Functional analysis

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Basic abstract spaces. Topology.

1. **Metric space.** E.g: discrete metric.
2. **Normed space.** Metric in a normed space (P). Special norms in $\mathbb{R}^n$.
3. **Inner product space.** Relation between inner product and norm (P).
4. **Sequence spaces**: c_0, ℓ^∞, ℓ^p, $p \geq 1$. Relation of ℓ^p and ℓ^q. (P).
5. **Function spaces**: $C([a, b])$ with possible norms.
6. The topology of metric spaces. **Open set, closed set.** Their properties. (P).
7. **Convergence of a sequence in a metric space.**
8. **Continuity** of functions between metric spaces.
9. **Compact sets**. Examples in finite and infinite dimension. Properties (P).
10. Compact sets in finite dimension (Heine-Borel theorem) (P).
11. The unit ball in $C([0, 1])$ is not compact.
12. **Completeness of a metric space.** Hilbert space, Banach space.
13. $C([0, 1])$ is complete with one the norm. (P).
14. $C([0, 1])$ is not complete with another norm. (P).

Measure spaces. Integral.

15. Measurable space. **Measure.** Examples: counting measure, probability measure.
16. Introduction of the Lebesgue measure on $\mathbb{R}$.
17. Properties of Lebesgue-measurable sets. **Null sets**. **Cantor set** in $[0, 1]$, properties (P).

18. Measurable functions, characterization of them (P). **Simple functions.**

19. *"Almost everywhere"*, as an equivalence relation (P).

20. Lebesgue integral, properties. **Condition of integrability** (P).

21. Comparison of the Lebesgue integral and the Riemann integral. Convergence theorems.

22. Lebesgue's $\mathcal{L}^{\mathbf{p}}(\mathbf{R})$ **spaces**, where $1 \leq p < \infty$. Elements and norms.

23. Connection between $\mathcal{L}^p(R)$ and $\mathcal{L}^q(R)$ when $m(R) < \infty$.

24. Essential supremum of a real function. The $\mathcal{L}^{\infty}(\mathbf{R})$ **space.**

25. Connection of it with $\mathcal{L}^p(R)$ when $m(R) < \infty$. Riesz theorem on Lebesgue $\mathcal{L}^p$ spaces.

General Fourier series.

26. $\mathcal{L}^2(R)$ as a Hilbert space. **Orthonormal sequence**, example in $\mathcal{L}^2[-\pi, \pi]$.

27. **Complete ON system.** Method for orthonormalising (P).

28. Legendre polynomials, their construction.

29. A complete ON system: Haar functions. Dimension of a vector space. Examples.

30. Fourier analysis in $\mathcal{L}^2(R)$. Fourier coefficients (P).

31. Parseval's theorem (P). **Riesz-Fisher theorem.** Isometry of $\mathcal{L}^{\mathbf{2}}(\mathbf{R})$ **and** ℓ^2**.**

32. General $\mathcal{L}^2_\rho(R)$ spaces with ρ weighting functions.

33. **ON systems of polynomials.** E.g. Chebyshev polynomials (P), Hermite polynomials.

Linear operators

34. Abstract linear operators. **Continuity**, properties (P).

35. Boundedness, and continuity (P). **Operator norm.**

36. Examples of bounded linear operators in $\mathbb{R}^n$, in ℓ^2 and in $C([a, b])$.

37. $\mathcal{B}(X, Y)$ **as a normed space.** Completeness.

38. Bounded linear operators in a *Banach space*. Multiplication of operators. $B(X)$.

39. **Inverse of an operator. A condition on the existence of the inverse operator.** (P).

40. Basic properties of the inverse operator.

41. **Spectrum** of a bounded linear operator. Connection with the eigenvalues.

42. Properties of the spectrum (P). Examples.

43. Linear functionals. **Norm of a bounded linear functional**. Examples in function spaces.

44. **Dual space.** Examples: $\mathbb{R}^n$ with different norms (P), ℓ^p.

45. Weak and strong convergence, their connection (P).

46. Linear functionals in Hilbert space. Riesz representation thm. **Dual space of a Hilbert space**.

47. **Adjoint of a bounded linear operator** in a Hilbert space, existence (P).

48. Examples of adjoint operator in finite and infinite dimension.

49. Self adjoint operator. E.g. orthogonal projection (P).

50. **An example.** Hilbert space methods in QM: Heisenberg's uncertainty principle (P).