Functional analysis

Lesson 11.

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$\mathcal{B}(X)$ Banach algebra

Let X be a Banach space.

$$\mathcal{B}(X) = \{T : X \to X \text{ bounded, linear}\}, \qquad \|T\| = \sup_{\|x\|=1} \|Tx\|$$

The multiplication is $(T \cdot S)(x) = T(S(x))$.

If $X = \mathbf{R}^n$, then $\mathcal{B}(X)$ is the set of *quadratic matrices*.

$T \in \mathcal{B}(X)$ is INVERTIBLE, if $\exists S \in \mathcal{B}(X)$ s.t. $TS = ST = I$. (Both!)

Example. $X = \ell^2$. T is the left shift, S is the right shift operator.

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots), \qquad S(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

Then $TS = I$, but $ST \neq I$. None of them is invertible.

Eigenvalue of a quadratic matrix. Reminder

Let $A \in \mathbb{C}^{n \times n}$ be a matrix . $\lambda \in \mathbb{C}$ is an EIGENVALUE, if

$$\exists v \neq 0 : \quad Av = \lambda v.$$

In other words $\exists v \neq 0$: $(A - \lambda I)\, v = 0$. v is an EIGENVECTOR OF A.

Equivalently: $\lambda \in \mathbb{C}$ eigenvalue $\iff$ $(A - \lambda I)$ is *non-invertible*.

$\longrightarrow$ *Extension* of this notion to *linear operators*: $\surd$

Spectrum of an operator

Let X be a Banach space. $T \in \mathcal{B}(X)$ is an operator.

Definition. The SPECTRUM of T is:

$$\sigma(T) = \{\lambda : T - \lambda I \text{ is not invertible}\}$$

If $\dim X = n < \infty$, then $T \in \mathcal{B}(X) \iff \exists A \in \mathbb{R}^{n \times n}$ s.t. $Tx = A \cdot x$

($\mathcal{B}(X) \equiv$ *square matrices*.) Here spectrum = set of eigenvalues.

$$\lambda \in \sigma(T) \iff \lambda \text{is eigenvalue of } A$$

If $\dim X = \infty$, and $T \in \mathcal{B}(X)$, then t*he spectrum might be larger*:

$$\sigma(T) = \{\text{eigenvalues}\} \cup \{\text{continuous spectrum}\}.$$

Example 1. $X = \mathbb{C}^3$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5-i \end{pmatrix} \implies \sigma(A) = ??? \quad = \{1, 2, 5-i\}$$

Moreover $\forall \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{C} \implies \exists A : \quad \sigma(A) = \{\lambda_1, \lambda_2, \lambda_3\}$

Example 2. $X = \ell^2$.

Let $D = \mathrm{diag}\,(\lambda_n : n \in \mathbb{N})$. Then the mapping $D : x \mapsto Dx$ is linear.

But linearity is not enough!

Question: $x \in \ell^2 \implies Dx \in \ell^2?$

Exc. Show that: (λ_n) is bounded $\iff \forall x \in \ell^2 \Rightarrow Dx \in \ell^2$.

$D = \mathrm{diag}\,(\lambda_n : n \in \mathbb{N})$. $\sigma(D) = ?$

If $\lambda = \lambda_n$, then $(D - \lambda_n I)$ has a row full of 0-s $\quad\Longrightarrow\quad \nexists (D - \lambda_n I)^{-1}$

Thus $\lambda = \lambda_n$ is an eigenvalue. $\{\lambda_n : n \in \mathbb{N}\} \subset \sigma(D)$.

Anything else?

If $\lambda \in \mathbb{C}$, then $(D - \lambda I) = \mathrm{diag}\,(\lambda_n - \lambda,\ n \in \mathbb{N})$. The "potential inverse":

$$\text{"}(D - \lambda I)^{-1}\text{"} = S = \mathrm{diag}\left(\frac{1}{\lambda_n - \lambda}, n \in \mathbb{N}\right).$$

$$S \in \mathcal{B}(\ell^2) \quad\Longleftrightarrow\quad \left(\frac{1}{\lambda_n - \lambda}\right) \text{ is bounded.}$$

If λ is an accumulation point of (λ_n), then $\left(\frac{1}{\lambda_n - \lambda}\right)$ is not bounded.

$$\sigma(D) = \{\lambda_n : n \in \mathbb{N}\} \cup \{\text{accumulation points}\}$$

Properties of the spectrum

Theorem. X is a Banach space, $T \in \mathcal{B}(X)$. Then

1. The spectrum is closed.
2. The spectrum is bounded.
3. The spectrum is not empty.

Proof. (Sketch)

1. The complementary of $\sigma(T)$ is open.
2. It can be shown, that if $|\lambda| > \|T\|$, then $\exists (T - \lambda I)^{-1} \Rightarrow \lambda \notin \sigma(T)$.
3. *Hard* proof, using theory of complex functions.

Spectral radius

Definition. The SPECTRAL RADIUS of $T \in \mathcal{B}(X)$ is defined by:

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

From the previous proof we get: $r(T) \leq \|T\|$.

Theorem. X is a Banach space, $T \in \mathcal{B}(X)$. Then

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Spectrum of the *left shift operator* in ℓ^2

$T(x_1, x_2, \ldots, x_n, \ldots) = (x_2, x_3, \ldots)$. $\Longrightarrow$ $r(T) \le \|T\| = 1$.

It can be written as infinite dimensional matrix-vector product:

$$D = \begin{pmatrix} 0 & 1 & 0 & . & . & 0 & . \\ 0 & 0 & 1 & 0 & . & 0 & . \\ 0 & . & . & . & . & 0 & . \\ 0 & . & . & . & 0 & 1 & . \\ 0 & . & . & . & 0 & 0 & . \end{pmatrix} \Longrightarrow Tx = D \cdot x.$$

$\sigma(T) = ?$ Assume $|\lambda| < 1$. *Is it eigenvalue?* Solve $\lambda x = Tx$?

$$\left.\begin{array}{lcl} \lambda x_1 & = & x_2 \\ \ldots & & \\ \lambda x_{n-1} & = & x_n \\ \ldots & & \end{array}\right\} \Longrightarrow x_n = \lambda^n x_1.$$

Define $x_\lambda = (1, \lambda, \lambda^2, \ldots), \in \ell^2$ $Tx_\lambda = \lambda x_\lambda$. Thus $|\lambda| < 1 \Rightarrow \lambda \in \sigma(T)$.

$\sigma(T)$ is closed $\Rightarrow$ $\boxed{\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}.}$ Is $\lambda_0 = 1$ an eigenvalue?

Special case II.

$$\mathcal{B}(X, \mathbb{R}) = X^*$$

Dual space

$(X, \|\cdot\|)$ is a normed space.

An operator $T : X \to \mathbb{K}$ is called FUNCTIONAL.

Notations. Functional f, g, e.t.c. (small letters)

$$f \ : \ x \quad \mapsto \quad f(x) \quad \approx \text{function}$$

Definition. The (*real*) DUAL SPACE of $(X, \|\cdot\|)$ is

$$X^* = \mathcal{B}(X, \mathbf{R}) = \{f : X \to \mathbf{R}, \text{ linear \& bounded}\}.$$

There is a norm in X^*: $\|f\| = \sup\{|f(x)| \ : \|x\| = 1\}$.

Corollary. X^* is *always* a Banach space.

Dual space of $\mathbb{R}^n$

Proposition. $f : \mathbb{R}^n \to \mathbb{R}$ is linear $\iff \exists a \in \mathbb{R}^n$ s.t. $f(x) = a^T x$.

1. Dual space of $(\mathbb{R}^n, \|\cdot\|_2)$?

$$|f(x)| = |\sum_{j=1}^{n} a_j x_j| \leq \sqrt{\sum_{j=1}^{n} a_j^2} \sqrt{\sum_{j=1}^{n} x_j^2} = \|a\|_2 \cdot \|x\|_2.$$

Since $f(a) = \|a\|_2^2 \implies \|f\| = \|a\|_2$.

Thus $\boxed{(\mathbb{R}^n, \|\cdot\|_2)^* = (\mathbb{R}^n, \|\cdot\|_2)}$

Dual space of $\mathbb{R}^n$ (cont.)

2. Dual space of $(\mathbb{R}^n, \|\cdot\|_\infty)$? Then

$$|f(x)| = |\sum_{j=1}^{n} a_j x_j| \leq \max_k |x_k| \sum_{j=1}^{n} |a_j| = \|x\|_\infty \|a\|_1 .$$

With $x_j = \mathrm{sign}(a_j)$: *"equality"*. $\implies$ $\|f\| = \|a\|_1$.

Thus $\boxed{(\mathbb{R}^n, \|\cdot\|_\infty)^* = (\mathbb{R}^n, \|\cdot\|_1)}$

Dual space of $\mathbb{R}^n$ and ℓ^p

Summary.

$$(\mathbb{R}^n, \|\cdot\|_2)^* = (\mathbb{R}^n, \|\cdot\|_2)\,, \qquad (\mathbb{R}^n, \|\cdot\|_\infty)^* = (\mathbb{R}^n, \|\cdot\|_1)\,.$$

In general: $\left(\mathbb{R}^n, \|\cdot\|_p\right)^* = \left(\mathbb{R}^n, \|\cdot\|_q\right)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

p, q are HÖLDER CONJUGATES.

Similarly $(\ell^2)^* = \ell^2$, and $(\ell^p)^* = \ell^q$, if p and q are Hölder conjugates.

Dual space of a HILBERT SPACE

Let $(H, \langle \cdot, \cdot \rangle)$ be *Hilbert space*. E.g. $(\mathbf{R}^n, \|\cdot\|_2)$, ℓ^2, $\mathcal{L}^2([a, b])$

Example of a functional. Let $y \in H$ be a fixed element.

Define $f_y \in H^*$ by $f_y(x) := \langle x, y \rangle$. $\implies$ $\|f_y\| = \|y\|$. HW

A convenient (and surprising) description of the dual:

Theorem. *Riesz representation theorem*

For any $f \in H^*$ $\exists! y \in H$, such that $f(x) = \langle x, y \rangle$, and $\|f\| = \|y\|$

$\implies$ a H Hilbert space and its dual space H^* are isomorph.

Weak and *strong* convergence

X is a normed space. $(x_n) \subset X$ converges to x_0, if $\|x_n - x_0\| \to 0$.

This is CONVERGENCE IN NORM. $\equiv$ STRONG convergence

(x_n) CONVERGES WEAKLY to x_0, if $\lim_{n\to\infty} f(x_n) = f(x_0)\ \forall f \in X^*$.

Remark. Instead of $(x_n) \subset X$ we consider **many** $(f(x_n)) \subset \mathbf{R}$.

Proposition. STRONG convergence $\implies$ WEAK convergence.

Proof. Assume (x_n) convergences strongly. Let $f \in X^*$. $\implies$

$$|f(x_n) - f(x_0)| = |f(x_n - x_0)| \leq \|f\| \cdot \|x_n - x_0\| \ \to\ 0 \quad \checkmark$$

WEAK convergence $\not\Longrightarrow$ STRONG convergence

Example. $X = \ell^\infty$. Then $X^* = \ell^1$.

$$e^n := (0, \dots, 0, \overset{n}{\smile}{1}, 0, \dots), \qquad n \in \mathbb{N}.$$

Weak convergence of (e^n)?

$$f \in X^* = \ell^1 \qquad \Longleftrightarrow \qquad \exists a = (a_k) \in \ell^1 : \quad f(x) = \sum_{k=1}^{\infty} a_k x_k.$$

$$\lim_{n\to\infty} f(e^n) = \lim_{n\to\infty} a_n = 0, \quad \Longrightarrow \quad e^n \overset{w}{\longrightarrow} 0.$$

BUT $\|e^n\|_\infty = 1 \quad \Longrightarrow \quad$ *does not converge strongly*. Can You see it?

Adjoint of an operator in $\mathcal{B}(H)$

Let $(H, \langle\cdot,\cdot\rangle)$ be *Hilbert space*. $A \in \mathcal{B}(H)$ is a linear operator.

The ADJOINT OPERATOR OF A is the $A^* \in \mathcal{B}(H)$ such that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \forall x, y \in H. \quad \text{Is the def. correct?}$$

We'll see, that the definition is correct. Let $y \in H$. *What is* A^*y*?*

Trick: Let's define $f(x) := \langle Ax, y\rangle$. $\Rightarrow f \in H^*$.

By Thm. of Riesz $\exists! y^* \in H$ such that $f(x) = \langle x, y^*\rangle$.

Thus there is a mapping $y \mapsto y^*$. This is the A^* operator:

$$f(x) := \langle Ax, y\rangle = \langle x, y^*\rangle.$$

Some properties of the adjoint operator

Theorem.

1. $I^* = I$.
2. $(A + B)^* = A^* + B^*$.
3. $(\alpha A)^* = \overline{\alpha} A^*$.
4. $(AB)^* = B^* A^*$.
5. $\|A^*\| = \|A\|$.

Check these properties yourself.

Adjoint operator in finite dimension.

Example. Let $H = \mathbb{R}^n$. The norm is (... finish...) the $\|\cdot\|_2$.

Elements of $\mathcal{B}(H)$ are identified with *matrices of dimension $n \times n$.*

Let $A \in \mathcal{B}(\mathbb{R}^n)$.

Then $A^* = A^T$, the *transpose of the original matrix*, since

$$\langle Ax, y\rangle = (Ax)^T\, y = x^T A^T y = \langle x, A^T y\rangle. \quad \surd$$

Adjoint operator in infinite dimension

Let $H = \mathcal{L}^2[0,1]$, where the inner product is $\langle u, v \rangle = \int_0^1 u(t)v(t)dt.$

$H_0 \subset H$ is the subspace of $u(t)$ functions,

that are infinitely differentiable, and $u(0) = u(1) = 0$.

In H_0 define the differential operator: $Au = u'$. Adjoint of A?

$$\langle Au, v \rangle = \int_0^1 u'(t)v(t)dt = u(t)v(t)\Big|_0^1 - \int_0^1 u(t)v'(t)dt = (*),$$

while integrating by parts. Let's continue:

$$(*) = 0 - \int_0^1 u(t)v'(t)dt = \langle u, -v' \rangle = \langle u, A^* v \rangle, \quad \Longrightarrow \quad A^* v = -v'.$$

Orthogonal projection

Let $E \subset H$ be a *closed* subspace.

For any $x \in H$ define $Px := x_E$, such that

$$\|x - x_E\| = \min_{u \in E} \|x - u\|$$

Then $\forall x \in H$ can be written as: $x = x_E + x_0$, such that

$$x_E \in E \quad \text{and} \quad \langle x_0, y \rangle = 0 \ \ \forall y \in E$$

$P : H \to H$ is the operator of ORTHOGONAL PROJECTION onto E.

What is the adjoint of P?

Orthogonal projection. (Cont.)

(From the previous slide): $x = Px + x_0$, with $Px \in E$ and $x_0 \perp E$. $P^* = ?$

$$\langle Px, y\rangle = \langle Px, Py + y_0\rangle = \langle Px, Py\rangle + \langle Px, y_0\rangle = (**),$$

where $\langle Px, y_0\rangle = 0$, since $Px \in E$ and $y_0 \perp E$. Let's continue:

$$(**) = \langle Px, Py\rangle + \langle x_0, Py\rangle = \langle x, Py\rangle.$$

Finally the result is $P = P^*$.

The other direction is also true:

If $\quad P = P^* \quad \Longrightarrow \quad P$ is an orthogonal projection.

Self-adjoint operator

The operator A is SELF-ADJOINT, if $A = A^*$.

Theorem. If A is self-adjoint, then

1. $\|A^n\| = \|A\|^n$.
2. It's spectral radius is: $r(A) = \|A\|$.
3. The spectrum is real: $\sigma(A) \subset \mathbb{R}$.

Self-adjoint operators in infinite dimension are extensions of

symmetric matrices in finite dimension.