

$$\implies \quad \|u\|^2 = \langle u, u\rangle = \langle v, v\rangle + \langle v, w\rangle^2 \cancel{\langle w, w\rangle} - 2\langle v, w\rangle \langle v, w\rangle =$$

$$= \langle v, v\rangle - \langle v, w\rangle^2 \geq 0 \quad \implies \quad \|v\| \geq |\langle v, w\rangle|.$$

In general, $w = \lambda w_0$ with $\|w_0\| = 1$. Finish the proof.

Important special spaces I.
Sequence spaces

Sequence spaces

V is a LINEAR SPACE of sequences with elements

$$x = (x_1, x_2, \dots, x_n, \dots) = (x_n), \qquad x_n \epsilon \mathbb{R}.$$

The subspace of *bounded* sequences: ℓ^∞ *"little ell infinity"*

$x \epsilon \ell^\infty$ if $\exists K \epsilon \mathbb{R} : \ |x_n| \leq K \quad \forall n \epsilon \mathbb{N}.$

$c_0 \subset \ell^\infty$ is the linear space of *null sequences*.

Proposition. ℓ^∞ (and c_0) is a NORMED SPACE with norm:

$$\|x\|_\infty := \sup\{|x_i|,\ i = 1, 2, \dots\} \qquad (\approx \max).$$

Examples

1. $x_n = (1 + \frac{1}{n})^n$, $\|x\| =?$
2. $y_n = 1 + \frac{1}{n}$, $\|y\| =?$
3. $z_n = \frac{(-1)^n}{n}$, $\|z\| =?$

Answers: $\|x\| = e, \quad \|y\| = 2, \quad \|z\| = 1$

In $\mathbb{R}^n$ we had norms $\|x\|_1$ and $\|x\|_2$:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Extend these norms for <u>sequences</u>? What is the problem?

Need to consider *special subspaces* of c_0, with certain

SUMMABILITY condition.

"*little ell one*" $\qquad \ell^1 = \left\{ (x_n) \ : \ \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$

("*little ell 1*") $\ell^1 = \{(x_n) \ : \ \sum_{n=1}^{\infty} |x_n| < \infty\}$

Is it a vector space?

1. $x \epsilon \ell^1 \implies \lambda x \epsilon \ell^1 \surd$
2. $x, y \epsilon \ell^1 \implies x + y \epsilon \ell^1 ?$

$$\forall N \qquad \sum_{n=1}^{N} |x_n + y_n| \leq \sum_{n=1}^{N} |x_n| + \sum_{n=1}^{N} |y_n| \surd$$

Let us define a norm in ℓ^1: $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$

This is really a norm. Check it!

Question: $\|x\|_1$ v.s. $\|x\|_\infty$?

ℓ^p (*"little ell p"*) for $1 \le p < \infty$

A possible linear space is: ℓ^p ("*little ell p*"):

$$\ell^p = \left\{ (x_n) \ : \ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} \subset c_0.$$

Proposition. ℓ^p is a normed space with norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Proof. Not trivial!

Examples. To which ℓ^p do these sequences belong to?

1. $x_n = \frac{1}{n}, \quad p = ?$
2. $y_n = \frac{1}{2^n}, \quad p = ?$
3. $v_n = (-1)^n \frac{1}{n}, \quad p = ?$
4. $z_n = 1, \quad p = ?$

The most important of all ℓ^p is ℓ^2. *(Why?)*

For $p = 2$ it is possible to define an INNER PRODUCT:

$$\langle x, y \rangle := \sum_{k=1}^{\infty} x_k \overline{y_k}$$

This is *the only norm*, that comes from an inner product.

Important special spaces II.
Function spaces

Bounded functions

$[a, b] \subset \mathbb{R}$ is a bounded interval.

X: linear space of *all functions* defined on $[a, b]$.

V is the linear space of *bounded functions* on $[a, b]$.

$$V = \{f : [a, b] \to \mathbb{R}, \quad \exists B : |f(x)| \leq B, \quad \forall x\}.$$

Proposition. V is a normed space with norm

$$\|f\|_\infty = \sup\{|f(x)| \ : \ x \epsilon [a, b]\}.$$

A subspace of V is $C[a, b]$, the set of continuous functions:

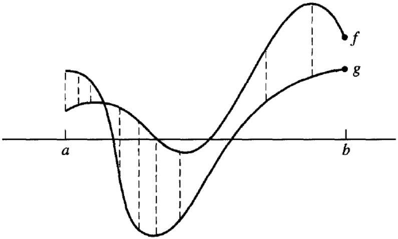
$$C[a, b] = \{f : [a, b] \to \mathbb{R}, \quad \text{continuous}\}.$$

In $C[a,b]$ we have the norm as before:

$$\|f\|_\infty = \sup\{|f(x)| \ : \ x \epsilon [a,b]\}.$$

This is a *metric space* also, with metric:

$$d(f,g) = \sup_{x \epsilon [a,b]} |f(x) - g(x)|$$



By Thm. of Weierstrass we have in $C[a,b]$

$$d(f,g) = \max_{x \epsilon [a,b]} |f(x) - g(x)|$$

$C[a, b]$ with inner product?

In $C[a, b]$ it is possible to define *inner product*.

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx \quad \left(\int_a^b f(x)\overline{g(x)}dx \right) \qquad \text{(Explain it.)}$$

$\Longrightarrow$ Orthogonality of functions can be defined this way.

This inner product induces a norm: $\|f\| = \left(\int_a^b f^2(x)dx \right)^{1/2}$

<u>New norm!</u> It is the *quadratic norm*. Notation: $\| \cdot \|_2$.

The previous norm is called *sup-norm*, the notation is $\| \cdot \|_\infty$.

Notation. $C_2[a, b]$. In general $\|f\|_\infty \neq \|f\|_2$.

Question: Is there an *inner product* in $C[a, b]$ for *sup*-norm?