Financial time series

Gerencsér László   Vágó Zsuzsanna   Gerencsér Balázs

Solutions to Exercises

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Chapter 1

Basic concepts

1.1 Wide sense stationary processes

Exercise 1.1 *Let (y_n) be a wide sense stationary process and let us define*

$$u_n = a_1 y_{n-1} + \cdots + a_p y_{n-p}, \qquad \text{with } a_k \text{ real}, \quad k = 1, \ldots p.$$

Show that (u_n) is also a wide sense stationary process.

Solution: The mean value:

$$\mathrm{E}(u_n) = \mathrm{E}\left(\sum_{k=1}^{p} a_k y_{n-k}\right) = \sum_{k=1}^{p} a_k \mathrm{E}\left(y_{n-k}\right) = 0.$$

The covariance of u_n and $u_{n+\tau}$ is the following:

$$\mathrm{E} u_{n+\tau} u_n = \mathrm{E}\left(\sum_{k=1}^{p} a_k y_{n-k+\tau} \sum_{l=1}^{p} a_l y_{n-l}\right) =$$

$$\sum_{k=1}^{p}\sum_{l=1}^{p} a_k a_l \mathrm{E}(y_{n-k+\tau} y_{n-l}) = \sum_{k=1}^{p}\sum_{l=1}^{p} a_k a_l r(l-k+\tau).$$

It is indeed independent of n. □

Exercise 1.2 *Define the $p \times p$ matrix $R = (R_{k,l})$ by*

$$R_{k,l} = r(l-k), \qquad k, l = 1, \ldots p.$$

Set

$$Y = (y_{n-1}, \ldots, y_{n-p})^T.$$

Prove that the matrix R defined above can also be written as $R = \mathrm{E}(YY^T)$.

Solution: We have

$$YY^T = \begin{pmatrix} y_{n-1} \\ \vdots \\ y_{n-p} \end{pmatrix} \cdot (y_{n-1} \ldots y_{n-p})^T,$$

thus $(YY^T)_{k,l} = y_{n-k}y_{n-l}$. Then

$$\mathrm{E}(YY^T)_{k,l} = \mathrm{E}y_{n-k}y_{n-l} = r(n-k-n+l) = r(l-k).$$

□

Exercise 1.3 *Using the representation* $R = \mathrm{E}(YY^T)$ *prove that* R *is symmetric and positive semidefinite.*

Solution: As the autocovariance function is symmetric, $(r(\tau) = r(-\tau))$, the matrix R is symmetric. We also have to prove, that for all $a \in \mathbb{R}^p$, the quadratic form $a^T Ra \geq 0$. Indeed,

$$a^T Ra = a^T \mathrm{E}(YY^T)\, a = \mathrm{E}a^T\, (YY^T)\, a = \mathrm{E}(a^T Y)\, (a^T Y)^T \geq 0.$$

□

Exercise 1.4 *Let* $y = (y_n)$ *be a complex-valued w.s.st. process with autocovariance function* $r^y(\cdot)$. *Show that the matrix* $R = (R_{k,l})$ *defined by*

$$R_{k,l} = r^y(l-k), \qquad k,l = 1, \ldots p$$

is a Hermitian, positive semi-definite Toeplitz matrix.

Solution: As the entry $R_{l,k}$ only depends on $l-k$, it is a Toeplitz matrix. For complex w.s.st. processes we have

$$R_{k,l} = r^y(l-k) = \overline{r^y(k-l)} = \overline{R_{l,k}},$$

thus R is Hermitian. For all $a \in \mathbb{C}^p$ we have

$$\overline{a}^T Ra = \overline{a}^T \mathrm{E}(\overline{Y}Y^T)\, a = \mathrm{E}\overline{a}^T\, (\overline{Y}Y^T)\, a = \mathrm{E}\overline{(a^T Y)}\, (a^T Y)^T \geq 0.$$

□

1.2 Orthogonal processes and their transformations

Exercise 1.5 *Show that the sequence $(y_{n,N})$ is a Cauchy sequence in L_2 norm.*

Solution: As $y_{n,N} = \sum_{k=0}^{N} h_k e_{n-k}$, we have by the orthogonality of (e_k) for $N < M$

$$\|y_{n,M} - y_{n,N}\|^2 = \sum_{k=N+1}^{M} h_k^2 < \varepsilon,$$

if N is large enough, since by assumption $(h_k) \in \ell_2$. □

Exercise 1.6 *Define y_n by*

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k}.$$

Show that $y = (y_n)$ is a wide sense stationary process.

Solution: We know from the previous exercise, that

$$y_n = \lim_{N\to\infty} y_{n,N}.$$

Then

$$\mathrm{E} y_n = \lim_{N\to\infty} \mathrm{E} y_{n,N} = 0.$$

Also, for the autocovariance function we have

$$\lim_{N\to\infty, M\to\infty} \mathrm{E} y_{n+\tau,N} y_{n,M} = \sum_{k=0}^{\infty} h_k h_{k+\tau} \cdot \sigma^2.$$

□

1.3 Prediction

Exercise 1.7 *Show that if R is singular then y_n can be predicted with 0 error.*

Solution: If R is singular, then there is a nonzero vector $v = (v_1, \ldots v_p)^T$, such that $v^T R = 0$, and also $v^T R v = 0$. We can assume, that the first non-zero element of v equals to 1. Also, we can assume that $v = (1, v_2, \ldots, v_p)^T$. Define

$$\hat{y}_n = \sum_{k=1}^{p} (-v_{k+1}) y_{n-k}.$$

Then

$$\begin{aligned}
\mathrm{E}(y_n - \hat{y}_n)^2 &= \mathrm{E}\left(y_n + \sum_{k=1}^{p-1} v_{k+1} y_{n-k}\right)^2 = \\
&= v^T \mathrm{E}(y_n, y_{n-1}, \dots y_{n-p+1})^T (y_n, y_{n-1}, \dots y_{n-p+1})\, v = v^T R v = 0,
\end{aligned}$$

thus the prediction error is 0. □

Exercise 1.8 * *Show that for the process* (y_n^r) *we have*

$$H_{-\infty}^{y^r} = \{0\}.$$

Solution: Let us recall that

$$y_n^r = \sum_{k=0}^{\infty} c_k e_{n-k}, \qquad c_0 = 1,$$

where $e = (e_n)$ is the innovation process of $y = (y_n)$. Therefore $y_n^r \in H_n^e$ for all n implying

$$H_n^{y^r} \subset H_n^e \qquad \text{for all} \quad n.$$

It follows that

$$H_{-\infty}^{y^r} \subset H_{-\infty}^e.$$

Now if $v \in H_{-\infty}^e$ then $v \in H_{-n}^e$ for all n. Since $e = (e_n)$ is an orthogonal process we have

$$e_m \perp H_{-n}^e \text{for any} \quad m > -n.$$

In particular

$$e_m \perp v.$$

Note that the latter claim holds independently of n, for all n. It follows that

$$v \perp H^e = H_\infty^e.$$

But, obviously, $v \in H_\infty^e$. Thus we must have $v = 0$.

Exercise 1.9 * *Show that*

$$H_{-\infty}^{y} = H_{-\infty}^{y^s}.$$

Solution: We have

$$y_n = y_n^r + y_n^s, \qquad \text{for all} \quad n.$$

The orthogonality $y^r \perp y^s$ implies

$$H_\infty^{y^r} \perp H_\infty^{y^s},$$

and thus

$$H_{-n}^{y^r} \perp H_{-n}^{y^s}, \qquad \text{for all} \quad n.$$

First we show that

$$H_{-\infty}^{y} \supset H_{-\infty}^{y^s}.$$

To see this note that the definition

$$e_n = y_n - (y_n | H_{n-1}^y)$$

implies $e_n \in H_n^y$ for all n, which in turn implies

$$H_n^e \subset H_n^y \qquad \text{for all} \quad n.$$

But then $y_n^r \in H_n^y$ implies

$$y^r{}_n \in H_n^y, \qquad \text{for all} \quad n,$$

and hence $y_n^s = y_n - y_n^r$ we also have

$$y^s{}_n \in H_n^y, \qquad \text{for all} \quad n.$$

We conclude from here that

$$H_n^{y^s} \subset H_n^y \qquad \text{for all} \quad n,$$

which implies

$$H_{-\infty}^{y^s} \subset H_{-\infty}^{y}.$$

Now to prove the opposite inclusion let $v \in H_{-\infty}^y$. Then $v \in H_{-n}^y$ for all n. The decomposition of y as $y_n = y_n^r + y_n^s$ with $y^r \perp y^s$ implies

$$H_{-n}^y = H_{-n}^{y^r} \oplus H_{-n}^{y^s}$$

for all n, where $\oplus$ denotes orthogonal direct sum. Let us write v as

$$v = v_{-n}^r + v_{-n}^s$$

with $v_{-n}^r \in H_{-n}^{y^r}$ and $v_{-n}^s \in H_{-n}^{y^s}$. Then

$$v_{-n}^r = (v | H_{-n}^{y^r}).$$

Now letting n tend to infinity we get that

$$\lim_{n\to\infty} v^r_{-n} = \lim_{n\to\infty}(v|H^{y^r}_{-n}) = (v|H^{y^r}_{-\infty})$$

But $H^{y^r}_{-\infty} = \{0\}$, hence $v^r_{-n} \to 0$ and we conclude that

$$v = \lim_{n\to\infty} v^s_{-n}.$$

Obviously, the right had side belongs to $H^{y^s}_{-m}$ for all m and hence it belongs to $H^{y^s}_{-\infty}$. $\square$

Chapter 2

Prediction, innovation and the Wold decomposition

2.1 Prediction using the infinite past

Exercise 2.1 *Prove that*

$$(y_n|H^y_{n-1}) = \lim_{p\to\infty}(y_n|H^y_{n-1,n-p}) \qquad \text{in} \quad L_2(\Omega, \mathcal{F}, P).$$

Solution: Let us denote

$$\hat{y}_n = (y_n|H^y_{n-1}), \quad \text{and} \quad \tilde{y}_n = y_n - \hat{y}_n.$$

Thus

$$y_n = \tilde{y}_n + \hat{y}_n,$$

where $\tilde{y}_n \perp H^y_{n-1}$, and also $\tilde{y}_n \perp (y_n|H^y_{n-1,n-p}$. Thus projecting both sides onto $H^y_{n-1,n-p})$ we get

$$(y_n|H^y_{n-1,n-p}) = (\hat{y}_n|H^y_{n-1,n-p}).$$

But $\hat{y} \in \text{cl}\ (\cup_p H^y_{n-1,n-p})$ implies

$$\lim_{p\to\infty}\left((\hat{y}_n|H^y_{n-1,n-p}) - \hat{y}\right) = 0.$$

□

2.2 Singular processes

Exercise 2.2 *Show that if $H^y_n = H^y_{n-1}$ for a single n, then $H^y_n = H^y_{n-1}$ for all n.*

Solution: We know, that the process (e_n) with $e_n = y_n - (y_n | H_{n-1}^y)$ is a w.s.st. orthogonal process. If $H_n^y = H_{n-1}^y$, then one of these random variable is zero, thus all of them are zero. □

Exercise 2.3

$$y_n = \xi e^{in\omega}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

Show that the above process is wide sense stationary.

Solution:

$$\mathrm{E} y_n = \mathrm{E} \xi e^{in\omega} = 0,$$

and

$$\mathrm{E} y_{n+\tau} \overline{y_n} = \mathrm{E} \xi e^{i(n+\tau)\omega} \overline{\xi} e^{-in\omega} = \sigma^2 e^{i\tau\omega}$$

is independent of n. □

Exercise 2.4 *Show that y is singular, i.e.*

$$H_n^y = H_{n-1}^y \qquad \text{for all} \quad n.$$

Solution: A simple proof is obtained by applying the result given in the following exercise. □

Exercise 2.5 *Let y be a w.s.st. process such that* $\dim(H_n^y) < \infty$ *for some n. Then y is singular.*

Solution: Let $\dim(H_n^y) = p < \infty$. Consider is a basis (y_{n-n_k}), $k = 1, \ldots p$. This is a generator system, thus

$$y_n = \sum_{k=1}^{p} c_k y_{n-n_k},$$

with some $c_k \in \mathbb{R}$. It means, y_n can be predicted with zero error. □

Exercise 2.6

$$y_n = \sum_{k=1}^{m} \xi_k e^{in\omega_k}$$

Prove that the process y defined above is singular, i.e.

$$H_n^y = H_{n-1}^y \qquad \text{for all} \quad n.$$

Solution: We apply previous Exercise. Let us consider $m+1$ random variables in H_n^y: $y_{n_0}, y_{n_1}, \dots y_{n_m}$. Then we can write

$$\begin{pmatrix} y_{n_0} \\ y_{n_1} \\ \vdots \\ y_{n_m} \end{pmatrix} = \begin{pmatrix} e^{in_0\omega_1} & e^{in_0\omega_2} & \dots & e^{in_0\omega_m} \\ e^{in_1\omega_1} & e^{in_1\omega_2} & \dots & e^{in_1\omega_m} \\ \vdots & \vdots & & \vdots \\ e^{in_m\omega_1} & e^{in_m\omega_2} & \dots & e^{in_m\omega_m} \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix},$$

or in compact form $Y = A \cdot Z$. Here Y and Z are random variables, and A is a matrix with constant elements from $\mathbb{C}$. The dimension of the matrix is $(m+1) \times m$, thus the rows are linearly dependent. There is an $\alpha \in \mathbb{C}^{m+1}$ vector such that $\alpha^T A = 0$ and thus $\alpha^T Y = 0$. It means, that any $m+1$ element in H_n^y is linearly dependent, thus $\dim(H_n^y) \leq m < \infty$, and the process is singular. □

Exercise 2.7 *A simple example for a real-valued singular process is given by*

$$y_n = \cos(\omega n + \varphi) \qquad \omega \neq 0,$$

where φ is a random phase with uniform distribution on $[0, 2\pi]$. Show that (y_n) is a wide sense stationary process.

Solution: As φ is a random variable with uniform distribution on $[0, 2\pi]$, its density function is

$$f(x) = \begin{cases} \dfrac{1}{2\pi} & x \in [0, 2\pi] \\ \\ 0 & \text{otherwise} \end{cases}$$

Thus the mean value

$$\mathrm{E} y_n = \int_0^{2\pi} \cos(n\omega + x) \frac{1}{2\pi} dx = 0.$$

Applying the identity

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

we get

$$y_{n+\tau} y_n = \cos\left(\omega(n+\tau) + \varphi\right)\cos(\omega n + \varphi) = \frac{\cos(\omega\tau) + \cos(\omega(2n+\tau) + 2\varphi)}{2}.$$

Taking expectation we get using the above argument, that

$$\mathrm{E} y_{n+\tau} y_n = \frac{\cos(\omega\tau) + 0}{2},$$

which is indeed independent of n. □

Exercise 2.8 *Show that (y_n) is singular.*

Solution: Apply the identity $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. We get

$$y_{n+1} = \cos\left((\omega n + \varphi) + \omega\right) = \cos(\omega n + \varphi)\cos(\omega) - \sin(\omega n + \varphi)\sin(\omega),$$

thus

$$y_{n+1} = y_n \cos(\omega) \pm \sqrt{1 - y_n^2}\,\sin(\omega) \in H_n^y.$$

It means, that

$$H_n^y = H_{n-1}^y \qquad \text{for all} \quad n.$$

$\square$

2.3 Wold decomposition

Lemma 1 *For any random variable $\xi \in L_2(\Omega, \mathcal{F}, P)$ we have*

$$\lim_{m\to\infty} (\xi | H_{-m}) = (\xi | H_{-\infty}).$$

Exercise 2.9 * *Prove Lemma 1.*

Solution: We shall apply the dual of this lemma, Lemma 2 bellow. Let us denote the orthogonal complement of H_{-m} in $L_2(\Omega, \mathcal{F}, P)$ by H_m^+, i.e.

$$L_2(\Omega, \mathcal{F}, P) = H_{-m} \oplus H_m^+.$$

Then

$$(\xi | H_{-m}) = \xi - (\xi | H_m^+).$$

Applying Lemma 2 we get

$$\lim_{m\to\infty} (\xi | H_m^+) = (\xi | H_\infty^+), \qquad H_\infty^+ = \mathrm{cl}\ (\cup_m H_m^+).$$

As H_∞^+ is the orthogonal complement of $H_{-\infty}$, Lemma 1 is proved. $\square$

Lemma 2 *Let $H_n \subset L_2(\Omega, \mathcal{F}, P)$ be a monotone increasing sequences of Hilbert subspaces, i.e. $H_n \subset H_{n+1}$. Let*

$$H_\infty = \mathrm{cl}\ (\cup_n H_n),$$

with cl *denoting the closure. Then for any $x \in L_2(\Omega, \mathcal{F}, P)$ we have*

$$\lim_{n\to\infty} (x | H_n) = (x | H_\infty).$$

Exercise 2.10 * *Prove Lemma 2.*

Solution: Let $(x|H_\infty) = y$, and write

$$x = y + \Delta x,$$

where $\Delta x \perp H_\infty$. Then also $\Delta x \perp H_n$ for any n. Thus projecting both sides of the above equality onto H_n we get

$$(x|H_n) = (y|H_n).$$

But $y \in \mathrm{cl}\,(\cup H_n)$ implies that

$$\lim_{n\to\infty} (y - (y|H_n)) = 0,$$

thus $(y|H_n)$ converges to y, and the proposition follows. □

Exercise 2.11 *Show that the processes (y_n^s) and (y_n^r) are orthogonal, $y^s \perp y^r$, meaning that*

$$y_n^s \perp y_m^r \qquad \text{for all} \quad n,\ m.$$

Solution: For any $v \in H_{-\infty}^y$ and any k we have $v \perp e_k$, since $v \in H_{k-1}^y$. By definition $y_n^s) \in H_{-\infty}^y$ (since $y_n^s = (y_n|H_{-\infty}^y)$), and $y_n^r = \sum_{k=0}^\infty c_k e_{n-k}$, Thus for all n and m, y_n^s and y_m^r are indeed orthogonal. □

Exercise 2.12 *Show that for the process $y^r = (y_n^r)$ we have*

$$H_{-\infty}^{y^r} = \{0\}.$$

Solution: First, by definition $y_n^r = \displaystyle\sum_{k=0}^\infty c_k e_{n-k}$, thus $H_n^{y^r} \subset H_n^e$ for all n. On the other hand $H_{-\infty}^e = \{0\}$, since $e_n \perp H_{-\infty}^e$ for all n.. □

Chapter 3

Spectral theory I.

3.1 The need for a spectral theory

Let

$$y_n = e_n + ce_{n-1}.$$

Then

$$e_n = -ce_{n-1} + y_n,$$

and iterating this equation we get, assuming $|c| < 1$,

$$e_n = \sum_{k=0}^{\infty} (-c)^k y_{n-k}.$$

Exercise 3.1 *Show that the right hand side above is well defined.*

Solution: Let us define

$$e_n^N = \sum_{k=0}^{N} (-c)^k y_{n-k}.$$

Then for $N < M$ we have

$$e_n^M - e_n^N = \sum_{k=N+1}^{M} (-c)^k y_{n-k},$$

thus

$$\mathrm{E}(e_n^M - e_n^N)^2 = r^y(0) \cdot \sum_{k=N+1}^{M} c^{2k} - 2r^y(1) \cdot \sum_{k=N+2}^{M} (c^{2k+1} + c^{2k-1}),$$

since $r^y(k) = 0$ if $|k| > 1$. It follows, that (e_n^M) is a Cauchy sequence as $M \to \infty$, thus the limit is well defined. $\square$

3.2 Fourier methods for w.s.st. processes

Proposition 1 *We have*

$$\begin{aligned}\lim_{N\to\infty} \xi_N(\omega_k) &= \xi_k, \\ \lim_{N\to\infty} \xi_N(\omega) &= 0 \qquad \text{for} \quad \omega \neq \omega_k\end{aligned}$$

in the sense of $L_2(\Omega, \mathcal{F}, P)$ *and also w.p.1.*

Exercise 3.2 *Prove Proposition 1.*

Solution:

$$\xi_N(\omega) = \frac{1}{2N+1}\sum_{n=-N}^{N} y_n e^{-in\omega} = \frac{1}{2N+1}\sum_{n=-N}^{N}\left(\sum_{k=1}^{m}\xi_k e^{in\omega_k}\right)e^{-in\omega} =$$

$$= \sum_{k=1}^{m}\xi_k\left(\frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(\omega_k-\omega)}\right) = \sum_{k=1}^{m} c_k\xi_k.$$

If $\omega = \omega_k$ for some k, then on the right hand side of the equation the coefficient of ξ_k is the following:

$$\frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(\omega_k-\omega_k)} = \frac{1}{2N+1}\sum_{n=-N}^{N} 1 = 1,$$

thus this term of the sum is exactly ξ_k.

If $\omega \neq \omega_k$, the coefficient of ξ_k is the following:

$$c_k = \frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(\omega_k-\omega)} = \frac{1+2\sum_{n=1}^{N}\cos(n(\omega_k-\omega))}{2N+1}.$$

We shall use the formula for the Dirichlet kernel:

$$1+2\sum_{k=1}^{n}\cos(kx) = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(x/2\right)}.$$

Thus

$$c_k = \frac{1}{2N+1}\frac{\sin\left((N+\frac{1}{2})(\omega_k-\omega)\right)}{\sin\left((\omega_k-\omega)/2\right)} \rightarrow 0 \qquad \text{for} \qquad N \rightarrow \infty.$$

It proves the Proposition. □

Corrolary 1 *The spectral weights* σ_k^2 *can be obtained as*

$$\sigma_k^2 = \mathrm{E}|\xi_k|^2 = \lim_{N\to\infty}\mathrm{E}\left|\frac{1}{2N+1}\sum_{n=-N}^{+N} y_n e^{-in\omega_k}\right|^2.$$

Exercise 3.3 *Prove the above corollary.*

Solution: This follows simply from the fact that $\xi_N(\omega_k) \to \xi_k$ in $L_2(\Omega, \mathcal{F}, P)$. □

3.3 Herglotz's theorem

Exercise 3.4 *Show that the truncated $r_N(\tau)$ sequence itself is an auto-covariance sequence.*

Solution: Obviously, $r_N(\tau)$ is a positive semi-definite sequence. Hence it is an auto-covariance sequence.

Chapter 4

Spectral theory II.

4.2 Random orthogonal measures. Integration

Exercise 4.1 *Prove, that for any* $0 \le a < b < 2\pi$ *we have*

$$F(b) - F(a) = \mathrm{E}\left|\zeta(b) - \zeta(a)\right|^2,$$

thus F *is monotone nondecreasing.*

Solution: Write $[0, b)$ as the union of $[0, a)$ and $[a, b)$. Then by definition

$$\zeta(b) - \zeta(a) \ \perp \ \zeta(a) - \zeta(0).$$

Apply Pythagoras theorem:

$$\mathrm{E}|\zeta(b) - \zeta(a)|^2 + \mathrm{E}|\zeta(a) - \zeta(0)|^2 = \mathrm{E}|\zeta(b) - \zeta(0)|^2.$$

Substituting $\zeta(0) = 0$, and $\mathrm{E}|\zeta(a)|^2 =: F(a)$ we get

$$\mathrm{E}|\zeta(b) - \zeta(a)|^2 + F(a) = F(b).$$

$\square$

Exercise 4.2 *Let* f, g *be two left continuous step functions on* $[0,\, 2\pi]$. *Then*

$$\mathrm{E}I(f)\overline{I(g)} = \int_0^{2\pi} f(\omega)\overline{g(\omega)}\, dF(\omega). \tag{4.1}$$

Solution: Take a common subdivision for f and g, denote the intervals by $[\alpha_k, \beta_k)$, $k = 1, 2, \ldots n$. Then

$$\begin{aligned} f(\omega) &= \sum_{k=1}^{n} \lambda_k \chi_{[\alpha_k,\beta_k)}, \\ g(\omega) &= \sum_{k=1}^{n} \eta_k \chi_{[\alpha_k,\beta_k)}, \end{aligned}$$

and the product function:

$$f(\omega)\overline{g(\omega)} = \sum_{k=1}^{n} \lambda_k \overline{\eta_k} \chi_{[\alpha_k,\beta_k)}.$$

Thus the integral of $f\overline{g}$ on the right hand side of (4.1) is the following:

$$\int_0^{2\pi} f(\omega)\overline{g(\omega)}\, dF(\omega) = \sum_{k=1}^{n} \lambda_k \overline{\eta_k}(F(\beta_k) - F(\alpha_k)).$$

Let us compute the left hand side of (4.1). We have

$$I(f) = \sum_{k=1}^{n} \lambda_k(\zeta(\beta_k) - \zeta(\alpha_k)), \qquad I(g) = \sum_{k=1}^{n} \eta_k(\zeta(\beta_k) - \zeta(\alpha_k)).$$

The random variables $(\zeta(\beta_k) - \zeta(\alpha_k))$ $k = 1, 2, \ldots n$ are independent, thus

$$\mathrm{E}I(f)\overline{I(g)} = \sum_{k=1}^{n} \lambda_k \overline{\eta_k} \mathrm{E}|\zeta(\beta_k) - \zeta(\alpha_k)|^2 = \sum_{k=1}^{n} \lambda_k \overline{\eta_k}(F(\beta_k) - F(\alpha_k)).$$

□

Exercise 4.3 *Prove that*

$$y_n = \int_0^{2\pi} e^{in\omega} d\zeta(\omega)$$

is w.s.st.

Solution: $\mathrm{E}y_n = 0$. Applying the isometry we get

$$\mathrm{E}y_n\overline{y_{n+\tau}} = \int_0^{2\pi} e^{in\omega}\overline{e^{i(n+\tau)\omega}}\, dF(\omega) = \int_0^{2\pi} e^{i(-\tau)\omega}\, dF(\omega),$$

which is independent of n. □

4.3 Representation of a wide sense stationary process

Exercise 4.4 *Prove the above implication.*

Solution: Let $g(\omega) = \sum_{n\in N} c_n e^{in\omega}$, $c_n \in \mathbb{C}$. Then $g = 0$ in $L_2^c(dF) = L_2^c([0, 2\pi), dF)$ means, that

$$\int_0^{2\pi} |g(\omega)|^2 dF(\omega) = 0.$$

The corresponding random variable is $I(g) = \sum_{n\in N} c_n y_n$. Then

$$\mathrm{E}I(g) = 0, \qquad \mathrm{E}I(g)\overline{I(g)} = \sum_n \sum_m c_n \overline{c_m} \int_0^{2\pi} e^{i(n-m)\omega} dF(\omega) = \int_0^{2\pi} |g(\omega)|^2 dF(\omega) = 0.$$

Thus $I(g) = 0$ a.s. in $L_2^c(\Omega, \mathcal{F}, P)$. □

Exercise 4.5 *Show that the structure function of the random orthogonal measure ζ is F, predetermined by the spectral distribution function of y.*

Solution: Indeed, we have

$$dF(\omega) = dF^y(\omega) \cdot \frac{1}{2\pi},$$

since $y_n = \displaystyle\int_0^{2\pi} e^{in\omega} d\zeta(\omega)$, thus

$$\mathrm{E}(y_{n+\tau}\overline{y_n}) = \int_0^{2\pi} e^{i(n+\tau)\omega} e^{-in\omega} dF(\omega) = \int_0^{2\pi} e^{in\tau} dF(\omega).$$

□

4.4 Change of measure

Let $d\zeta(\omega)$ be a random orthogonal measure on $[0, 2\pi)$ with the structure function $F(\omega)$. Let $g \in L_2^c(dF)$, and define

$$\eta(\omega) = \int_0^{\omega} g(\omega') d\zeta(\omega') \qquad 0 \le \omega < 2\pi.$$

Exercise 4.6 *Show that $d\eta(\omega)$ is a random orthogonal measure, with the structure function*

$$dG(\omega) = |g(\omega)|^2 dF(\omega).$$

Solution: We have $\eta(0) = 0$. For any two non-overlapping intervals $[a, b)$ and $[c, d)$ contained in $[0, 2\pi)$ we have

$$\eta(d) - \eta(c) = \int_c^d g(\omega')d\zeta(\omega'), \qquad \eta(b) - \eta(a) = \int_a^b g(\omega')d\zeta(\omega').$$

As the domains of integration are non-overlapping,

$$\eta(d) - \eta(c) \ \perp \ \eta(b) - \eta(a).$$

The structure function can be computed using the isometry between $L_2^c(dF)$ and $L_2^c(\Omega, \mathcal{F}, P)$:

$$\mathrm{E}|\eta(a)|^2 = \mathrm{E}\eta(a)\overline{\eta(a)} = \int_0^{2\pi} g(\omega)\overline{g(\omega)}dF(\omega) = \int_0^{2\pi} |g(\omega)|^2 dF(\omega).$$

□

4.5 Linear filters

Exercise 4.7 *Show that the spectral representation process of y is given by*

$$d\zeta^y(\omega) = H(e^{-i\omega})d\zeta^u(\omega),$$

where

$$H(e^{-i\omega}) = \sum_{k=0}^{m} h_k e^{-ik\omega}.$$

Solution: The spectral representation of (u_n) gives

$$u_n = \int_0^{2\pi} e^{in\omega} d\zeta^u(\omega).$$

Thus

$$u_{n-k} = \int_0^{2\pi} e^{i(n-k)\omega} d\zeta^u(\omega), \quad \text{and} \quad h_k u_{n-k} = \int_0^{2\pi} h_k e^{i(n-k)\omega} d\zeta^u(\omega).$$

It follows, that

$$\begin{aligned} y_n &= \sum_{k=0}^{m} h_k u_{n-k} = \int_0^{2\pi} \sum_{k=0}^{m} h_k e^{i(n-k)\omega} d\zeta^u(\omega) = \\ &= \int_0^{2\pi} e^{in\omega} \left(\sum_{k=0}^{m} h_k e^{-ik\omega} d\zeta^u(\omega) \right) = \int_0^{2\pi} e^{in\omega} d\zeta^y(\omega). \end{aligned}$$

□

Exercise 4.8 *Show that the integrand on the right hand side converges to* $e^{in\omega}H(e^{-i\omega})$ *in* $L_2^c(dF^u)$.

Solution: We know, that

$$\lim_{m\to\infty} H^m(e^{-i\omega}) = \lim_{m\to\infty} \sum_{k=0}^{m} h_k e^{-ik\omega}. = y_n$$

in $L_2^c(\Omega, \mathcal{F}, P)$. Thus due to isometry

$$\lim_{m\to\infty} \int_0^{2\pi} e^{in\omega} H^m(e^{-i\omega}) d\zeta^u(\omega) = \int_0^{2\pi} e^{in\omega} H(e^{-i\omega}) d\zeta^u(\omega).$$

□

Chapter 5

AR, MA and ARMA processes

5.3 The AR(1) process

Exercise 5.1 *Prove that* (y_n) *defined by* $y_n = \sum_{k=0}^{\infty}(-a)^k e_{n-k}$ *does indeed satisfy* $y_n + ay_{n-1} = e_n$ *.*

Solution: We have

$$y_n = e_n + \sum_{k=1}^{\infty}(-a)^k e_{n-k} = e_n + (-a)\sum_{k=1}^{\infty}(-a)^{k-1} e_{n-1-(k-1)} =$$

$$= e_n + (-a)\sum_{k=0}^{\infty}(-a)^k e_{n-1-k} = e_n - ay_{n-1}.$$

Here we used the fact, that the coefficient-sequence is absolute convergent, and (e_{n-k}) is an orthogonal sequence in $L_2(\Omega, \mathcal{F}, P)$. $\square$

Exercise 5.2 *If* $|a| > 1$ *define*

$$y_n = \sum_{k=0}^{\infty}(-\frac{1}{a})^k \frac{1}{a} e_{n+k+1}.$$

Show that the r.h.s. is well-defined, and (y_n) *does indeed satisfy* $y_n + ay_{n-1} = e_n$*.*

Solution: The r.v-s (e_{n+k}), $k = 1, 2, \ldots$ form an orthogonal sequence in $L_2(\Omega, \mathcal{F}, P)$. As $|a| > 1$, the coefficient sequence on the right hand side is absolute convergent. Thus the right hand side is well-defined.

We have

$$y_n = \frac{1}{a}e_{n+1} + \frac{1}{a}\sum_{k=1}^{\infty}(-\frac{1}{a})^{k-1}\left(-\frac{1}{a}\right)e_{n+k+1} = \frac{1}{a}e_{n+1} - \frac{1}{a}\sum_{k=0}^{\infty}(-\frac{1}{a})^{k}\frac{1}{a}e_{(n+1)+(k+1)},$$

thus

$$y_n = \frac{1}{a}e_{n+1} - \frac{1}{a}y_{n+1},$$

and it can be rearranged as

$$e_{n+1} = y_{n+1} + ay_n.$$

□

5.5 MA processes

Thus let $e = (e_n)$ be a wide sense stationary orthogonal process and define

$$y_n = \sum_{k=0}^{m} c_k e_{n-k}$$

Exercise 5.3 *Show that (y_n) is a wide sense stationary process.*

Solution: The mean value and the covariances are independent of n, since $\mathrm{E}y_n = 0$, and

$$\mathrm{E}(y_n^2) = \sum_{k=0}^{m}\sum_{j=0}^{m} c_k c_j \mathrm{E}(e_{n-k}e_{n-j}) = \sum_{k=0}^{m} c_k^2\sigma^2.$$

For $0 < \tau \leq m$ we have:

$$\mathrm{E}(y_{n+\tau}y_n) = \sum_{k=0}^{m}\sum_{j=0}^{m} c_k c_j \mathrm{E}(e_{n-k+\tau}e_{n-j}) = \sum_{k=\tau}^{m} c_k c_{k-\tau}.$$

Similarly, for $\tau > m$ we get $\mathrm{E}(y_{n+\tau}y_n) = 0$. □

Exercise 5.4 *Show that $e_n = \sum_{k=0}^{\infty}(-1)^k b^k y_{n-k}$ is well-defined, if $\sum_{k=0}^{\infty} b^{2k} < \infty$, i.e. $|b| < 1$. Similarly, $e_n = \sum_{k=0}^{\infty}(-1)^k \frac{1}{b^k} z_{n-k}$ is well-defined if $|b| > 1$.*

Solution: We shall use the Cauchy-Schwartz-Buniakovskii inequality in $L_2(\Omega, \mathcal{F}, P)$. Thus

$$|\mathrm{E}(y_n y_{n-\tau})| = |\langle y_n, y_{n-\tau}\rangle| \leq \|y_n\|\,\|y_{n-\tau})\| = \sqrt{\mathrm{E}y_n^2\,\mathrm{E}y_{n-\tau}^2} = r^y(0).$$

Let us consider the $|b| < 1$ case. We have

$$e_n = \lim_{N\to\infty} e_n^N, \quad \text{with} \quad e_n^N = \sum_{k=0}^{N} (-1)^k b^k y_{n-k}.$$

Then using CSB we get

$$\mathrm{E}(e_n^N)^2 \leq \sum_{k=0}^{N} b^{2k} r^y(0) < r^y(0) \frac{1}{1-b^2}.$$

Thus (e_n^N), $N \in \mathbb{N}$ is a uniformly convergent Cauchy sequence, it is convergent. □

5.6 ARMA processes

Proposition 2 *Assume that $A(z^{-1}) \neq 0$ for $|z| = 1$. Then there is a unique w.s.st. process satisfying $A(q^{-1})y = C(q^{-1})e$.*

Exercise 5.5 *Prove the above proposition.*

Solution: Let us assume, that such a process does exist. As (y_n) is a wide sense stationary process, there is a unique random orthogonal measure $d\zeta^y(\omega)$, such that

$$y_n = \int_0^{2\pi} e^{in\omega} d\zeta^y(\omega).$$

The spectral representation of the process $u_n = A(q^{-1})y_n = C(q^{-1})e_n$ is

$$u_n = \int_0^{2\pi} A(e^{-in\omega}) d\zeta^y(\omega) = \int_0^{2\pi} C(e^{-in\omega}) d\zeta^e(\omega).$$

From the uniqueness of the spectral representation it follows, that

$$d\zeta^y(\omega) = \frac{C(e^{-in\omega})}{A(e^{-in\omega})} d\zeta^e(\omega).$$

As (e_n) has spectral density function, (y_n) also has spectral density function::

$$f^y(\omega) = \frac{C(e^{in\omega})C(e^{-in\omega})}{A(e^{in\omega})A(e^{-in\omega})} f^y(\omega) = \frac{1}{2\pi} |H(e^{-in\omega})|^2,$$

where

$$H(e^{-in\omega}) = \frac{C(e^{-in\omega})}{A(= e^{in\omega})}.$$

Now, since $A(-e^{in\omega}) \neq 0$ for $\omega \in [0, 2\pi]$, obviously $f(\omega) \geq 0$. Thus there exist a process with this spectral density function. □

Proposition 3 *Assume that $A(z^{-1})$ and $C(z^{-1})$ are stable polynomials. Then (e_n) is the innovation process of (y_n).*

Exercise 5.6 *Prove the above proposition.*

5.8 ARMA processes with unstable zeros

Exercise 5.7 *Let G be an all-pass transfer function, and let e' be a w.s.st. orthogonal process. Then the process e defined by*

$$d\zeta^e(\omega) = G(e^{-i\omega})d\zeta^{e'}(\omega)$$

is also a w.s.st. orthogonal process.

Solution: The spectral density of e is

$$f(\omega) = |G(e^{-i\omega})|^2 f_{e'}(\omega) = f_{e'}(\omega),$$

since $|G(e^{-i\omega})|^2 = 1$. Thus the spectral density functions of e and e' are the same, e' is also a w.s.st. orthogonal process $\square$

Chapter 6

Multivariate time series

6.1 Vector valued wide sense stationary processes

Exercise 6.1 *Show that* (z_n) *is an* $\mathbb{R}$*-valued wide sense stationary process and we have*

$$\mathrm{E}z_n^2 = \sum_{k=1}^{p}\sum_{l=1}^{p} a_k^\top R(k-l)a_l \geq 0. \tag{6.1}$$

Solution:

$$\mathrm{E}z_n^2 = \mathrm{E}\sum_{k=1}^{p} a_k^\top y_{n-k}\left(\sum_{l=1}^{p} a_l^\top y_{n-l}\right)^\top = \mathrm{E}\sum_{k=1}^{p} a_k^\top y_{n-k}\cdot\sum_{l=1}^{p} y_{n-l}^\top a_l =$$
$$= \sum_{k=1}^{p}\sum_{l=1}^{p} a_k^\top\left(\mathrm{E}y_{n-k}y_{n-l}^\top\right)a_l = \sum_{k=1}^{p}\sum_{l=1}^{p} a_k^\top\left(R(n-k-(n-l))\right)a_l.$$

□

Exercise 6.2 *Prove the converse statement: let* $R(\tau)$, $-\infty < \tau < +\infty$ *be a positive definite sequence of real-valued,* $s \times s$ *matrices. Then it is the auto-covariance sequence of an* $\mathbb{R}^s$*-valued, wide sense stationary Gaussian process.*

Solution: (Sketch) First truncate the sequence as

$$R_N(\tau) = \begin{cases} R(\tau) & \text{if} \quad |\tau| \leq N \\ 0 & \text{if} \quad |\tau| > N \end{cases}$$

Then there are random variables $\xi_{-N}, \dots \xi_0, \dots \xi_N$ with Gaussian distribution, such that the autocovariances are the given $R_N(\tau)$.

Now we can apply Kolmogorov extension theorem, since the measures satisfy the consistency conditions as $N \to \infty$. This theorem ensures, that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process with the above marginal distributions. □

6.2 Prediction and the innovation process

Exercise 6.3 *Show that $e_n \equiv 0$ is indeed equivalent to*

$$H_n^y = H_{n-1}^y \text{ for all } n.$$

Solution: It is the same, as in the scalar case.

6.3 Spectral theory

Exercise 6.4 *Prove that a quadratic form $\alpha^\top F \alpha$, with F symmetric, determines the bilinear form corresponding to F uniquely as*

$$\beta^\top F \gamma = \frac{1}{4}\left((\beta+\gamma)^\top F(\beta+\gamma) - (\beta-\gamma)^\top F(\beta-\gamma)\right).$$

Solution: We are in a Euclidian space. This formula gives the well-known connection between the norm an the scalar product. Especially, for $\beta = \gamma$ we get back the quadratic form:

$$\beta^\top F \beta = \frac{1}{4}\left((2\beta)^\top F(2\beta) - (\beta-\beta)^\top F(\beta-\beta)\right) = \frac{1}{4}\left((2\beta)^\top F(2\beta)\right).$$

Exercise 6.5 * *Let $F(\cdot)$ be an $s \times s$ matrix-valued function on $[0, 2\pi]$ such that the increments $F(\cdot)$ are symmetric and positive semidefinite. Then for any $k, l = 1, \ldots, s$ the elements $F_{k,l}(\omega)$ are of finite variations.*

Solution: For any fixed $\alpha \in \mathbb{R}^s$ the function $F_\alpha(\omega) = \alpha^\top F(\omega)\alpha$ is monotone non-decreasing. Recall the expression of the bilinear form $\beta^\top F(\omega)\gamma$ via the corresponding quadratic form.

Exercise 6.6 *Show that for an $\mathbb{R}^s$-valued orthogonal wide sense stationary process (e_n) with covariance matrix $\Lambda = \mathrm{E} e_n e_n^\top$ we have*

$$f(\omega) = \Lambda \quad \text{for} \quad \forall \omega.$$

Solution: Using the given $f(\omega)$ function we get

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\tau\omega} f(\omega) d\omega = \Lambda \frac{1}{2\pi}\int_0^{2\pi} e^{i\tau\omega} d\omega = \begin{cases} \Lambda & \text{if} \quad \tau = 0 \\ 0 & \text{if} \quad \tau \neq 0 \end{cases},$$

and this is the autocovariance matrix function of an orthogonal wide sense stationary process with covariance matrix Λ. $\square$

6.4 Filtering

Proposition 4 *The spectral distribution of the process v is given by*

$$dF^v(\omega) = H(e^{-i\omega})dF^y(\omega)H(e^{i\omega})^\top.$$

Exercise 6.7 *Prove Proposition 4.*

Solution: We have

$$\mathrm{E}y_{n+\tau}y_n^T = R(\tau) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\tau\omega}dF(\omega).$$

Then

$$\begin{aligned}
\mathrm{E}v_{n+\tau}v_n^T &= \mathrm{E}\sum_{k=0}^{p} h_k y_{n+\tau-k}\left(\sum_{j=0}^{p} h_j y_{n-j}\right)^T = \\
&= \sum_{k=0}^{p}\sum_{j=0}^{p} h_k \mathrm{E}\left(y_{n+\tau-k}y_{n-j}^T\right)h_j^T == \sum_{k=0}^{p}\sum_{j=0}^{p} h_k\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(\tau-k+j)\omega}dF(\omega)\right)h_j^T \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\tau\omega}\left(\sum_{k=0}^{p} h_k e^{-ik\omega}\right)dF(\omega)\left(\sum_{j=0}^{p} h_j^T e^{ij\omega}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\tau\omega}dF^v(\omega).
\end{aligned}$$

□

Proposition 5 *Assume that the right hand side of*

$$H(e^{-i\omega}) = \sum_{k=0}^{\infty} h_k e^{-i\omega k}$$

converges in $L_2^{c,r\times s}(dF)$. Then the process (v_n) under

$$v_n = \sum_{k=0}^{\infty} h_k y_{n-k}$$

is well-defined, i.e. the right hand side converges in $L_2^{c,r}(\Omega, \mathcal{F}, P)$, and the spectral distribution of (v_n) is given by

$$dF^v(\omega) = H(e^{-i\omega})dF^y(\omega)H^\top(e^{i\omega}).$$

Exercise 6.8 *Prove the above proposition following the proof for the scalar case.*

Solution: Truncate the infinite sum defining v_n at m, i.e. define

$$v_n^m = \sum_{k=0}^{m} h_k y_{n-k}.$$

Then the spectral representation of $v^m = (v^m_n)$ is given as

$$v^m_n = \int_0^{2\pi} e^{in\omega} H^m(e^{-i\omega}) d\zeta^u(\omega),$$

where

$$H^m(e^{-i\omega}) = \sum_{k=0}^{m} h_k e^{-ik\omega}.$$

Now letting m tend to infinity, the l.h.s. of the above equality converges to v_n in $L_2^{c,r}(\Omega, \mathcal{F}, P)$. Similarly, the integrand on the right hand side converges to $e^{in\omega} H(e^{-i\omega})$ in $L_2^{c,r\times s}(dF)$ □

6.5 Multivariate random orthogonal measures

Theorem 1 *Let $d\zeta(\omega)$ be a $\mathbb{C}^s$-valued random orthogonal measure, with structure function $dF(\omega)$. Then the $\mathbb{C}^s$-valued process*

$$y_n = \int_0^{2\pi} e^{in\omega} d\zeta(\omega)$$

is wide sense stationary, and its spectral distribution function is given by

$$dF^y(\omega) = 2\pi \, dF(\omega).$$

Exercise 6.9 *Prove Theorem 1.*

Solution: Apply previous Theorem about the isometry, stating

$$\mathrm{E} \; I(g) \; I(h)^* = \int_0^{2\pi} g(\omega) \; \overline{h}(\omega) dF(\omega)$$

where

$$I(g) = \int_0^{2\pi} g(\omega) \cdot I \; d\zeta(\omega), \qquad I(h) = \int_0^{2\pi} h(\omega) \cdot I \; d\zeta(\omega).$$

Then

$$y_n = I(e^{in\omega}), \qquad y_{n+\tau} = I(e^{i(n+\tau)\omega}).$$

Thus

$$\mathrm{E} \; y_n \; y^*_{n+\tau} = \mathrm{E} \; I((e^{in\omega}) \; I(e^{i(n+\tau)\omega})^* = \int_0^{2\pi} e^{in\omega} \; \overline{e}^{i(n+\tau)\omega} dF(\omega) =$$

$$= \int_0^{2\pi} e^{-i\tau\omega} dF(\omega) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\tau\omega} \; 2\pi dF(\omega) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\tau\omega} dF^y(\omega).$$

□

6.7 Linear filters

Exercise 6.10 *Show that the spectral representation process of v is given by*

$$d\zeta^v(\omega) = H(e^{-i\omega})d\zeta^y(\omega).$$

Solution: The spectral representation of (y_n) gives

$$y_n = \int_0^{2\pi} e^{in\omega} d\zeta^y(\omega).$$

Thus

$$y_{n-k} = \int_0^{2\pi} e^{i(n-k)\omega} d\zeta^y(\omega), \quad \text{and} \quad h_k y_{n-k} = \int_0^{2\pi} h_k e^{i(n-k)\omega} d\zeta^y(\omega).$$

It follows, that

$$\begin{aligned} v_n &= \sum_{k=0}^{p} h_k y_{n-k} = \int_0^{2\pi} \sum_{k=0}^{p} h_k e^{i(n-k)\omega} d\zeta^y(\omega) = \\ &= \int_0^{2\pi} e^{in\omega} \left(\sum_{k=0}^{p} h_k e^{-ik\omega} d\zeta^y(\omega) \right) = \int_0^{2\pi} e^{in\omega} d\zeta^v(\omega). \end{aligned}$$

□

Exercise 6.11 *Assume, that the infinite series*

$$H(e^{-i\omega}) = \sum_{k=0}^{\infty} h_k e^{-i\omega k}.$$

converges in $L_2^{c,r\times s}(dF^y)$. Then the spectral representation process of (v_n) is

$$d\zeta^v(\omega) = H(e^{-i\omega})d\zeta^y(\omega).$$

Solution: Truncate the infinite sum defining v_n at p, i.e. define

$$v_n^p = \sum_{k=0}^{p} h_k y_{n-k}.$$

Then the spectral representation of $v^p = (v_n^p)$ is given as

$$v_n^p = \int_0^{2\pi} e^{in\omega} H^p(e^{-i\omega}) d\zeta^u(\omega),$$

where

$$H^p(e^{-i\omega}) = \sum_{k=0}^{p} h_k e^{-ik\omega}.$$

Now letting p tend to infinity, the l.h.s. of the previous equality converges to v_n in $L_2^{c,s}(\Omega, \mathcal{F}, P)$. □

Exercise 6.12 *Re-derive the formula for the spectral distribution measure of v:*

$$dF^v(\omega) = H(e^{-i\omega})dF^y(\omega)H^\top(e^{i\omega})$$

using the exercise above.

Solution: The spectral representation process of (v_n) is

$$d\zeta^v(\omega) = H(e^{-i\omega})d\zeta^y(\omega),$$

which means that the process can be represented as

$$v_n = \int_0^{2\pi} e^{in\omega} d\zeta^v(\omega) = \int_0^{2\pi} e^{in\omega} H(e^{-i\omega})d\zeta^y(\omega).$$

Then also we have

$$v_{n-\tau} = \int_0^{2\pi} e^{i(n-\tau)\omega} H(e^{-i\omega})d\zeta^y(\omega).$$

The covariance of these two random vectors can be written using the isometry:

$$\mathrm{E} v_n v_{n-\tau}^T = \int_0^{2\pi} e^{in\omega} H(e^{-i\omega})dF^y(\omega)\overline{e^{i(n-\tau)\omega} H(e^{-i\omega})} = \int_0^{2\pi} e^{i\tau\omega} H(e^{-i\omega})dF^y(\omega)H(e^{i\omega}).$$

Comparing it with the definition of the spectral distribution measure:

$$\mathrm{E} v_n v_{n-\tau}^T = \int_0^{2\pi} e^{i\tau\omega} dF^v(\omega)$$

we get the desired formula. □

6.8 Proof of the spectral representation theorem

Exercise 6.13 *Let $\zeta(a), 0 \leq a < 2\pi$ be an C^s-valued stochastic process such that for any $\alpha \in \mathbb{C}^s$ the complex-valued process $\alpha^\top \zeta(a), 0 \leq a < 2\pi$ has orthogonal increments. Then the process $\zeta(\cdot)$ itself has orthogonal increments.*

Solution: Indirect. Let us assume, that there exist $0 \leq a < b \leq c < d \leq 2\pi$, such that and the covariance matrix of the increments is not 0, i.e.

$$\mathrm{E}(\zeta(b) - \zeta(a))(\overline{\zeta}(d) - \overline{\zeta}(c))^\top = R \neq 0 \in \mathbb{R}^{s\times s}.$$

But on the other hand for any $\alpha \in \mathbb{R}^s$ $\alpha^* R\alpha = 0$. Then also $\alpha^* R^* \alpha = 0$, and thus the quadratic form

$$\alpha^*(R + R^*)\alpha = 0, \qquad \Longrightarrow \qquad (R + R^*) = 0.$$

It means, that R is antisymmetric. □

Chapter 7

State-space representation

7.1 From multivariate AR(1) to state-space equations

Proposition 6 *Let us consider the multivariate linear stochastic equation*

$$x_{n+1} = Ax_n + Bv_n.$$

Let A be a stable $s \times s$ matrix. Then this equation has a unique wide sense stationary solution (x_n), given by

$$x_{n+1} = \sum_{k=0}^{\infty} A^k B v_{n-k}.$$

It follows that x is a causal linear function of v, more exactly, for all n

$$H^x_{n+1} \subset H^v_n.$$

Exercise 7.1 *Prove the above proposition.*

Solution: The spectral representation of x is given by

$$d\zeta^x(\omega) = A(e^{-i\omega})^{-1} d\zeta^v(\omega),$$

where

$$A(e^{-i\omega})^{-1} = (e^{i\omega} I - A)^{-1} B.$$

As A is stable, $\det(zI - A)$ is not zero for $|z| \geq 1$. It follows, that

$$(zI - A)^{-1} = \sum_{k=0}^{\infty} A^k z^k$$

is well defined for $|z| \geq 1$, i.e.

$$\sum_{k=0}^{\infty} A^k z^{-k}$$

is well defined for $|z| < 1$. Then

$$\sum_{k=0}^{\infty} A^k B v_{n-k}$$

is well defined. $\square$

Lemma 3 *Let A be an $s \times s$ stable real matrix. Then*

$$(e^{i\omega} I - A)^{-1} = \sum_{k=1}^{\infty} h_k e^{-ik\omega}, \qquad h_1 = I$$

with some sequences of $s \times s$ real matrices h_k, where convergence in the right hand side is understood in the sense of $L_2^{c,s\times s}(d\omega)$. In fact, convergence is also uniform *in ω.*

Exercise 7.2 *Prove the Lemma 3.*

Solution: Note that the matrix-valued function $(zI - A)^{-1}$ is analytic in $\{z \ : \ |z| > 1-\varepsilon\}$ for some $\varepsilon > 0$. Then it is analytic for $|z| = 1$, it can be represented as a Taylor series for $z = e^{i\omega}$.

Exercise 7.3 *Prove the Proposition 6 using Lemma 3.*

Solution: It was already done in Exercise 7.1.

Proposition 7 *Assume, that $e^{i\omega} I - A$ is not singular for all $\omega \in [0, 2\pi]$. Then*

$$x_{n+1} = Ax_n + Bv_n$$

has a unique solution.

Exercise 7.4 *Prove the above Proposition 7.*

Solution: Assume that a wide sense stationary solution (x_n) exists. There is a unique random orthogonal measure $d\zeta^x(\omega)$, such that

$$x_n = \int_0^{2\pi} e^{in\omega} d\zeta^x(\omega).$$

Then also show that necessarily

$$x_{n+1} = \int_0^{2\pi} e^{i(n+1)\omega} d\zeta^x(\omega).$$

and

$$Ax_n + Bv_n = \int_0^{2\pi} e^{in\omega} \left(Ad\zeta^x(\omega) + B\right) d\zeta^v(\omega).$$

From the uniqueness of the orthogonal measure we get

$$d\zeta^x(\omega) = (e^{i\omega} I - A)^{-1} B \, d\zeta^v(\omega).$$

Then the spectral representation of the autocovariance matrices gives

$$dF^x(\omega) = (e^{i\omega} I - A)^{-1} B dF^y(\omega) \left((e^{-i\omega} I - A)^{-1} B\right) \top.$$

The condition of Proposition 7 ensures, that $dF^x(\omega)$ is well-defined. Thus the state-space equation has a unique solution. □

7.2 Auto-covariances and the Lyapunov-equation

Exercise 7.5 *Show that P can be written as*

$$P = \sum_{k=0}^{\infty} A^k B \Sigma_{vv} B^T (A^T)^k.$$

Solution: We can use the expression of (x_n) given by

$$x_{n+1} = \sum_{k=0}^{\infty} A^k B v_{n-k}.$$

The solution follows from the orthogonality of (v_n), i.e.

$$\mathrm{E} v_n^T v_m = \begin{cases} \Sigma_{vv} & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

□

Exercise 7.6 *Show directly, with purely algebraic arguments, that, if A is stable, the Lyapunov-equation*

$$P = APA^T + B\Sigma_{vv} B^T$$

has a unique solution P, and show that it can be written in the form

$$\sum_{k=0}^{\infty} A^k B \Sigma_{vv} B^T (A^T)^k.$$

Prove that the solution P, given by this formula, is positive semi-definite.

Solution: Iterate equation

$$P = APA^T + B\Sigma_{vv}B^T.$$

We get

$$P = APA^T + B\Sigma_{vv}B^T = A^2P(A^T)^2 + AB\Sigma_{vv}B^TA^T + B\Sigma_{vv}B^T = \dots$$

$$= A^{n+1}P(A^T)^{n+1} + \sum_{k=0}^{n} A^kB\Sigma_{vv}B^T(A^T)^k.$$

The first term on the last expression tends to zero with exponential rate, since A is stable. Thus we get the given representation. □

Exercise 7.7 *Consider two Lyapunov equations*

$$P = APA^T + B\Sigma_{vv}B^T$$

with a **common** *stable A and such that*

$$B_1\Sigma_{1,vv}B_1^T \leq B_2\Sigma_{2,vv}B_2^T.$$

Let the solutions be denoted by P_1 and P_2. Show that $P_1 \leq P_2$.

Solution: If A is stable, then

$$B_1\Sigma_{1,vv}B_1^T \leq B_2\Sigma_{2,vv}B_2^T \qquad \Longrightarrow \qquad AB_1\Sigma_{1,vv}B_1^TA^T \leq AB_2\Sigma_{2,vv}B_2^TA^T.$$

Thus

$$P_1 = \sum_{k=0}^{\infty} A^kB_1\Sigma_{vv}B_1^T(A^T)^k \leq \sum_{k=0}^{\infty} A^kB_2\Sigma_{vv}B_2^T(A^T)^k = P_2.$$

Exercise 7.8 *Prove that for $\tau < 0$ we have $R(\tau) = R(-\tau)^T$.*

Solution:

$$R(\tau) = \mathrm{E}y_{n+\tau}y_n^T = (\mathrm{E}y_ny_{n+\tau}^T)^T = (\mathrm{E}y_{n-\tau}y_n^T)^T = R(-\tau)^T.$$

□

Exercise 7.9 *Prove the validity of the recursion*

$$P_{n+1} = AP_nA^T + BB^T$$

for P_n.

To get the covariance function of (x_n) take the dyadic product of

$$x_{n+1} = Ax_n + Bv_n$$

with itself:

$$x_{n+1}x_{n+1}^T = Ax_n x_n^T A^T + Bv_n v_n^T B^T + Ax_n v_n^T B^T + Bv_n x_n^T A^T.$$

Now as $x_n \in H_{n-1}^v$, we have

$$x_n \perp v_n.$$

Then taking expectation on both sides of the above equation we get given recursion. $\square$

Exercise 7.10 *Show that if A is stable, then P_n converges to the unique solution of*

$$P = APA^T + B\Sigma_{vv}B^T.$$

Solution: Reiterating the previous recursion, we get

$$P_{n+1} = AP_nA^T + BB^T = A\left(AP_{n-1}A^T + BB^T\right)A^T + BB^T =$$

$$= A^2P_{n-1}A^{2T} + ABB^TA^T + BB^T = \cdots = A^nP_0(A^n)^T + \sum_{k=0}^{n-1} A^kBB^T(A^T)^k.$$

As A is stable:

$$\lim_{n\to\infty} A^nP_0(A^n)^T = 0,$$

thus

$$\lim_{n\to\infty} P_n = \sum_{k=0}^{\infty} A^kBB^T(A^T)^k = P_\infty.$$

In Exercise 7.6 we have shown that this is the solution of the discrete Lyapunov equation. $\square$

7.3 State space representation of ARMA processes

Exercise 7.11 *Prove that R_N is non-singular by taking a state-space representation of y.*

Chapter 8

Kalman filtering

8.1 The filtering problem

Exercise 8.1 *Provide an argument for the validity of*

$$H_n^y = H_{n-1}^y \oplus \mathcal{L}(\nu_n).$$

Solution: We have $\nu_n = y_n - (y_n | H_{n-1}^y)$, thus ν_n is the innovation process of y_n. $\square$

8.2 The Kalman-gain matrix

Exercise 8.2 *Prove that the projection of the random vector $x \in L_2^s(\Omega, \mathcal{F}, P)$ onto the finite dimensional subspace of $L_2(\Omega, \mathcal{F}, P)$ spanned by the components of ν is given by*

$$\widehat{x} = \mathrm{E}(x\nu^T)(\mathrm{E}(\nu\nu^T))^{-1}\nu.$$

Solution: Obviously, $\widehat{x}$ is an element of the finite dimensional subspace of $L_2(\Omega, \mathcal{F}, P)$ spanned by the components of ν. Also, $x - \widehat{x}$ is orthogonal to the components of ν, since

$$\mathrm{E}\left((x-\widehat{x})\nu^T\right) = \mathrm{E}x\nu^T - \mathrm{E}\widehat{x}\nu^T = \mathrm{E}x\nu^T - \mathrm{E}\left(\mathrm{E}(x\nu^T)(\mathrm{E}(\nu\nu^T))^{-1}\nu\right)\nu^T =$$

$$= \mathrm{E}x\nu^T - \mathrm{E}(x\nu^T)(\mathrm{E}(\nu\nu^T))^{-1}\mathrm{E}\left(\nu\nu^T\right) = 0.$$

Thus $\widehat{x}$ is the projection. $\square$

Exercise 8.3 *Derive the above expression*

$$H^{-1}(q^{-1}) = I - C(qI - A + KC)^{-1}K$$

from $H(q^{-1}) = C(qI - A)^{-1}K + I$ *using the matrix inversion lemma.*

Solution: THE MATRIX INVERSION LEMMA is the following: *Let*

$$F = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$$

be a 2×2 *block-matrix with* A_0 *and* D_0 *being square matrices. Assume that* A_0, D_0 *are non-singular and so is* $A_0 - B_0 D_0^{-1} C_0$. *Then*

$$\left(A_0 - B_0 D_0^{-1} C_0\right)^{-1} = A_0^{-1} + A_0^{-1} B_0 \left(D_0 - C_0 A_0^{-1} B_0\right)^{-1} C_0 A_0^{-1}.$$

Now we apply this lemma with the following choices:

$$A_0 = I, \quad B_0 = C, \quad C_0 = -K, \quad D_0 = qI - A.$$

Then we get

$$\left(I + C(qI - A)^{-1}K\right)^{-1} = I - C\left(qI - A + KC\right)^{-1} K.$$

□

Chapter 9

Identification of AR processes

9.1 Least Squares estimate of an AR process

Exercise 9.1 *Prove that R^* is non-singular.*

Solution: Obviously R^* is positive semidefinite, as it is the covariance matrix of $(y_{n-1}, \ldots y_{n-p})$. The singularity of R^* would imply the singularity of the process y. □

Exercise 9.2 *Prove that R^* is non-singular by taking a state-space representation of y.*

Hint: Show that the pair $(\tilde{A}, b)$, as defined in Section 7, is controllable.

Exercise 9.3 ** Show that Condition*

$$\lim_N (-r_N) = \lim_N \frac{1}{N} \sum_{n=1}^{N} \varphi_n y_n = \mathrm{E} \varphi_n y_n = -r^* \quad \text{a.s.}$$

implies that

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \varphi_n e_n = \mathrm{E} \varphi_n e_n = 0 \quad \text{a.s.} \tag{9.1}$$

9.2 The asymptotic covariance matrix of the LSQ estimate

Exercise 9.4 *Assume that (e_n) is Gaussian. Show that $(\sum_{n=1}^{N} y_{n-1} y_{n-1})^{-1}$ has no finite expectation.*

Solution: We have
$$\sum_{n=1}^{N} y_{n-1}y_{n-1} \leq C \sum_{n=1}^{N} y'_{n-1}y'_{n-1},$$
where y'_n is an i.i.d. sequence of standard normally distributed random variables, hence the latter sum has a χ^2 distribution with N degrees of freedom. □

9.3 The recursive LSQ method

Exercise 9.5 *Assume that A and D are non-singular. Then*
$$\frac{d}{d\varepsilon}\left(A - \varepsilon BD^{-1}C\right)^{-1}\bigg|_{\varepsilon=0} = A^{-1}BD^{-1}CA^{-1}.$$

Solution: If $A = A(\varepsilon)$ is nonsingular, differentiable function of ε, then
$$\frac{d}{d\varepsilon}\left(A(\varepsilon)\right)^{-1} = \left(A(\varepsilon)\right)^{-1}\frac{d}{d\varepsilon}A(\varepsilon)\left(A(\varepsilon)\right)^{-1}.$$
Now $A(\varepsilon) = A - \varepsilon BD^{-1}C$, thus
$$\frac{d}{d\varepsilon}\left(A - \varepsilon BD^{-1}C\right)^{-1} = \left(A - \varepsilon BD^{-1}C\right)\left(A - \varepsilon BD^{-1}C\right)\left(A - \varepsilon BD^{-1}C\right).$$
Substituting $\varepsilon = 0$ we get the proposition of the exercise. □

Exercise 9.6 *Prove the Sherman-Morrison lemma.*

Solution: This is a special case of the matrix inversion lemma. We consider the following block-matrix:
$$F = \begin{pmatrix} A & b \\ c^T & -1 \end{pmatrix}.$$
Then we have
$$\left(A - b(-1)c^T\right)^{-1} = A^{-1} + A^{-1}b\left((-1) - c^TA^{-1}b\right)^{-1}c^TA^{-1} =$$
$$= A^{-1} + A^{-1}bc^TA^{-1}\frac{-1}{1 + +c^TA^{-1}b}.$$
□

Chapter 10

Identification of MA and ARMA models

10.1 Identification of MA models

Exercise 10.1 *Prove that the stability of $C(z^{-1})$, implying the stability of $\widetilde{C}$, yields that*

$$\mathrm{E}\left|x_n^*(\theta) - x_n(\theta)\right|^2 = O(\gamma^n)$$

with any γ such that $\gamma > \varrho(\widetilde{C})$, with $\varrho(\widetilde{C})$ denoting the spectral radius of $\widetilde{C}$ (known to be less then 1).

Solution: Use the state-space representations for both processes, and take the difference of the two state-space equations. Exploit linearity to see the effect of the differences in initial values.

Exercise 10.2 *Provide an expression of the coefficients c_l in terms of the roots, say γ_l, and express $\frac{\partial}{\partial \gamma}$ via $\frac{\partial}{\partial \theta}$.*

Solution: The connection can be expressed via the following equation:

$$C(z^{-1}) = 1 + \sum_{l=1}^{r} c_l z^{-l} = \prod_{l=1}^{r} (1 - z^{-1}\gamma_l).$$

The derivative w.r.t. γ_l is:

$$\frac{\partial}{\partial \gamma_l} C(z^{-1}) = z^{-1} \prod_{j \neq l} (1 - z^{-1}\gamma_j) = \frac{z^{-1}}{1 - z^{-1}\gamma_l} C(z^{-1}).$$

10.2 The asymptotic covariance matrix of $\widehat{\theta}_N$

Exercise 10.3 *Show that the first term on the r.h.s. of the equality*

$$\lim_N \frac{1}{N} V_{\theta\theta N}(\theta^*) = \mathrm{E}\bigg(\varepsilon^*_{\theta\theta n}(\theta^*)e_n + \varepsilon^*_{\theta n}(\theta^*)\varepsilon^{*T}_{\theta n}(\theta^*)\bigg).$$

is zero.

Solution: As we have seen before $\varepsilon^*_{\theta\theta n}(\theta)$ is in the subspace spanned by $\{\varepsilon_{n-k}(\theta) \ : \ k = 1, 2, \ldots.\}$. For $\theta = \theta^*$ this set is H^e_{n-1}, and $e_n \perp H^y_{n-1}$. $\square$

10.3 Identification of ARMA models

Exercise 10.4 *Show that if $A^*(z^{-1})$ and $C^*(z^{-1})$ have no common factor then R^* is non-singular.*

Solution: Write $R^* = \mathrm{E}\varepsilon^{*T}_{\theta n}(\theta^*)\varepsilon^*_{\theta n}(\theta^*)$, and assume that R^* is singular. Then there exists a pair of vectors, say (u^T, v^T) such that

$$(u^T, v^T)\varepsilon^{*T}_{\theta n}(\theta^*) = 0$$

a.s. Defining the polynomials $U(q^{-1})$ and $V(q^{-1})$ with their coefficients being equal to the components of u and v, resp., we arrive to the equality

$$\frac{U}{A^*}e - \frac{V}{C^*}e = 0$$

a.s. Rearranging this we get

$$UC^* - VA^* = 0,$$

which contradicts to the assumption that $A^*(z^{-1})$ and $C^*(z^{-1})$ have no common factor.

Exercise 10.5 *Show that if $A^*(z^{-1})$ and $C^*(z^{-1})$ have a common factor then R^* is singular.*

Solution: Apply the previous arguments in reverse order.

Exercise 10.6 *Compute the gradient process for the following models: $MA(1)$, $AR(1)$, $ARMA(1,1)$.*

Solution:

MA(1) : The process is

$$y_n = e_n + c^* e_{n-1}.$$

The inverse process for $|c| < 1$ is:

$$\varepsilon_n + c\varepsilon_{n-1} = y_n.$$

The gradient process can be written as

$$\varepsilon_{c,n} + c\varepsilon_{c,n-1} + \varepsilon_{n-1} = 0, \quad \Longrightarrow \quad \varepsilon_{c,n} + c\varepsilon_{c,n-1} = -\varepsilon_{n-1}.$$

AR(1) : The process is

$$y_n + a^* y_{n-1} = e_n.$$

The inverse process for $|a| < 1$ is:

$$\varepsilon_n = y_n + a y_{n-1}.$$

The gradient process can be written as

$$\varepsilon_{a,n} = y_{n-1}.$$

ARMA(1,1) : The process is

$$y_n + a^* y_{n-1} = e_n + c^* e_{n-1}.$$

The inverse process for $|a| < 1$ and $|c| < 1$ is:

$$\varepsilon_n + c\varepsilon_{n-1} = y_n + a y_{n-1}.$$

The gradient process can be written as

$$\varepsilon_{c,n} + c\varepsilon_{c,n-1} = -\varepsilon_{n-1},$$

$$\varepsilon_{a,n} + c\varepsilon_{a,n-1} = y_{n-1}.$$

$\square$

Exercise 10.7 *Show that for $\theta = \theta^*$ we have*

$$\varepsilon_\theta(\theta)_{|\theta=\theta^*} = (\frac{1}{A^*}\left[q^{-1}\ldots q^{-p}\right] e, -\frac{1}{C^*}\left[q^{-1}\ldots q^{-r}\right] e)^T.$$

Solution: In **Lemma 10.10.** it was proved, that

$$C(q^{-1})\ \varepsilon_\theta(\theta) = -\phi(\theta).$$

Substituting $\theta = \theta^*$ we get

$$C^*(q^{-1})\ \varepsilon_\theta(\theta^*) = -\phi(\theta^*) = -(-y_{n-1}, ..., -y_{n-p}, e_{n-1}, ..., e_{n-r})^T. \tag{10.1}$$

Here we used $\varepsilon_n(\theta^*) = e_n$. The first part of the vector can be written as

$$(y_{n-1}, ..., y_{n-p})^T = (\left[q^{-1}\ldots q^{-p}\right] y)^T = (\left[q^{-1}\ldots q^{-p}\right] \frac{C^*}{A^*}e)^T.$$

The second part of the vector is simply:

$$-(e_{n-1}, ..., e_{n-r})^T = -(\left[q^{-1}\ldots q^{-r}\right] e)^T$$

Then dividing equation (10.1) by $C^*(q^{-1})$ we get the statement of the exercise. □

Chapter 12

Stochastic volatility: ARCH and GARCH models

12.2 Stochastic volatility models

Exercise 12.1 *Show that that the triplet $(\varepsilon_n, y_n, \sigma_n)$ itself is jointly strictly stationary.*

Solution: First we show that (y_n, σ_n) is jointly strictly stationary. Once we know $y_n, y_{n-1}, y_{n-2}, \ldots$, any σ_k is a deterministic function of them for $k \leq n+1$. Moreover, the equation $\varepsilon_n = y_n/\sigma_n$ allows a deterministic computation of the error process if we know the other two. The process (y_n) is strictly stationary, thus the joint distribution of the history $y_n, y_{n-1}, y_{n-2}, \ldots$ does not change under time shift. $(\varepsilon_n, y_n, \sigma_n)$ is obtained by a deterministic function from (y_n), consequently their joint distributions are also insensitive to time shifts. $\square$

Exercise 12.2 *Show that under the conditions above*

$$\mathrm{E}[y_n|\mathcal{F}_{n-1}^y] = 0 \quad \text{a.s.}$$

In other words (y_n) is a martingale difference process.

Solution: Write

$$\mathrm{E}[y_n|\mathcal{F}_{n-1}^y] = \mathrm{E}[\sigma_n\varepsilon_n|\mathcal{F}_{n-1}^y] = \sigma_n\mathrm{E}[\varepsilon_n|\mathcal{F}_{n-1}^y] = 0.$$

The first equality is just the definition of y_n, the second uses the fact that σ_n is a function of the history of (y_n). The is the result of (ε_n) being i.i.d. $\square$

Exercise 12.3 *Show that under the conditions above*

$$\mathrm{E}[y_n^2|\mathcal{F}_{n-1}^y] = \sigma_n^2 \quad \text{a.s.}$$

Solution: Following the lines of the previous solution we get

$$\mathrm{E}[y_n^2|\mathcal{F}_{n-1}^y] = \mathrm{E}[\sigma_n^2\varepsilon_n^2|\mathcal{F}_{n-1}^y] = \sigma_n^2\mathrm{E}[\varepsilon_n^2|\mathcal{F}_{n-1}^y] = \sigma_n^2.$$

□

Exercise 12.4 *Show that under the conditions above* (y_n) *is a w.s.st. orthogonal process.*

Solution: For $n > m$ we have

$$\mathrm{E}[\sigma_n\varepsilon_n\sigma_m\varepsilon_m|\mathcal{F}_{n-1}^y] = \mathrm{E}[\sigma_n\varepsilon_n\sigma_m\varepsilon_m|\mathcal{F}_{n-1}^\varepsilon] = \sigma_n\sigma_m\varepsilon_m\mathrm{E}[\varepsilon_n|\mathcal{F}_{n-1}^\varepsilon] = 0.)$$

□

12.4 State space representation

Exercise 12.5 *Show that* X_n *defined by the right hand side of*

$$X_n^* = u_n + \sum_{k=1}^{\infty} A_n A_{n-1} \dots A_{n-k+1} u_{n-k}$$

satisfies the state equation

$$X_{n+1}^* = A_{n+1}^* X_n^* + u_{n+1}, \quad n \in \mathbb{Z}$$

corresponding to the GARCH process.

Solution:

$$\begin{aligned} A_{n+1}X_n^* &= A_{n+1}u_n + A_{n+1}\sum_{k=1}^{\infty} A_n A_{n-1} \dots A_{n-k+1} u_{n-k} \\ &= \sum_{k=0}^{\infty} A_{n+1}A_n \dots A_{n-k+1} u_{n-k} = X_{n+1}^* - u_{n+1} \end{aligned}$$

□

Exercise 12.6 *Prove uniqueness as stated in the theorem.*

Hint: Let (X_{1n}) and (X_{2n}) be two solutions. Consider the dynamics for the difference process $(X_{1n} - X_{2n})$.

12.5 Existence of a strictly stationary solution

Exercise 12.7 *Show that (u_n) is sub-additive, i.e. we have for any $n, m > 0$ the inequality $u_{n+m} \leq u_n + u_m$.*

Solution: We have

$$u_{n+m} = \mathrm{E} \log \|A_{n+m} \ldots A_{n+1} A_n \ldots A_1\|.$$

The norm is sub/multiplicative, i.e.

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|.$$

Thus

$$\begin{aligned} \log \|A_{n+m} \ldots A_{n+1} A_n \ldots A_1\| &\leq \log(\|A_{n+m} \ldots A_{n+1}\| \cdot \|A_n \ldots A_1\|) \\ &= \log \|A_{n+m} \ldots A_{n+1}\| + \log \|A_n \ldots A_1\|. \end{aligned}$$

Taking expectation we get the statement of the Exercise. □

Lemma 4 *Let (u_n) be a sub-additive sequence. Then the limit*

$$\lambda = \lim_{n \to \infty} \frac{1}{n} u_n$$

exists, where $-\infty \leq \lambda < +\infty$, moreover

$$\lambda = \inf_{n \to \infty} \frac{1}{n} u_n.$$

Exercise 12.8 *Prove the above lemma.*

Solution: Let $\lambda = \inf_{n \to \infty} \frac{1}{n} u_n$. Take $\varepsilon > 0$ and choose a k such that $u_k / k < \lambda + \varepsilon$. Then for any integer n we have

$$u_{nk} \leq n u_k, \qquad \Longrightarrow \qquad \frac{u_{nk}}{nk} < \lambda + \varepsilon.$$

Moreover, for any $0 \leq r < k$ and n large enough we get

$$\frac{1}{nk + r} u_{nk+r} \leq \frac{1}{nk + r}(u_{nk} + u_r) \leq \frac{1}{nk + r}(u_{nk} + \max_{0 \leq s < k} u_s) < \lambda + 2\varepsilon.$$

Consequently $\limsup_{n \to \infty} \frac{1}{n} u_n < \lambda + 2\varepsilon$. □

Exercise 12.9 *Prove that for any $\varepsilon > 0$ there exist finite r.v.-s $C_n(\omega, \varepsilon)$ such that for any n we have*

$$||A_n A_{n-1} \dots A_{n-k+1}|| \leq C_n(\omega, \varepsilon) e^{(\lambda(\mathcal{A})+\varepsilon)(n-k)}.$$

Show that $C_n(\omega, \varepsilon)$ can be assumed to be a stationary sequence.

Solution: After taking the logarithm we get the following constraint for $C_n(\omega, \varepsilon)$:

$$\log ||A_n A_{n-1} \dots A_{n-k+1}|| - (\lambda(\mathcal{A}) + \varepsilon)(n-k) \leq \log C_n(\omega, \varepsilon).$$

By the definition of the top-Lyapunov exponent the left hand side tends to $-\infty$ as $k \to \infty$. Therefore

$$\sup_k \left(\log ||A_n A_{n-1} \dots A_{n-k+1}|| - (\lambda(\mathcal{A}) + \varepsilon)(n-k)\right) < \infty.$$

Choose this value as $C_n(\omega, \varepsilon)$. $\square$

Exercise 12.10 *Prove that for any sequence of r.v.-s (u_n) the condition*

$$\sup_n \mathrm{E} \log^+ |u_n| < +\infty$$

implies that for any $\varepsilon > 0$ there exists a r.v. $C(\omega, \varepsilon)$ such that

$$|u_n| \leq C(\omega, \varepsilon) e^{\varepsilon n}.$$

Hint: Show that for any fix $C > 0$ the probability that the events $\{|u_n| > Ce^{\varepsilon n}\}$ occur infinitely often is 0. For this use the Borel-Cantelli lemma.

Chapter 13

High-frequency data. Poisson processes

13.3 Poisson point processes on a general state space

Exercise 13.1 *Show that μ is non-atomic, i.e. for all $x \in S$ we have $\mu(\{x\}) = 0$.*

Solution: Clearly $N(\{x\}) = |\Pi \cap \{x\}|$ can only be 0 or 1. On the other hand, if we had $\mu(\{x\}) = \varepsilon > 0$, then $N(\{x\})$ would follow a Poisson distribution with a positive parameter. But this can take values larger than one with positive probability, which is a contradiction. $\square$

Exercise 13.2 *Show that for any $A_1, A_2 \in \mathcal{G}$ with $\mu(A_i) < \infty$ we have*

$$\operatorname{cov}(N(A_1), N(A_2)) = \mu(A_1 \cap A_2).$$

Solution: Let $B = (A_1 \cap A_2)$. We may write the covariance as follows:

$$\operatorname{cov}(N(A_1), N(A_2)) = \operatorname{cov}(N(A_1 \setminus B) + N(B), N(A_2 \setminus B) + N(B)).$$

Using the bi-linearity of the covariance we can expand the r.h.s., and observe that all but one term vanishes due to independence. Thus

$$\operatorname{cov}(N(A_1), N(A_2)) = \operatorname{cov}(N(B), N(B)).$$

But the variable $N(B)$ follows a Poisson distribution with parameter $\mu(B)$, consequently its variance is indeed $\mu(B)$. $\square$

Theorem 2 *Let* Π *be a Poisson process with mean measure* μ *on* S *and let* $S_1 \subset S$ *be measurable. Then*

$$\Pi_1 = \Pi \cap S_1$$

is a Poisson process on S *with mean measure*

$$\mu_1(A) = \mu(A \cap S_1).$$

Exercise 13.3 *Prove the above theorem.*

Solution: We verify the requirements of the definition. For Π, (i) holds for any $A_1 \cap S_1, A_2 \cap S_1, \ldots, A_n \cap S_1 \subset S$ provided that A_i are disjoint. But

$$N(A_i \cap S_1) = N_1(A_i),$$

thus (i) holds for Π_1. Similarly

$$N_1(A) = N(A \cap S_1) \stackrel{\mathcal{L}}{=} P(\mu(A \cap S_1)) = P(\mu_1(A)).$$

$\square$

Theorem 3 *Let* Π *be a Poisson process in* $\mathbb{R}^D$ *with rate function* $\lambda(x_1, ..., x_D)$. *Let* $d < D$ *and let*

$$\Pi_d$$

be the projection of Π *on the first* d *coordinates. Then* Π_d *is a Poisson-process with rate function*

$$\lambda^*(x_1, ..., x_d) = \int ... \int \lambda(x_1, ..., x_D) dx_{d+1} ... dx_D.$$

Exercise 13.4 *Prove the above projection theorem.*

Solution: Apply the mapping theorem with

$$f(x_1, ..., x_D) = (x_1, ..., x_d).$$

$\square$

13.4 Construction of Poisson processes

Proposition 8 *Let $X_1, ..., X_n$ be i.i.d. on S according to the probability measure η. Assume that η has no atom. Then*

$$\Pi = \{X_1, .., X_n\}$$

is a Bernoulli process.

Exercise 13.5 *Prove the above proposition.*

Solution: Verify

$$P(N(A_0) = n_0, ..., N(A_k) = n_k) = \frac{n!}{n_0!...n_k!}\eta(A_0)^{n_0}...\eta(A_k)^{n_k}$$

for the counts

$$N(A) = \#\{r : X_r \in A\}.$$

Use the fact that all X_i are different almost surely.

Exercise 13.6 *Work out the details of the proof of the existence of a Poisson process.*

Solution: First assume that $\mu(X) = \lambda < \infty$. Define $\eta(A) = \mu(A)/\mu(X)$. Choose the number of points in X, say N according to $P(\lambda)$. With N being fixed, construct the realization of a Bernoulli process. Then the resulting process will be a Poisson point process with intensity measure μ. The case $\mu(X) = \infty$ is handled by first writing

$$X = \cup_{i=1}^{\infty} X_i$$

so that $\mu(X_i) = \lambda_i < \infty$, (assuming that μ is σ-finite.)

□

13.5 Sums and integrals over Poisson processes

Exercise 13.7 *Prove the two identities:*

$$\mathrm{E}(\Sigma) = \int_S f(x)\mu(dx).$$

and, if

$$\Sigma_1 = \sum_{x \in \Pi} f_1(x), \qquad \Sigma_2 = \sum_{x \in \Pi} f_2(x),$$

then

$$\mathrm{Cov}(\Sigma_1, \Sigma_2) = \int_S f_1(x) f_2(x) \mu(dx).$$

Solution: Both identities are true for step-functions. From there the final result follows by a simple limiting argument. □

Exercise 13.8 *Provide a formal proof for the above expression of* $\mathrm{var}(\Sigma)$ *by differentiating the master equation*

$$\mathrm{E}(e^{it\Sigma}) = \exp\{\int_S (e^{itf(x)} - 1)\mu(dx)\}$$

w.r.t. t *once and twice, and setting* $t = 0$*.*

Solution: Formal differentiation of the l.h.s. gives

$$\frac{d}{dt}\mathrm{E}(e^{it\Sigma}) = \mathrm{E}i\Sigma(e^{it\Sigma}) = \mathrm{E}i\Sigma,$$

when setting $t = 0$. Similarly, formal differentiation of the r.h.s. gives

$$\exp\{\int_S (e^{itf(x)} - 1)\mu(dx)\} \cdot \int_S if(x)e^{itf(x)}\mu(dx) = \int_S if(x)\mu(dx)$$

when setting $t = 0$, which proves the first claim. We proceed with the second derivatives similarly.

□

Exercise 13.9 *Provide a formal proof for the above expression of* $\mathrm{cov}(\Sigma_1, \Sigma_2)$ *by considering the master equation*

$$\mathrm{E}(e^{it_1\Sigma_1 + it_2\Sigma_2}) = \exp\{\int_S (e^{it_1 f_1(x) + it_2 f_2(x)} - 1)\mu(dx)\}.$$

and taking mixed second order partial derivatives

$$\frac{\partial^2}{\partial t_1 \partial t_2},$$

and setting $t_1 = t_2 = 0$*.*

Solution: Follow the proof of the previous exercise. □

Chapter 14

High-frequency data. Lévy Processes

14.1 Motivation and basic properties

Exercise 14.1 *Show that the characteristic function of L_t can be written in the form*

$$\mathrm{E}e^{iuL_t} = e^{t\psi(u)}.$$

Solution: Write $\mathrm{E}e^{iuL_t} = e^{\rho_t(u)}$. Since the increments are i.i.d. we have

$$\rho_t(u) + \rho_{\Delta t}(u) = \rho_{t+\Delta t}(u).$$

Since $\rho_t(u)$ is continuous in t, it follows that $\rho_t(u) = t\psi(u)$ with some $\psi(u)$, as stated.

Theorem 4 *Let (L_t) be a pure-jump Lévy process (having no Brownian motion component), with finite variation trajectories, defined by*

$$L_t = \int_0^t \int_{\mathbf{R}^1} xN(ds, dx). \tag{14.1}$$

Then

$$\mathrm{E}\left[e^{iuL_t}\right] = \exp\left[t\left(ibu + \int_{\mathbf{R}^1} \left(e^{iux} - 1\right)\nu(dx)\right)\right].$$

Exercise 14.2 *Provide a derivation of the above simplified version of the Lévy-Khintchine formula for processes with finite variation trajectories, using Campbell's theorem.*

Solution: For pure-jump Lévy processes with finite variation trajectories given by (14.1) the integrability condition

$$\int_{\mathbf{R}^1} \min(|x|, 1)\nu(dx) < \infty,$$

implies the integrability condition present in Campbell's theorem with $f(t', x) = x$ for $0 \leq t' \leq t$. Then we will have $\Sigma_f = L_t$. The expression given for $\mathrm{E}\left[e^{iuL_t}\right]$ given by Campbell's theorem is then just the simplified form of the Lévy-Khintchine formula:

$\square$