Functional analysis

Lesson 1.

February 11, 2021

Basics

Lectures: Thursday 14:15. Online, (may be offline later)

Practical classes: Friday 12:15. Online

Tutorials: To be set up

REFERENCES:

K. Saxe: Beginning Functional Analysis. Springer-Verlag 2002.

"What everyone should know about functional analysis."

Vágó Zsuzsanna: Funkcionálanalízis (In Hungarian)

Requirements and grading

(All details are on webpage)

1. We'll start Lectures with a *20 minutes Moodle test.*

   Thursday at 14:15. They are worth *10 points* each.

2. 1-2 HW every week. To be solved in writing within a week.

   $\geq$ 8 good solutions. The extra solutions will be credited.

3. There will be an exam at the end of the term.

   A recommended mark is offered.

4. An option: brief bibliographies of "key players" – a possible project. (Next two weeks?)

Requirements and grading (cont.)

1. The total score:

$$\sum \text{written test (110 pts)} + 50\% \text{ excess HW.}$$

2. Requirement for end-term signature is 40 points.
3. A recommended mark is offered as follows:
   - 60–84: *recommended 2*
   - 85–99: *recommended 3*
   - 100–110: *recommended 4*
   - 111–: *recommended 5*

History

Functional analysis:

last years of 19th century – first decades of 20th century

(Paul Levy)

Response to ”*understand physical phenomena*”

$\Longrightarrow$ study DIFFERENTIAL EQ and INTEGRAL EQ.

PREREQUISITES:

First courses in *linear algebra* and *analysis* (calculus).

(You may have to review these topics! Ask at Tutorial classes.)

PURPOSE:

It is a unifying approach

to view *functions* as *points* in some abstract vector space

and to study DE-s and IE-s in terms of

linear transformations on these spaces.

Review. $\mathbb{R}^n$ – the most important space in Analysis

$\mathbb{R}^n$ is a *vector space* or *linear space*.

Elements: *vectors.* $x = (x_1, x_2, \ldots, x_n)$, $x_k \epsilon \mathbb{R}$.

Operations: *addition* and *multiplication by a scalar*

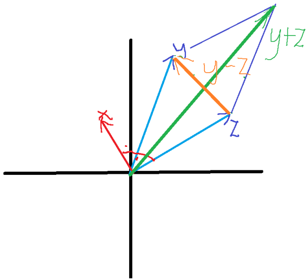
with certain properties $\surd$

Question: What is a *subspace*?

Some properties:

- Distance of two elements
- Length of a vector
- Orthogonality

(draw for $n = 2$)



$\mathbb{R}^n$, $n \to \infty$?

V is a linear space with elements $x = (x_1, x_2, \dots, x_n, \dots)$

- *Addition*:

$$x + y = (x_1 + y_1,\ x_2 + y_2,\ \dots, x_n + y_n, \dots).$$

- *Multiplication by a scalar*:

$$\lambda x = (\lambda x_1,\ \lambda x_2,\ \dots, \lambda x_n, \dots).$$

The well-known properties of these operations $\surd$

V is the *linear space of all sequences*.

(Please repeat the definition of vector space.)

Sequence spaces. Some important *linear subspaces* of V.

1. $\ell^\infty \subset V$ ("*little ell infinity*"): *bounded* sequences.

$$x \epsilon \ell^\infty \qquad \text{if} \qquad \exists K \epsilon \mathbb{R} : \ |x_n| \leq K \quad \forall n \epsilon \mathbb{N}.$$

2. c: *convergent* seq. $x \epsilon c$ if $\exists \lim_{n\to\infty} x_n$. *Ex.* $a_n = \dfrac{n+1}{n+2}$

3. c_0: *null* seq. $x \epsilon c_0$ if $\lim_{n\to\infty} x_n = 0$. *Ex.* $b_n = \dfrac{1}{n+2}$.

4. ℓ^1 ("*little ell one*"). $\ell^1 = \{(x_n) \ : \ \sum_{n=1}^{\infty} |x_n| < \infty\}$.

Are they subspaces, indeed?

Abstract spaces. Should be known from LA

Metric space $\ll$ Normed space $\ll$ Inner product space

Generally called *Topologic space*

They are given in order of *increasing structure*.

From left to right:

From *simple* structure to *more complicated* structure.

From left to right:

All definitions and all *theorems* remain true.

It is ”easiest” to be a metric space.

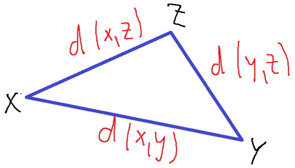
It is is ”easiest” to work with the inner product space.

Metric space

Goal: defining distance in an abstract way.

M is a set, $d : M \times M \to \mathbb{R}$ is a function with these properties:

- $d(x, y) \geq 0$, *nonnegative*,
- $d(x, y) = 0 \iff x = y$, *non degenerate*,
- $d(x, y) = d(y, x)$, *symmetric*,
- $d(x, y) + d(y, z) \geq d(x, z)$, *triangle inequality*.



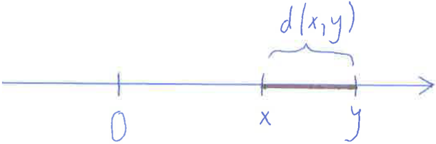
Then (M, d) is a METRIC SPACE.

The function $d(x, y)$ is the METRIC.

Metric space. Examples.

0. $M = \mathbb{R}$,

   $d(x, y) = |x - y|$

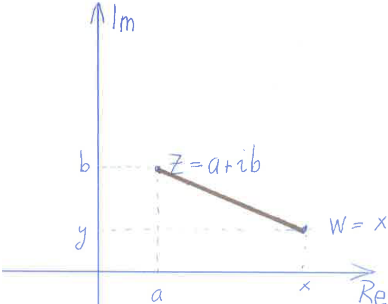


1. $M = \mathbb{R}^n$.

   $$d(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} \qquad x = (x_1, ..., x_n),\ y = (y_1, ..., y_n).$$

2. $M = \mathbb{C}$,

   $d(z, w) = |z - w|$



3. Let $n \epsilon \mathbb{N}$ be a fixed number

$$M = \{x = (x_1, \ldots, x_n) \mid x_i \epsilon \{0, 1\}\}, \quad d(x, y) = \#\{i \mid x_i \neq y_i\}.$$

E.g. $n = 5$.

$x = (0, 0, 0, 1, 1)$, $y = (1, 0, 0, 1, 0)$.

$d(x, y) =?$

4. *Discrete metric*. M is an arbitrary set. The metric:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}, \qquad \forall x, y \epsilon M.$$

E.g. In previous example?

Topological notions of $\mathbb{R}$ in a metric space

Definítion. (M, d) is a metric space. $(x_n) \subset M$ is a sequence.

(x_n) is *convergent* and *the limit of* (x_n) *is* $x^* \epsilon M$, if

$\forall \varepsilon > 0 \; \exists N$, such that

$$d(x_n, x^*) < \varepsilon \qquad \text{if} \qquad n \geq N.$$

Example. Let d be the *discrete metric on* $\mathbb{R}$.

What are convergent sequences in $(\mathbb{R}, d)$? n E.g. Is the sequence $x_n = \dfrac{1}{n}$ convergent?

Topological notions from $\mathbb{R}$ in a metric space

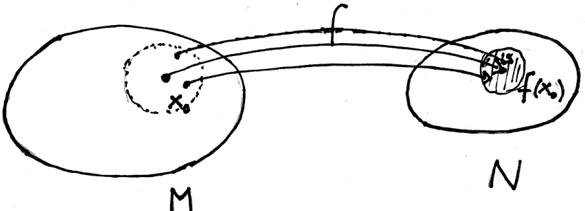
Definítion.

Let (M, d_M) and (N, d_N) be metric spaces.

f is a function $f : M \to N$. Let $x_0 \epsilon M$ be an arbitrary point.

f is *continuous at x_0*, if $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$\text{if} \quad d_M(x, x_0) < \delta \quad \Longrightarrow \quad d_N(f(x), f(x_0)) < \varepsilon$$



Normed space

Goal: defining length in an abstract way.

V a linear space over $\mathbb{K}$. (It may be $\mathbb{R}$ or $\mathbb{C}$.)

The *norm* is a $\|\cdot\| : V \to \mathbb{R}$ function with these properties:

1. $\|v\| \geq 0$, *nonnegative*,
2. $\|v\| = 0 \iff v = 0$, *non degenerate*,
3. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$, $\forall \lambda \epsilon \mathbb{K}$, *multiplicative*,
4. $\|v + w\| \leq \|v\| + \|w\|$, *triangle inequality*.

Then $(V, \|\cdot\|)$ is a normed space.

Normed space. Examples

1. $V = \mathbb{R}$, $\|x\| = |x|$.
2. $V = \mathbb{R}^n$,

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

   This is *Euclidean norm*, or *quadratic norm*

3. More norms in $\mathbb{R}^n$:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_\infty = \max_{i=1,\dots,n} |x_i|$$

Normes in $\mathbb{R}^2$

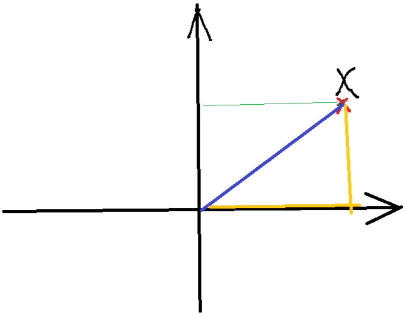
It is possible to define many norms in $\mathbb{R}^2$.

For any $x = (x_1, x_2)$ we define

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|_1 = |x_1| + |x_2|,$$

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}$$



Normed space $\longrightarrow$ Metric space

Theorem. Norms always give rise to metrics.

Specifically, assume $(V, \|\cdot\|)$ is a normed space.

$$\Downarrow$$

Then it is also a metric space with the following metric:

$$d(x,y) := \|x-y\|, \qquad x, y \epsilon V.$$

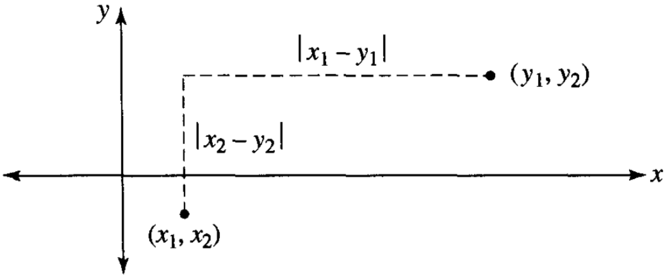
(HW: check it!)

Norm $\longrightarrow$ Metric in $\mathbb{R}^2$

$$\|x\|_1 = |x_1| + |x_2|,$$

$$\Downarrow$$

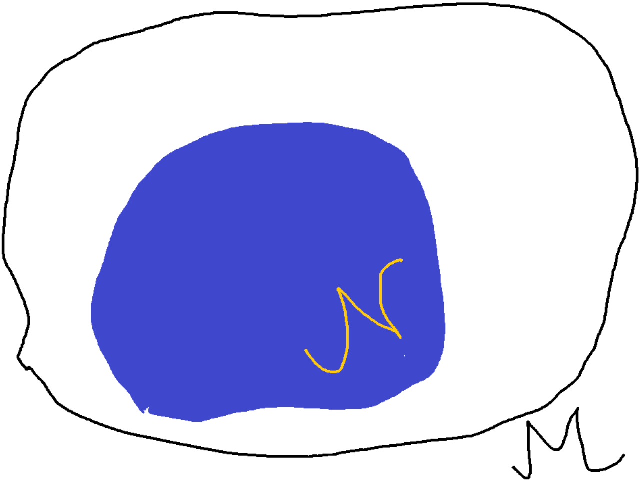
$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$$



Think about $d(x, y)$ and $d(x, y)$

Normed space ⟶ Metric space

The other direction is not true! Why?



Review. $\mathbb{R}^n$

Up to this point you have done *Math* in $\mathbb{R}^n$. Why?

$\mathbb{R}^n$ is a *vector space* , elements are *vectors*:

$$x = (x_1, x_2, \ldots, x_n), \qquad x_k \epsilon \mathbb{R}$$

Some Important properties:

- Distance of two elements $\longrightarrow$ *Metric space* $\surd$
- Length of a vector $\longrightarrow$ *Normed space* $\surd$
- Orthogonality $\longrightarrow$?

In $\mathbb{R}^n$ *what is orthogonality*?

Inner product space over $\mathbb{R}$

Goal: defining orthogonality in an abstract way.

V is a linear space over $\mathbb{R}$.

The inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is an operation:

1. $\langle v, v \rangle \geq 0$, nonnegative,

   and $\langle v, v \rangle = 0 \quad \Longleftrightarrow \quad v = 0$, nondegenerate.
2. $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$, $\lambda \epsilon \mathbb{R}$, multiplicative.
3. $\langle v, w \rangle = \langle w, v \rangle$, symmetric.
4. $\langle v, w + u \rangle = \langle v, w \rangle + \langle v, u \rangle$, distributive property.

Then $(V, \langle \cdot, \cdot \rangle)$ is a real INNER PRODUCT SPACE.

Inner product space over $\mathbb{C}$

Some changes in the definition:

The inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is an operation:

1. $\langle v, v \rangle \geq 0$, nonnegative,

   and $\langle v, v \rangle = 0 \iff v = 0$, nondegenerate.
2. $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$, $\lambda \epsilon \mathbb{C}$, multiplicative.
3. $\langle v, w \rangle = \overline{\langle w, v \rangle}$
4. $\langle v, w + u \rangle = \langle v, w \rangle + \langle v, u \rangle$, distributive.

Then $(V, \langle \cdot, \cdot \rangle)$ is a complex INNER PRODUCT SPACE.

Orthogonality and examples

Defin͡ition. v and w are orthogonal, if $\langle v, w \rangle = 0$.

1. **Example.** $V = \mathbb{R}^n$, n-dimensional real vectors with

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

2. **Example.** $V = \mathbb{C}^n$, n-dimensional complex vectors with

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w_i}$$

There are very few examples, not by accident.

Inner product space $\longrightarrow$ Normed space