Functional analysis

Lesson 10.

April 29. 2021

Linear operator

X and Y are *vector spaces* over $\mathbb{R}$ or $\mathbb{C}$.

A function $T : X \to Y$ is called OPERATOR.

This is a LINEAR OPERATOR, if

$$T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 \quad \forall x_1, x_2 \in X, \ \ \forall \alpha, \beta \in \mathbb{K}.$$

(Notation: $T(x)$ is written as Tx, without brackets.)

Corollary. If T is linear, then $T0 = 0$. Why?

Examples

1. $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$.

$T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* $\iff$ $\exists A \in \mathbb{R}^{m\times n}$ matrix, s.t. $Tx = Ax$

(On the left hand side there is a ”matrix-vector” product)

2. $X = C[a,b]$, $Y = \mathbb{R}$. The ”Integral is linear”.

$$T : C[a,b] \to \mathbb{R}, \qquad Tf := \int_a^b f(x)dx = \int_{[a,b]} f \, dm.$$

More examples

3. $X = Y = \ell^2$.

$T : \ell^2 \to \ell^2$ is the LEFT SHIFT OPERATOR, defined as:

$$T(x_1, x_2, \dots, x_n, \dots) = (x_2, x_3, \dots, x_{n+1}, \dots)$$

$S : \ell^2 \to \ell^2$ is the RIGHT SHIFT OPERATOR, defined as:

$$S(x_1, x_2, \dots, x_n, \dots) = (0, x_1, x_2, \dots, x_{n-1}, \dots)$$

Both of them are *linear*. Check it yourself.

$(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are *normed spaces* over $\mathbb{R}$ or $\mathbb{C}$.

Definition. (*review*) $T : X \to Y$ is continuous at $x_0 \in X$, if

$$\forall (x_n) \subset X, \quad \lim x_n = x_0 \quad \Longrightarrow \quad \lim Tx_n = Tx_0.$$

Proposition. Assume T is linear. Then

T is continuous on the whole X $\Longleftrightarrow$ T is continuous at $x_0 = 0$.

(It is surprising!)

Proof. (Sketch)

$\Longrightarrow \surd$

T is continuous on X $\iff$ continuous at $x_0 = 0$.

$\Longleftarrow$ Let $y_0 \in X$. Let $(y_n) \subset X$ s.t. $\lim y_n = y_0$.

Is it true, that $\lim Ty_n = Ty_0$?

Trick: let $x_n := y_n - y_0$. Then $\lim x_n = 0$. By continuity $\lim Tx_n = 0$.

As T is linear, $Tx_n = T(y_n - y_0) = Ty_n - Ty_0 \to 0$.

Thus $Ty_n \to Ty_0$ $\surd$

Boundedness

Question. What would we expect?

Analogy. An $f : \mathbb{R} \to \mathbb{R}$ function is *bounded*, if

$$\exists K \geq 0 \quad \text{s.t.} \quad |f(x)| \leq K \quad \forall x.$$

BUT if T is linear, it is *impossible*. Why?

$\longrightarrow$ Because e.g. $T(2x) = 2 \cdot Tx \implies \nexists K$.

A bounded operator $\equiv$ "*on the unit ball*".

Bounded operator

A $T : X \to Y$ linear operator is BOUNDED, if $\exists M \geq 0$, such that

$$\|Tx\|_Y \leq M \cdot \|x\|_X, \quad \forall x. \tag{1}$$

Example.1. $X = Y = \ell^2$. Is the left shift operator bounded?

If $x = (x_1, x_2, \dots x_n, \dots) \in \ell^2$, then $\|x\| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}$.

Estimate the norm of $Tx = (x_2, x_3, \dots x_n, \dots)$ as:

$$\|Tx\| = (\sum_{i=2}^{\infty} x_i^2)^{1/2} \leq (\sum_{i=1}^{\infty} x_i^2)^{1/2} = \|x\|.$$

Thus (1) is true $\forall M \geq 1$. <u>T is bounded.</u>

Example. 2. $T : C[a,b] \to \mathbb{R}$ is the integral-operator:

$Tf = \int_a^b f(x)dx$. Is it bounded?

In $C[a,b]$ the norm is $\|f\| = \max_{x\in[0,1]} |f(x)|$. Then

$$\|Tf\| = \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \le \int_a^b \|f\|dx = (b-a)\cdot\|f\|.$$

Thus (1) is true $\forall M \ge b-a$. <u>T is bounded.</u>

Remark. In both examples there was a *smallest* M satisfying (1).

Operator norm

The NORM of a bounded linear operator T is:

$$\|T\| := \inf\{M \ : \|Tx\| \le M \cdot \|x\|, \ \forall x \in X\}.$$

Example.1. (Cont.) In ℓ^2 $T$is the *left shift operator*. $\|T\| \le 1$

If $x = (0, x_2, x_3, \dots)$, then $\|Tx\| = \|x\|$.

Thus $M < 1$ can not be. $\|T\| = 1$.

Example. 2. (Cont.) T is the *integral operator*. $\|T\| \le b - a$.

Is there an element, where $\|Tf\| = (b - a) \cdot \|f\|$? "$=$" ?

Guess? $f \equiv c\surd$ Thus: $\|T\| = b - a$.

Normed space of linear bounded operators

Definition. $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are *normed spaces* over $\mathbb{R}$ or $\mathbb{C}$.

The set of bounded linear operators

$$\mathcal{B}(X, Y) = \{T : X \to Y \quad \text{bounded and linear}\}$$

is a NORMED SPACE with norm:

$$\|T\| := \inf\{M \ : \|Tx\|_Y \leq M \cdot \|x\|_X, \ \forall x \in X\}. \qquad (2)$$

Theorem. (2) is a norm, indeed.

We do not prove the Thm.

An equivalent form of the operator norm

If $x = 0$, then $\forall M \geq 0$ is true: $\|Tx\| \leq M\|x\|$. Why? Because $T0 = 0$

If $x \neq 0$, then

$$\|Tx\| \leq M \cdot \|x\| \quad \Longleftrightarrow \quad \frac{\|Tx\|}{\|x\|} \leq M.$$

Corollary.

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|$$

$\|T\|$ depends on the norm in X and Y.

Example. 3. $X = \mathbb{R}^2$, $Y = \mathbb{R}$. $T(x_1, x_2) := x_1 + 2x_2$.

1st case. $T : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$. $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$.

$$|Tx| = |x_1 + 2x_2| \le |x_1| + |2x_2| \le 2(|x_1| + |x_2|) \implies M \le 2.$$

On the other hand if $x_1 = 0$, then $|Tx| = 2\|x\|$. $\Rightarrow$ $\|T\|_1 = 2$.

2nd case. $T : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$. $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$.

$$|Tx| \le |x_1| + |2x_2| \le \max\{|x_1|, |x_2|\}(1 + 2) \implies M \le 3.$$

If $x_1 = x_2$, then "equality". $\Rightarrow$ $\|T\|_\infty = 3$.

Example 4.

Let $C^1[a,b] = \{f : [a,b] \to \mathbf{R}, \ \text{cont. differentiable}\}$

$D : C^1[a,b] \to C[a,b]$ is the DIFFERENTIAL OPERATOR:

$$Df := f'.$$

It is linear, but it is not bounded.

Why? Not easy.

Example 5.

The *Fredholm operator* $T : C[a,b] \to C[a,b]$ is defined as

$$f \mapsto Tf \qquad Tf(s) = \int_a^b k(s,t)f(t)dt, \qquad s \in [a,b],$$

with $k : [a,b] \times [a,b] \to \mathbf{R}$ piecewise continuous in s and t.

$\approx$ *"matrix-vector product"*. $(Ax)_j = \sum_{k=1}^{n} a_{jk}x_k$.

Question: $\|T\| = ?$

Connection between continuity and boundedness

Theorem. $T : X \to Y$ is a linear operator. Then

$$T \text{ is bounded} \iff T \text{ is continuous.}$$

Remark. The two main properties are equivalent.

Stop for a while, and memorize it.

Proof. $\Rightarrow$

If T is bounded, then $\exists M > 0$: $\|Tx\| \leq M\|x\|$.

Then if $\lim x_n = 0 \implies \lim Tx_n = 0$. T is *continuous*. $\surd$

Boundedness $\iff$ Continuity (cont.)

$\Leftarrow$ If T is *continuous at* $x_0 = 0$, then for $\varepsilon = 1$ $\exists\delta$:

$$\|x - 0\| \le \delta \qquad \Rightarrow \qquad \|Tx - 0\| \le 1.$$

Let $x \in X$, $x \neq 0$. Define $y = \frac{\delta}{\|x\|}x$. The norm of it:

$$\|y\| = \frac{\delta}{\|x\|} \cdot \|x\| = \delta.$$

Thus due to continuity $\|Ty\| \le 1$. Then

$$Ty = T\left(\frac{\delta}{\|x\|} \cdot x\right) = \frac{\delta}{\|x\|} \cdot Tx \qquad \Rightarrow \qquad \|Ty\| = \frac{\delta}{\|x\|} \cdot \|Tx\| \le 1.$$

Then $\|Tx\| \le \frac{1}{\delta}\|x\|$. Thus $M = \frac{1}{\delta}$ is good constant in boundedness.

The set of Bounded linear operators.

Reminder. $\mathcal{B}(X,Y) := \{T : X \to Y \mid \text{bounded, linear}\}$,

with $\|T\| = \sup_{\|x\|=1} \|Tx\|$. It is a normed space.

Theorem. If Y is Banach space, then $\mathcal{B}(X,Y)$ is a Banach space.

Definition. If the image of the operator is $\mathbb{R}$ or $\mathbb{C}$, then

$T : X \to \mathbb{K}$ is called

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Special case I.

$$\mathcal{B}(X, X) = \mathcal{B}(X)$$

$\mathcal{B}(X)$ is a Banach algebra

Let X be a Banach space. $\mathcal{B}(X) = \{T : X \to X, \text{ bounded, linear}\}$

$\mathcal{B}(X)$is a BANACH ALGEBRA, with the following properties.

- Linear space ✓
- Normed space, "*length*" is $\|T\| = \sup_{\|x\|=1} \|Tx\|$ ✓
- There is multiplication: $(T \cdot S)(x) = T(S(x))$, algebra ✓
- Unity of multiplication: $I : X \to X$, $Ix = x$.✓
- Norm + multiplication: $\|T \cdot S\| \leq \|T\| \cdot \|S\|$. ✓

Invertible operator

$T \in \mathcal{B}(X)$ is INVERTIBLE, if $\exists S \in \mathcal{B}(X)$ s.t. $TS = ST = I$. (Both!)

Theorem. If $T \in \mathcal{B}(X)$ satisfies $\|T\| < 1$, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$. (Similar to $\approx \frac{1}{1-q} = \sum_{k=0}^{\infty} q^k$.)

Corollary. Let $T \in \mathcal{B}(X)$ be *invertible.*

Assume $S \in \mathcal{B}(X)$ s.t. $\|S\| < \frac{1}{\|T^{-1}\|}$. Then $T + S$ is also invertible.

Corollary. $G = \{T \in \mathcal{B}(X) \mid \exists T^{-1}\}$ is the set of invertible operators.

It is an OPEN SET in $\mathcal{B}(X)$.