ANALÍZIS II. Példatár

Fourier sorok

2010. február

Feladatok

Tekintsük az alábbi függvényeket. Ahol másképp nem jelezzük, ott a függvény a megadott tartományon kívül 2π szerint periodikus, vagyis $f(x) = f(x + k2\pi)$, $k = \pm 1, \pm 2.....$

Írja el ezek Fourier sorát!

F.1

$$f(x) = \begin{cases} \pi, & \text{ha} \quad -\pi < x \leq 0 \\ \pi - x, & \text{ha} \quad 0 < x < \pi \\ \frac{\pi}{2}, & \text{ha} \quad x = \pm\pi \end{cases}$$

A felírt sor segítségével számítsuk ki az alábbi számsor összegét:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

F.2

$$f(x) = \begin{cases} +1, & \text{ha} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1, & \text{ha} \quad \frac{\pi}{2} < x < \frac{3\pi}{2} \\ 0, & \text{ha} \quad x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}$$

Számítsuk ki az alábbi sor összegét:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + -... = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}.$$

F.3

$$f(x) = \begin{cases} x + \frac{\pi}{2}, & \text{ha} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \frac{3\pi}{2} - x, & \text{ha} \quad \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

F.4 $f(x) = \dfrac{\text{ch }(x)}{2}$, ha $-\pi \leq x \leq \pi$.

F.5

$$f(x) = \begin{cases} \dfrac{\pi}{2} - x, & \text{ha} \quad 0 < x \leq \dfrac{\pi}{2} \\ x - \dfrac{\pi}{2}, & \text{ha} \quad \dfrac{\pi}{2} < x \leq \pi \\ x - \dfrac{3\pi}{2}, & \text{ha} \quad \pi < x \leq \dfrac{3\pi}{2} \\ \dfrac{3\pi}{2} - x, & \text{ha} \quad \dfrac{3\pi}{2} \leq x < 2\pi \\ 0, \ \text{ha} \quad x = k\pi, \ k = 0, \pm 1 \end{cases}$$

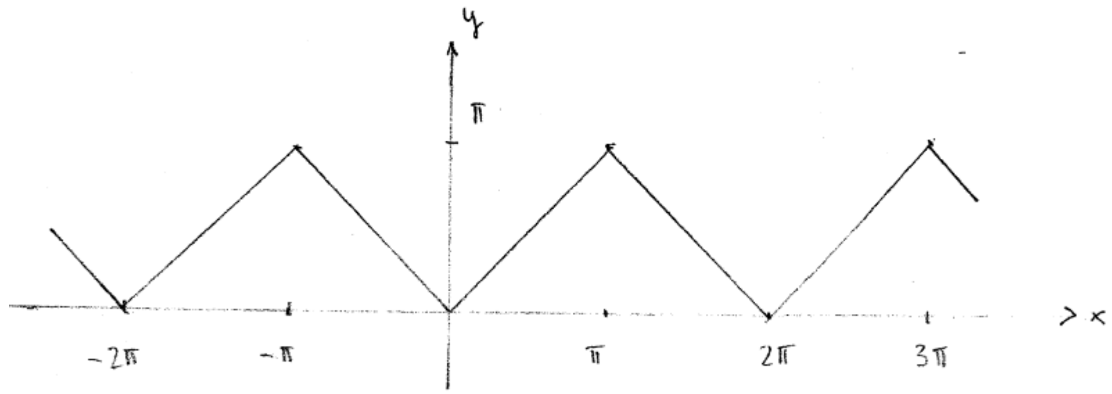
F.6

$$f(x) = \begin{cases} x, & \text{ha} \quad -\dfrac{\pi}{3} \leq x \leq \dfrac{\pi}{3} \\ \dfrac{\pi}{3}, & \text{ha} \quad \dfrac{\pi}{3} \leq x \leq \dfrac{2\pi}{3} \\ \pi - x, & \text{ha} \quad \dfrac{2\pi}{3} \leq x \leq \dfrac{4\pi}{3} \\ -\dfrac{\pi}{3}, & \text{ha} \quad \dfrac{4\pi}{3} \leq x \leq \dfrac{5\pi}{3} \end{cases}$$

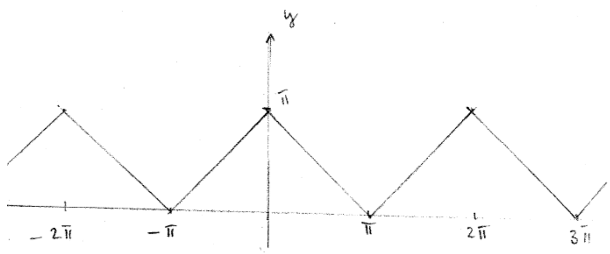
F.7

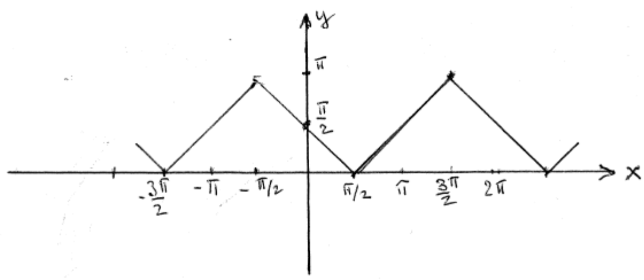
$$f(x) = \begin{cases} x, & \text{ha} \quad 0 \leq x \leq \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \text{ha} \quad \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \\ 2\pi - x, & \text{ha} \quad \dfrac{3\pi}{2} \leq x \leq 2\pi \end{cases}$$

F.8 Határozzuk meg a görbével megadott függvény Fourier-sorát:

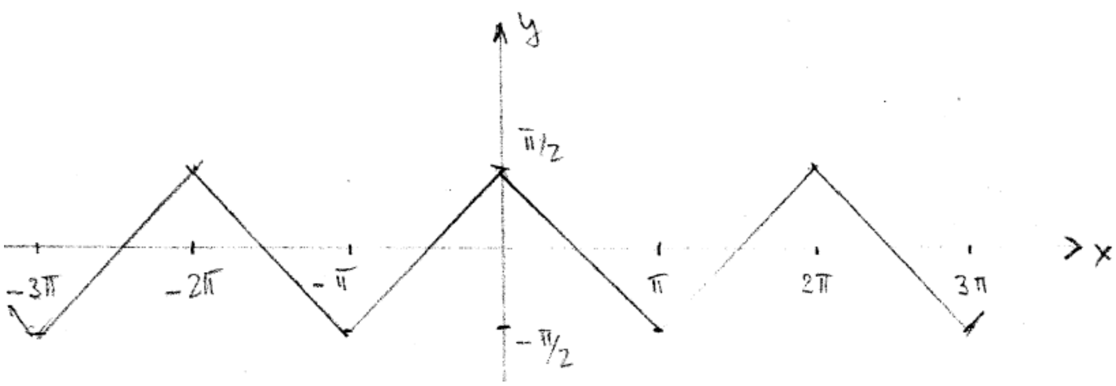


F.9 -F.10 Határozzuk meg a következő példákban görbével megadott függvények Fourier-sorát:

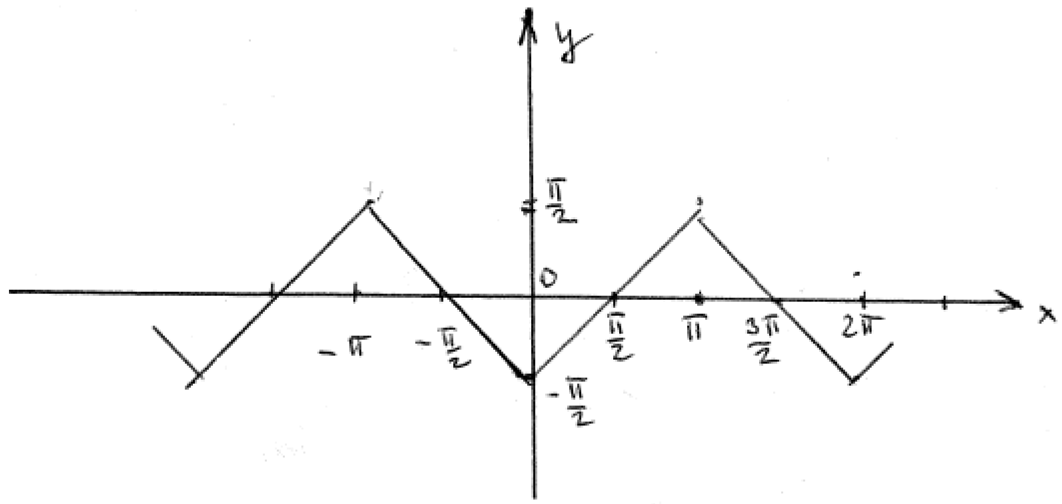




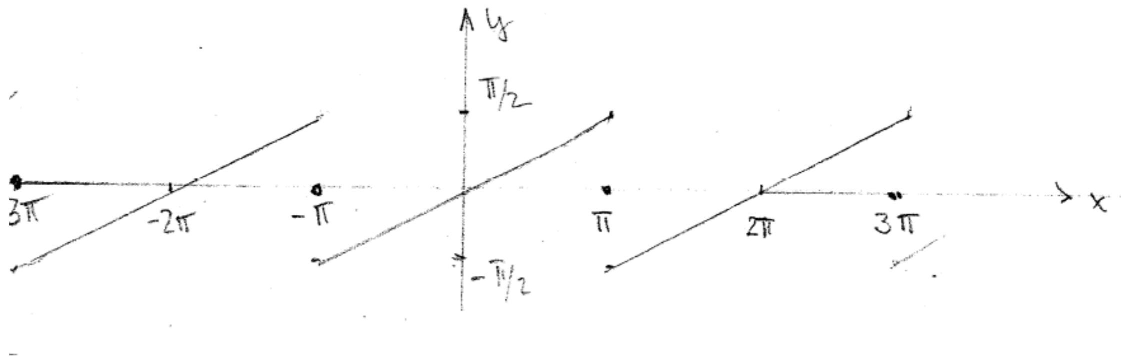
F.11 Határozzuk meg a következő példában görbével megadott függvény Fourier-sorát:



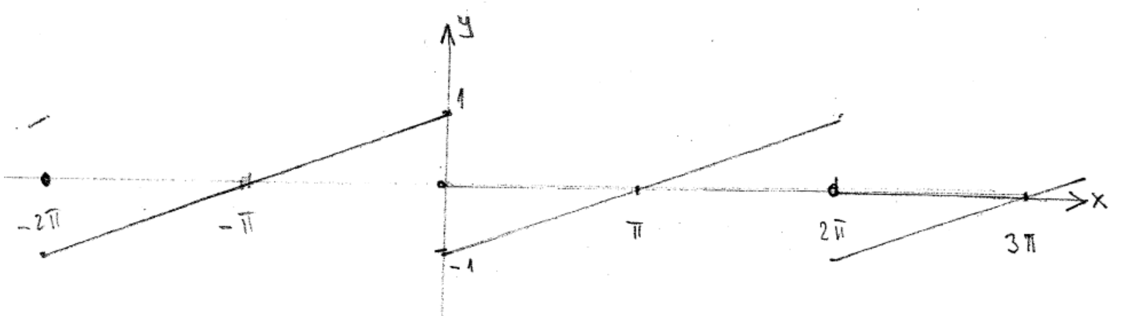
F.12 Határozzuk meg a következő példában görbével megadott függvény Fourier-sorát:



F.13 Határozzuk meg a következő példában görbével megadott függvény Fourier-sorát:



F.14 Határozzuk meg a következő példában görbével megadott függvény Fourier-sorát:



Határozzuk meg a következő példákban adott függvények Fourier-sorát!

(Ahol a függvényeket egy L hosszú intervallumon definiáljuk ott ezután L szerint periodikusan kiterjesztjük - ezt külön nem jelezzük.)

F.15

$$f(x) = \begin{cases} \dfrac{x}{2} + \dfrac{\pi}{4}, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\[2ex] \dfrac{3\pi}{4} - \dfrac{x}{2}, & \text{ha} \quad \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \end{cases}$$

F.16

$$f(x) = \begin{cases} -\dfrac{x}{2} - \dfrac{\pi}{4}, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\[2ex] \dfrac{x}{2} - \dfrac{3\pi}{4}, & \text{ha} \quad \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \end{cases}$$

F.17

$$f(x) = \begin{cases} \dfrac{\pi}{2}, & \text{ha} \quad -\pi \leq x \leq -\dfrac{\pi}{2} \\[2ex] -x, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\[2ex] -\dfrac{\pi}{2}, & \text{ha} \quad \dfrac{\pi}{2} \leq x < \pi \\[2ex] 0, & \text{ha} \quad x = \pm\pi \end{cases}$$

F.18

$$f(x) = \begin{cases} -\dfrac{\pi}{2}, & \text{ha} \quad -\pi \leq x \leq -\dfrac{\pi}{2} \\[2ex] x, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\[2ex] \dfrac{\pi}{2}, & \text{ha} \quad \dfrac{\pi}{2} \leq x < \pi \\[2ex] 0, & \text{ha} \quad x = \pm\pi \end{cases}$$

F.19

$$f(x) = \begin{cases} 2x + \dfrac{\pi}{2}, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq 0 \\ \dfrac{\pi}{2} - 2x, & \text{ha} \quad 0 \leq x \leq \dfrac{\pi}{2} \\ -\dfrac{\pi}{2}, & \text{ha} \quad \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \end{cases}$$

F.20

$$f(x) = \begin{cases} -\dfrac{\pi}{2} - 2x, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq 0 \\ 2x - \dfrac{\pi}{2}, & \text{ha} \quad 0 \leq x \leq \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \text{ha} \quad \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \end{cases}$$

F.21

$$f(x) = \begin{cases} x, & \text{ha} \quad -\pi \leq x \leq 0 \\ -x, & \text{ha} \quad 0 \leq x \leq \pi \end{cases}$$

F.22

$$f(x) = \begin{cases} x - \dfrac{\pi}{2}, & \text{ha} \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\ \dfrac{\pi}{2} - x, & \text{ha} \quad \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \end{cases}$$

F.23

$$f(x) = x^2, \quad \text{ha} \quad -\pi \leq x \leq \pi.$$

f Fourier-sorából számítsuk ki a $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}$ számsor összegét.

F.24

$$f(x) = \begin{cases} -\frac{2}{\pi}x - 1, & \text{ha} \quad -\pi \leq x \leq -\frac{\pi}{2} \\ \cos(x), & \text{ha} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \frac{2}{\pi}x - 1, & \text{ha} \quad \frac{-\pi}{2} \leq x \leq \pi \end{cases}$$

F.25 $f(x) = \sin^2(x)$

F.26 $f(x) = \cos^2(x)$

F.27 $f(x) = \frac{x^3}{9}, \qquad -\pi \leq x \leq \pi$

F.28 $f(x) = |\sin(x)|$

F.29 $f(x) = |\cos(x)|$

F.30 $f(x) = \sin(\frac{x}{2}),\ 0 \leq x \leq 2\pi$

F.31 $f(x) = x^2,\ -1 \leq x \leq 1$

F.32

$$f(x) = \begin{cases} 0 & \text{ha} \quad -2 < x \leq -1 \\ \frac{x}{2} + \frac{1}{2}, & \text{ha} \quad -1 \leq x \leq 1 \\ 1 & \text{ha} \quad 1 \leq x \leq 2 \\ \frac{1}{2}, & \text{ha} \quad x = 4k+2 \qquad (k = 0, \pm 1, \pm 2, ...) \end{cases}$$

és $f(x) = f(x + 4k), \quad k = \pm 1, \pm 2, ...$

F.33

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{ha} \quad 0 < x < 4 \\ 1 & \text{ha} \quad x = 4k \end{cases}$$

és $f(x) = f(x + 4k), \quad k = \pm 1, \pm 2, ...$

F.34 $f(x) = -x^2$, ha $-1 \leq x \leq 1$.

F.35

$$f(x) = \begin{cases} \dfrac{e^{-x} - 1}{2} & \text{ha} \quad -3 \leq x \leq 0 \\ \dfrac{e^{x} - 1}{2} & \text{ha} \quad 0 \leq x \leq 3 \end{cases}$$

F.36

$$f(x) = \begin{cases} \sin x, & \text{ha} \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0, & \text{ha} \quad x = (2k+1)\pi \qquad (k = 0, \pm 1, \pm 2, ...) \end{cases}$$

F.37

$$f(x) = \begin{cases} (x-1)^2, & \text{ha} \quad 0 \leq x \leq 2 \\ 1, & \text{ha} \quad 2 \leq x \leq \pi \end{cases}$$

Megoldások

F.05

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}\left(\frac{2}{2k+1}+(-1)^{k+1}\frac{4}{\pi(2k+1)^2}\right)\sin(2k+1)x = \\ &= 2\left[\left(1-\frac{2}{\pi}\right)\sin x+\left(\frac{1}{3}+\frac{2}{9\pi}\right)\sin 3x+\left(\frac{1}{5}-\frac{2}{25\pi}\right)\sin 5x+\dots\right]. \end{aligned}$$

F.06

$$f(x)=\frac{2\sqrt{3}}{\pi}\left(\sin x-\frac{\sin 5x}{5^2}+\frac{\sin 7x}{7^2}-\frac{\sin 11x}{11^2}+\frac{\sin 13x}{13^2}-\dots\right).$$

F.07

$$f(x)=\frac{3\pi}{8}-\frac{2}{\pi}\left(\cos x+\frac{2}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x++\frac{2}{6^2}\cos 6x+\frac{1}{7^2}\cos 7x+\dots\right).$$

F.08

$$f(x)=\begin{cases} x, & \text{ha } 0\le x<\pi \\ 2\pi-x, & \text{ha } \pi\le x<2\pi \end{cases}$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\frac{\cos 7x}{7^2}+\dots\right).$$

F.09

$$f(x)=\begin{cases} \pi+x, & \text{ha } -\pi\le x<0 \\ \pi-x, & \text{ha } 0\le x<\pi \end{cases}$$

$$f(x)=\frac{\pi}{2}+\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\frac{\cos 7x}{7^2}+\dots\right).$$

F.10

$$f(x)=\begin{cases} \frac{\pi}{2}-x, & \text{ha } -\frac{\pi}{2}<x\le\frac{\pi}{2} \\ x-\frac{\pi}{2}, & \text{ha } \frac{\pi}{2}<x\le\frac{3\pi}{2} \end{cases}$$

$$f(x)=\frac{\pi}{2}+\frac{4}{\pi}\left(\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\dots\right).$$

F.11

$$f(x) = \begin{cases} \dfrac{\pi}{2} + x, & \text{ha } -\pi < x \leq 0 \\ \\ \dfrac{\pi}{2} - x, & \text{ha } 0 < x \leq \pi \end{cases}$$

$$f(x) = \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots\right).$$

F.12

$$f(x) = \begin{cases} -(x + \dfrac{\pi}{2}), & \text{ha } -\pi < x \leq 0 \\ \\ -(\dfrac{\pi}{2} - x), & \text{ha } 0 < x \leq \pi \end{cases}$$

$$f(x) = -\frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots\right).$$

F.13

$$f(x) = \begin{cases} \dfrac{x}{2}, & \text{ha } -\pi < x < \pi \\ 0, & \text{ha } x = (2k+1)\pi \quad (k = 0, \pm 1, \pm 2, \dots) \end{cases}$$

$$f(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots$$

F.14

$$f(x) = \begin{cases} \dfrac{x}{\pi} - 1, & \text{ha } 0 < x < 2\pi \\ 0, & \text{ha } x = 2k\pi; \qquad (k = 0, \pm 1, \pm 2, \dots) \end{cases}$$

$$f(x) = -\frac{2}{\pi}\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \dots\right).$$

F.15

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \dots\right).$$

F.16

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \dots\right).$$

F.17

$$f(x) = -\frac{1}{\pi}\left[(\pi+2)\sin x - \frac{\pi}{2}\sin 2x + \frac{3\pi-2}{3^2}\sin 3x - \frac{\pi}{4}\sin 4x + \dots\right].$$

F.18

$$f(x) = \frac{1}{\pi}\left[(\pi+2)\sin x - \frac{\pi}{2}\sin 2x + \frac{3\pi-2}{3^2}\sin 3x - \frac{\pi}{4}\sin 4x + \dots\right].$$

F.19

$$f(x) = -\frac{\pi}{4} + \frac{4}{\pi}\Bigg(\cos x + \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + $$
$$+\frac{2}{6^2}\cos 6x + \frac{1}{7^2}\cos 7x + \dots\Bigg).$$

F.20

$$f(x) = \frac{\pi}{4} - \frac{4}{\pi}\Bigg(\cos x + \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + $$
$$+\frac{2}{6^2}\cos 6x + \frac{1}{7^2}\cos 7x + \dots\Bigg).$$

F.21

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots\right).$$

F.22

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \dots\right).$$

F.23

$$f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \dots\right).$$

Az $x = 0$ helyen $\cos(kx) = 1$ ezt helyettesítsük be. Így, mivel $f(0) = 0$, ezt kapjuk:

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right).$$

Azaz:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{1}{k^2} = \frac{\pi^2}{12}$$

F.24

$$f(x) = \frac{2}{\pi} + \frac{1}{2} + \frac{1}{\pi}\left[-\frac{4}{\pi}\cos x + \left(\frac{1}{2^2-1} + \frac{8}{\pi 2^2}\right)\cdot\cos 2x - \frac{4}{\pi 3^2}\cos 3x - \right.$$
$$\left. - \frac{1}{4^2-1}\cos 4x - \frac{4}{\pi 5^2}\cos 5x + \left(\frac{1}{6^2-1} + \frac{8}{\pi 6^2}\right)\cos 6x - \dots\right].$$

F.25

$$f(x) = \frac{1}{2} - \frac{\cos 2x}{2}$$

F.26

$$f(x) = \frac{1}{2} + \frac{\cos 2x}{2}$$

F.27

$$f(x) = \frac{2}{9}\left[(\pi^2-6)\sin x - \frac{2\pi^2-6}{2^2}\sin 2x + \frac{(3\pi-6)}{3^2}\sin 3x - \right.$$
$$\left. - \frac{4\pi^2-6}{4^2}\sin 4x + \dots\right].$$

F.28

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots\right)$$

Érdemes észrevenni, hogy ha $x = 0$, akkor:

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots\right)$$

azaz

$$\frac{4}{\pi}\cdot\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{2}{\pi} \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{1}{2}$$

F.29

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} - \dots\right)$$

$x = 0$-nál:

$$1 = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \dots\right).$$

Ebből következik, hogy

$$\frac{\pi-2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \dots \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{4k^2-1} = +\frac{\pi-2}{\pi}.$$

F.30

$$f(x)=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos x}{3}+\frac{\cos 2x}{15}+\frac{\cos 3x}{35}+\dots\right).$$

F.31 Ha az $f(x)$ függvény nem 2π, hanem általános $2l$ periódusú függvény, akkor a Fourier-sor ebben az esetben

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cdot\cos\frac{k\pi}{l}x+b_k\cdot\sin\frac{k\pi}{l}x\right)$$

alakú, ahol az együtthatókat az

$$\begin{aligned} a_0 &= \frac{1}{l}\int_a^{a+2l} f(x)dx \\ a_k &= \frac{1}{l}\int_a^{a+2l} f(x)\cdot\cos\frac{k\pi}{l}xdx \\ b_k &= \frac{1}{l}\int_a^{a+2l} f(x)\cdot\sin\frac{k\pi}{l}xdx \end{aligned}$$

összefüggések segítségével számítjuk ki.

Feladatunkban az $f(x)$ függvény páros, s így $b_k=0$. Továbbá itt is alkalmazható a fél-intervallumon integrálás:

$$a_0=\frac{2}{l}\int_0^1 f(x)dx,\qquad a_k=\frac{2}{l}\int_0^1 f(x)\ \cos\frac{k\pi}{l}xdx.$$

Most $2l=2$, és így $l=1$.

$$\begin{aligned} a_0 &= 2\int_0^1 x^2dx=2\left[\frac{x^3}{3}\right]_0^1=\frac{2}{3} \\ a_k &= 2\int_0^1 x^2\cos(k\pi x)\,dx=\frac{4}{k\pi}\left(\left[x\frac{\cos k\pi x}{k\pi}\right]_0^1-\int_0^1\frac{\cos(k\pi x)}{k\pi}\,dx\right)= \\ &= \frac{4}{k\pi}\left(\frac{\cos(k\pi)}{k\pi}-\left[\frac{\sin(k\pi x)}{k^2\pi^2}\right]_0^1\right)=(-1)^k\frac{4}{k^2\pi^2}. \end{aligned}$$

A Fourier sorfejtés:

$$f(x)=\frac{1}{3}+\frac{4}{\pi^2}\left(-\cos(\pi x)+\frac{1}{2^2}\cos(2\pi x)-\frac{1}{3^2}\cos(3\pi x)+\cdots\right).$$

F.32 Ha a függvény görbéjét $(-\frac{1}{2})$-del az y tengely mentén eltoljuk, akkor páratlan függvényt kapunk. Legyen tehát:

$$g(x) = f(x) - \frac{1}{2} = \begin{cases} -1/2, & \text{ha } -2 < x \le -1 \\ x/2, & \text{ha } -1 \le x \le 1 \\ 1/2, & \text{ha } 1 \le x < 2 \\ 0, & \text{ha } x = \pm 2; \end{cases}$$

A függvény páratlan, ezért $a_0 = a_k = 0$, $k \in \mathbb{N}$.

A függvény periódusa $2l = 4$, vagyis $l = 2$. Ezért a sinus-os tag együtthatói:

$$\begin{aligned}
b_k &= 2\,\frac{1}{2}\left(\int_0^1 \frac{x}{2}\sin(\frac{k\pi}{2}x)\,dx + \int_1^2 \frac{1}{2}\sin(\frac{k\pi}{2}x)\,dx\right) = \\
&= \left[\frac{x}{2}\cdot\frac{-\cos\frac{k\pi}{2}}{\frac{k\pi}{2}}\right]_0^1 + \frac{1}{k\pi}\int_0^1 \cos(\frac{k\pi}{2}x)\,dx - \frac{1}{2}\left[\frac{\cos(\frac{k\pi}{2})}{\frac{k\pi}{2}}\right]_1^2 = \\
&= -\frac{1}{k\pi}\cos(\frac{k\pi}{2}) + \frac{2}{k^2\pi^2}\left[\sin(\frac{k\pi}{2}x)\right]_0^1 - \frac{1}{k\pi}\cos(k\pi) + \frac{1}{k\pi}\cos(\frac{k\pi}{2}) = \\
&= -\frac{\cos(k\pi)}{k\pi} + \frac{2}{k^2\pi^2}\sin(\frac{k\pi}{2}) = \frac{1}{k\pi}\left[\frac{2}{k\pi}\sin(\frac{k\pi}{2}) - \cos(k\pi)\right]
\end{aligned}$$

Mivel

$$\sin(\frac{k\pi}{2}) = \begin{cases} 0, & \text{ha } k = 2n \\ (-1)^k, & \text{ha } k = 2n+1 \end{cases},$$

és

$$\cos(k\pi) = \begin{cases} 1, & \text{ha } k = 2n \\ -1, & \text{ha } k = 2n+1 \end{cases},$$

ezért:

$$\begin{aligned}
b_{2k} &= -\frac{1}{2k\pi} \\
b_{2k+1} &= \frac{1}{(2k+1)\pi}\left((-1)^k\frac{2}{(2k+1)\pi} + 1\right).
\end{aligned}$$

A keresett Fourier-sor: $f(x) = \frac{1}{2} + g(x)$. Ezért

$$\begin{aligned}
f(x) &= \frac{1}{2} + \frac{1}{\pi}\left[\left(1 + \frac{2}{\pi}\right)\frac{1}{\pi}\sin(\frac{\pi}{2}x) - \frac{1}{2\pi}\sin(\frac{2\pi}{2}x) + \left(1 - \frac{2}{3\pi}\right)\frac{1}{3\pi}\sin(\frac{3\pi}{2}x) - \right. \\
&- \left. \frac{1}{4\pi}\sin(\frac{4\pi}{2}x) + \left(1 + \frac{2}{5\pi}\right)\frac{1}{5\pi}\sin(\frac{5\pi}{2}x) - \frac{1}{4\pi}\sin(\frac{4\pi}{2}x) + \ldots\right]
\end{aligned}$$

F.33

$$f(x) = 1 - \frac{2}{\pi}\left(\sin(\frac{\pi}{2}x) + \frac{1}{2}\sin(2\,\frac{\pi}{2}x) + \frac{1}{3}\sin(3\,\frac{\pi}{2}x) + \dots\right).$$

F.34

$$f(x) = -\frac{1}{3} + \frac{4}{\pi^2}\left(\cos\pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x - \dots\right).$$

F.35

$$f(x) = \frac{e^3 - 4}{6} + 3\sum_{k=1}^{\infty}\frac{e^3(-1)^k - 1}{9 + k^2\pi^2}\cdot\cos\frac{k\pi}{3}x$$

F.36

$$f(x) = \frac{8}{\pi}\left(\frac{\sin 2x}{1\cdot 3} - \frac{2\sin 4x}{3\cdot 5} + \frac{3\sin 6x}{5\cdot 7} - \dots\right).$$

F.37

$$f(x) = 1 - \frac{4}{3\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\left(\frac{1+\cos 4k}{k^2} - \frac{\sin 4k}{2k^3}\right)\cos 2kx + \right.$$
$$\left. + \left(\frac{\sin 4k}{k^2} + \frac{\cos 4k - 1}{2k^3}\right)\sin 2kx\right]$$