Functional analysis

Lesson 12.

May 13. 2021

Adjoint of an operator in $\mathcal{B}(H)$. Review.

Let $(H, \langle \cdot, \cdot \rangle)$ be *Hilbert space*. $A \in \mathcal{B}(H)$ is a linear operator.

The ADJOINT OPERATOR OF A is the $A^* \in \mathcal{B}(H)$ such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x, y \in H.$$

Example. Let $H = \mathbb{C}^n$. The norm is the $\|\cdot\|_2$.

$$\mathcal{B}(H) \quad \equiv \quad \textit{complex matrices of dimension} \quad n \times n$$

Let $A \in \mathcal{B}(\mathbb{C}^n)$. Then $A^* = \overline{A}^T$, the *conjugate+transpose of A*

Orthogonal projection in H Hilbert space

Let $E \subset H$ be a *closed* subspace. Then $\forall x \in H$ can be written as

$$x = x_E + x_0 : \qquad x_E \in E \quad \text{and} \quad x_0 \perp E$$

Let $Px := x_E$.

Definition. $P : H \to H$ is the ORTHOGONAL PROJECTION onto E.

Proposition. Then $P = P^*$. The other direction is also true:

$$\text{If} \quad P = P^* \quad \Longrightarrow \quad P \text{ is an orthogonal projection.}$$

Self-adjoint operator

The operator A is SELF-ADJOINT, if $A = A^*$.

Theorem. If A is self-adjoint, then

1. $\|A^n\| = \|A\|^n$.
2. It's spectral radius is: $r(A) = \|A\|$.
3. The spectrum is real: $\sigma(A) \subset \mathbb{R}$.

Self-adjoint operators in infinite dimension are extensions of

symmetric matrices in finite dimension.

Example. $H = \mathcal{L}^2[a,b]$, $Mf(x) := x \cdot f(x)$. Then $\forall f,\ g \in H$:

$$\langle Mf, g\rangle = \int_{[a,b]} xf(x)g(x)\, dm = \int_{[a,b]} f(x)\, xg(x)\, dm = \langle f, Mg\rangle \ \Rightarrow\ M = M^*.$$

Hilbert space methods

in

Quantum Mechanics

AN EXAMPLE.

One particle (electron) is moving along a straight line.

Moving is described by: $f(x, t) \in \mathbb{C}$.

— t is the time

— x is the position

The probability, that the *position* is in $[a, b]$ at *time* t is :

$$\int_a^b |f(x,t)|^2 \, dx.$$

This $f(x, t)$ is the STATE FUNCTION. We expect:

$$\int_{-\infty}^{\infty} |f(x,t)|^2 \, dx = 1 \quad \forall t.$$

Now fix t. We use $f(x)$.

In *Math.* language

The state function $f \in \mathcal{L}^2(\mathbb{R})$, where $\mathcal{L}^2(\mathbb{R}) = H$ is a *Hilbert space*.

$$\|f\|^2 = 1.$$

The position x is an "OBSERVABLE", using QM terminology.

Remark. Another way: the *position* is a random variable.

The density function of this r.v. is $|f(x)|$.

Momentum

Another "observable" is the MOMENTUM. It is given by *FT of f:*

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixw} f(x)\, dx.$$

The probability of w is in $[a, b]$:

$$\int_a^b |\widehat{f}(w)|^2\, dw.$$

By Parseval equality

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\widehat{f}(w)|^2\, dw,$$

thus $\widehat{f} \in \mathcal{L}^2(\mathbb{R})$ and $\|\widehat{f}\| = 1.$

Let's denote $\overline{x}$ the mean value of the position :

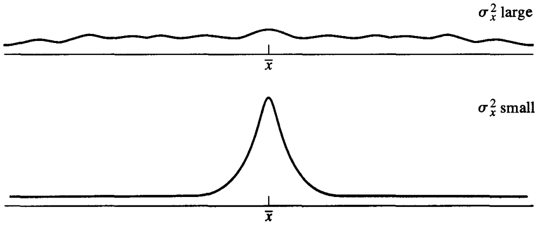
$$\overline{x} := \int_{-\infty}^{\infty} x \cdot |f(x)|^2 \, dx,$$

Let's denote $\overline{w}$ the mean value of the momentum:

$$\overline{w} := \int_{-\infty}^{\infty} w \cdot |\widehat{f}(w)|^2 \, dw.$$

The variances are:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \overline{x})^2 \cdot |f(x)|^2 \, dx, \qquad \sigma_w^2 = \int_{-\infty}^{\infty} (w - \overline{w})^2 \cdot |\widehat{f}(w)|^2 \, dw.$$



The HEISENBERG'S UNCERTAINTY PRINCIPLE states , that

$$\sigma_x^2 \cdot \sigma_w^2 \geq \frac{1}{4}.$$

(For simplicity Planck's constant is 1 here.)

Both of σ_x^2 and σ_w^2 can not be small at the same time,

"*position*" and "*momentum*" can not be localized simultaneously.

Sketch of the Proof

We assume $\overline{x} = 0$ and $\overline{w} = 0$. (By shifting...)

In the $\mathcal{L}^2(\mathbf{R})$ Hilbert space we consider two operators:

$$
\begin{aligned}
Mf(x) &:= x \cdot f(x) \\
Df(x) &:= f'(x).
\end{aligned}
$$

(*Remark.* M and D are defined in a subspace if H. No problem.

Our f is in a "good space".)

$\longrightarrow$ Up to this point $f, \widehat{f} \in \mathcal{L}^2(\mathbf{R})$, with unit norm.

Variance of the position

$$\|Mf\|^2 = \int_{-\infty}^{\infty} |x \cdot f(x)|^2 \, dx = \int_{-\infty}^{\infty} x^2 \cdot |f(x)|^2 \, dx = \sigma_x^2.$$

Thus we have proved :

Proposition.

$$\|Mf\|^2 = \sigma_x^2.$$

Variance of the momentum.

Proposition.

$$\|Df\|^2 = \sigma_w^2.$$

Proof. By the Parseval equality $\|Df\|^2 = \|\widehat{Df}\|^2$.

By the definition of the norm:

$$\|\widehat{Df}\|^2 = \int_{-\infty}^{\infty} |\widehat{Df}(w)|^2 \, dw.$$

An important property of the FT is:

$$\widehat{Df}(w) = iw\,\widehat{f}(w),$$

Thus

$$\|Df\|^2 = \|\widehat{Df}\|^2 = \int_{-\infty}^{\infty} w^2 \, |\widehat{f}(w)|^2 \, dw = \sigma_w^2.$$

$$\begin{aligned} Mf(x) &:= x \cdot f(x) \\ Df(x) &:= f'(x). \end{aligned}$$

Proposition. (*A special property*) M and D satisfy this equality:

$$DM - MD = I. \tag{1}$$

Proof. Apply the rule on the derivative of a product:

$$\big(x \cdot f(x)\big)' = f(x) + xf'(x),$$

or equivalently

$$D \circ M(f) = I(f) + M \circ D(f).$$

Remark. (1) is valid only in the right subspace.

Adjoint of M

Proposition. M is selfadjoint, i.e.

$$\langle Mf, g\rangle = \langle f, Mg\rangle.$$

Proof.

$$\langle Mf, g\rangle = \int_{-\infty}^{\infty} xf(x) \cdot g(x)\, dx = \int_{-\infty}^{\infty} f(x) \cdot xg(x)\, dx = \langle f, Mg\rangle.$$

Remeber?

Adjoint of D

Proposition. The adjoint of D is $-D$, i.e. $\forall f,\ g$

$$\langle Df, g\rangle = \langle f, -Dg\rangle \quad .$$

Proof. We use partial integration:

$$\langle Df, g\rangle = \int_{-\infty}^{\infty} f'g = fg\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} fg' = -\langle f, Dg\rangle.$$

Meanwhile we used the fact, that $\forall g \in \mathcal{L}^2(\mathbf{R})$:

$$\lim_{x\to\pm\infty} g(x) = 0.$$

FINALLY

$$
\begin{aligned}
1 = \|f\|^2 &= \langle f, f\rangle = \langle f, (DM - MD)f\rangle = \\
&= \langle f, DMf\rangle - \langle f, MDf\rangle = \\
&= -\langle Df, Mf\rangle - \langle Mf, Df\rangle
\end{aligned}
$$

Thus $1 = 2|\langle Df, Mf\rangle|$. Then using C-S-B inequality

$$\frac{1}{2} = |\langle Df, Mf\rangle| \leq \|Mf\| \cdot \|Df\| = \sigma_x \cdot \sigma_w,$$

+ rearrangement $\surd$