(ix) The *Navier–Stokes* equations for the viscous flow of an incompressible liquid connect the velocity components u_k and the pressure p:

$$\frac{\partial u_i}{\partial t} + \sum_k \frac{\partial u_i}{\partial x_k} u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \gamma \Delta u_i, \qquad (2.16a)$$

$$\sum_k \frac{\partial u_k}{\partial x_k} = 0, \qquad (2.16b)$$

where ρ is the constant density and γ the kinematic viscosity.

(x) An example of a third-order nonlinear equation for a function $u(x,t)$ is furnished by the *Korteweg–de Vries* equation

$$u_t + cuu_x + u_{xxx} = 0 \qquad (2.17)$$

first encountered in the study of water waves.

In general we shall try to describe the *manifold of solutions* of a P.D.E. The results differ widely for different classes of equations. Meaningful "well-posed" problems associated with a P.D.E. often are suggested by particular physical interpretations and applications.

3. Analytic Solution and Approximation Methods in a Simple Example*

We illustrate some of the notions that will play an important role in what follows by considering one of the simplest of all equations

$$u_t + cu_x = 0 \qquad (3.1)$$

for a function $u = u(x,t)$, where $c = \text{const.} > 0$. Along a line of the family

$$x - ct = \text{const.} = \xi \qquad (3.2)$$

("characteristic line" in the xt-plane) we have for a solution u of (3.1)

$$\frac{du}{dt} = \frac{d}{dt}u(ct+\xi, t) = cu_x + u_t = 0.$$

Hence u is constant along such a line, and depends only on the parameter ξ which distinguishes different lines. The general solution of (3.1) then has the form

$$u(x,t) = f(\xi) = f(x - ct). \qquad (3.3)$$

Formula (3.3) represents the general solution u uniquely in terms of its *initial values*

$$u(x,0) = f(x). \qquad (3.4)$$

Conversely every u of the form (3.3) is a solution of (3.1) with initial values f provided f is of class $C^1(\mathbb{R})$. We notice that the value of u at any point

*([18], [20], [25], [29])

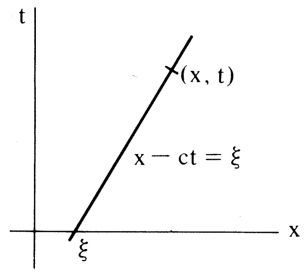

Figure 1.1

(x,t) depends only on the initial value f at the single argument $\xi = x - ct$, the abscissa of the point of intersection of the characteristic line through (x,t) with the initial line, the x-axis. The *domain of dependence* of $u(x,t)$ on the initial values is represented by the single point ξ. The *influence* of the initial values at a particular point ξ on the solution $u(x,t)$ is felt just in the points of the characteristic line (3.2). (Fig. 1.1)

If for each fixed t the function u is represented by its graph in the xu-plane, we find that the graph at the time $t = T$ is obtained by translating the graph at the time $t = 0$ parallel to the x-axis by the amount cT:

$$u(x,0) = u(x + cT, T) = f(x).$$

The graph of the solution represents a *wave* propagating to the right with velocity c without changing shape. (Fig. 1.2)

We use this example with its explicit solution to bring out some of the notions connected with the numerical solution of a P.D.E by the *method of finite differences*. One covers the xt-plane by a rectangular grid with mesh size h in the x-direction and k in the t-direction. In other words one considers only points (x,t) for which x is a multiple of h and t a multiple of k. It would seem natural for purposes of numerical approximation to replace the P.D.E. (3.1) by the difference equation

$$\frac{v(x,t+k) - v(x,t)}{k} + c\frac{v(x+h,t) - v(x,t)}{h} = 0. \qquad (3.5)$$

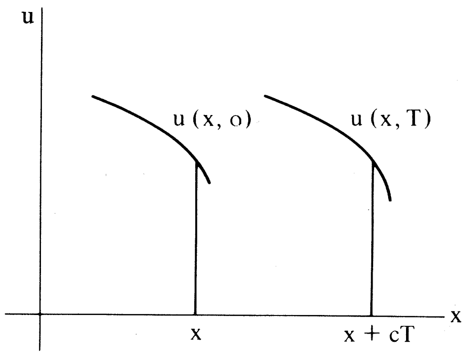

Figure 1.2

Formally this equation goes over into $v_t + cv_x = 0$ as $h,k \to 0$. We ask to what extent a solution v of (3.5) in the grid points with initial values

$$v(x,0) = f(x) \tag{3.6}$$

approximates for small h,k the solution of the initial-value problem (3.1), (3.4).

Setting $\lambda = k/h$, we write (3.5) as a recursion formula

$$v(x,t+k) = (1+\lambda c)v(x,t) - \lambda c v(x+h,t) \tag{3.7}$$

expressing v at the time $t+k$ in terms of v at the time t. Introducing the *shift operator* E defined by

$$Ef(x) = f(x+h), \tag{3.8}$$

(3.7) becomes

$$v(x,t+k) = ((1+\lambda c) - \lambda c E)v(x,t) \tag{3.8a}$$

for $t = nk$ this immediately leads by iteration to the solution of the initial-value problem for (3.5):

$$\begin{aligned} v(x,t) = v(x,nk) &= ((1+\lambda c) - \lambda c E)^n v(x,0) \\ &= \sum_{m=0}^{n} \binom{n}{m} (1+\lambda c)^m (-\lambda c E)^{n-m} f(x) \\ &= \sum_{m=0}^{n} \binom{n}{m} (1+\lambda c)^m (-\lambda c)^{n-m} f(x+(n-m)h). \end{aligned} \tag{3.9}$$

Clearly the domain of dependence for $v(x,t) = v(x,nk)$ consists of the set of points

$$x,\ x+h,\ x+2h,\ \ldots,\ x+nh \tag{3.10}$$

on the x-axis, all of which lie between x and $x+nh$. The domain of the differential equation solution consists of the point $\xi = x - ct = x - c\lambda nh$, which lies completely outside the interval $(x, x+nh)$. It is clear that v for $h,k \to 0$ cannot be expected to converge to the correct solution u of the differential equation, since in forming $v(x,t)$ we do not make use of any information on the value of $f(\xi)$, which is vital for determining $u(x,t)$, but only of more and more information on f in the interval $(x, x+(t/\lambda))$ which is irrelevant. The difference scheme fails the *Courant–Friedrichs–Lewy* test, which requires that the limit of the domain of dependence for the difference equation contains the domain of dependence for the differential equation.

That the scheme (3.5) is inappropriate also is indicated by its high degree of *instability*. In applied problems the data f are never known with perfect accuracy. Moreover, in numerical computations we cannot easily use the exact values but commit small round-off errors at every step. Now it is clear from (3.9) that errors in f of absolute value ε with the proper (alternating) sign can lead to a resulting error in $v(x,t) = v(x,nk)$ of size

$$\varepsilon \sum_{m=0}^{n} \binom{n}{m} (1+\lambda c)^m (\lambda c)^{n-m} = (1+2\lambda c)^n \varepsilon. \tag{3.11}$$

Thus for a fixed mesh ratio λ the possible resulting error in v grows exponentially with the number n of steps in the t-direction.

A more appropriate difference scheme uses "backward" difference quotients:

$$\frac{v(x,t+k) - v(x,t)}{k} + c\frac{v(x,t) - v(x-h,t)}{h} = 0 \tag{3.12}$$

or symbolically

$$v(x,t+k) = ((1-\lambda c) + \lambda c E^{-1})v(x,t). \tag{3.13}$$

The solution of the initial-value problem for (3.13) becomes

$$v(x,t) = v(x,nk) = \sum_{m=0}^{n} \binom{n}{m} (1-\lambda c)^m (\lambda c)^{n-m} f(x-(n-m)h). \tag{3.14}$$

In this scheme the domain of dependence for $v(x,t)$ on f consists of the points

$$x,\ x-h,\ x-2h,\ \ldots,\ x-nh = x - \frac{t}{\lambda} \tag{3.15}$$

Letting $h,k \to 0$ in such a way that the mesh ratio λ is held fixed, the set (3.15) has as its limit points the interval $[x-(t/\lambda), x]$ on the x-axis. The Courant–Friedrichs–Lewy test is satisfied, when this interval contains the point $\xi = x - ct$, that is when the mesh ratio λ satisfies

$$\lambda c \leqslant 1. \tag{3.16}$$

Stability of the scheme under the condition (3.16) is indicated by the fact that by (3.14) a maximum error of size ε in the initial function f results in a maximum possible error in the value of $v(x,t) = v(x,nk)$ of size

$$\varepsilon \sum_{m=0}^{n} \binom{n}{m} (1-\lambda c)^m (\lambda c)^{n-m} = \varepsilon((1-\lambda c) + \lambda c)^n = \varepsilon. \tag{3.17}$$

We can prove that the v represented by (3.14) actually converges to $u(x,t) = f(x-ct)$ for $h,k \to 0$ with $k/h = \lambda$ fixed, provided the stability criterion (3.16) holds and f has uniformly bounded second derivatives. For that purpose we observe that $u(x,t)$ satisfies

$$\begin{aligned} &|u(x,t+k) - (1-\lambda c)u(x,t) - \lambda c u(x-h,t)| \\ &= |f(x-ct-ck) - (1-\lambda c)f(x-ct) - \lambda c f(x-ct-h)| \leqslant Kh^2, \end{aligned} \tag{3.18}$$

where

$$K = \tfrac{1}{2}(c^2\lambda^2 + \lambda c)\sup|f''|, \tag{3.19}$$

as is seen by expanding f about the point $x-ct$. Thus, setting $w=u-v$ we have

$$|w(x,t+k)-(1-\lambda c)w(x,t)-\lambda c w(x-h,t)| \leqslant Kh^2$$

and hence

$$\begin{aligned}\sup_x |w(x,t+k)| &\leqslant (1-\lambda c)\sup_x |w(x,t)| + \lambda c \sup_x |w(x-h,t)| + Kh^2 \\ &= \sup_x |w(x,t)| + Kh^2. \end{aligned} \tag{3.20}$$

Applying (3.20) repeatedly it follows for $t=nk$ that

$$\begin{aligned}|u(x,t)-v(x,t)| &\leqslant \sup_x |w(x,nk)| \\ &\leqslant \sup_x |w(x,0)| + nKh^2 = \frac{Kth}{\lambda},\end{aligned}$$

since $w(x,0)=0$. Consequently $w(x,t)\to 0$ as $h\to 0$, that is, the solution v of the difference scheme (3.12) converges to the solution u of the differential equation.

Problems

1. Show that the solution v of (3.12) with initial data f converges to u for $h\to 0$ and a fixed $\lambda \leqslant 1/c$, under the sole assumption that f is continuous. (Hint: Use the fact that both u and v change by at most ε when we change f by at most ε.)

2. To take into account possible round-off errors we assume that instead of (3.13) v satisfies an inequality

$$|v(x,t+k)-(1-\lambda c)v(x,t)-\lambda c v(x-h,t)| < \delta.$$

Show that for a prescribed δ and for K given by (3.19) we have the estimate

$$|u(x,t)-v(x,t)| \leqslant \frac{Kth}{\lambda} + \frac{t}{\lambda h}\delta \tag{3.21}$$

assuming that (3.16) holds and that $v(x,0)=f(x)$. Find values for λ and h based on this formula that will guarantee the smallest maximum error in computing $u(x,t)$.

3. Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation. In particular let $f(x)=e^{\alpha x}$ with a complex constant α. Show that for fixed x, t and any fixed positive $\lambda = k/h$ whatsoever both the expressions (3.9) and (3.14) converge for $n\to\infty$ towards the correct limit $e^{\alpha(x-ct)}$. (This is consistent with the Courant–Friedrichs–Lewy test, since for an *analytic* f the values of f in any interval determine those at the point ξ uniquely.)

4. Quasi-linear Equations

The general first-order equation for a function $u=u(x,y,\ldots,z)$ has the form

$$f(x,y,\ldots,u,u_x,u_y,\ldots,u_z)=0. \tag{4.1}$$

Equations of this type occur naturally in the calculus of variations, in particle mechanics, and in geometrical optics. The main result is the fact that the general solution of an equation of type (4.1) can be obtained by solving systems of Ordinary Differential Equations (O.D.E.s for short). This is not true for higher-order equations or for systems of first-order equations. In what follows we shall mostly limit ourselves to the case of two independent variables x,y. The theory can be extended to more independent variables without any essential change.

We first consider the somewhat simpler case of a quasi-linear equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{4.2}$$

We represent the function $u(x,y)$ by a surface $z=u(x,y)$ in xyz-space. Surfaces corresponding to solutions of a P.D.E. are called *integral surfaces* of the P.D.E. The prescribed functions $a(x,y,z), b(x,y,z), c(x,y,z)$ define a field of vectors in xyz-space (or in a portion Ω of that space). Obviously only the direction of the vector, the *characteristic direction*, matters for the P.D.E. (4.2). Since $(u_x,u_y,-1)$ constitute direction numbers of the normal of the surface $z=u(x,y)$, we see that (4.2) is just the condition that the normal of an integral surface at any point is perpendicular to the direction of the vector (a,b,c) corresponding to that point. Thus integral surfaces are surfaces that at each point are tangent to the characteristic direction.

With the field of characteristic directions with direction numbers (a,b,c) we associate the family of *characteristic curves* which at each point are tangent to that direction field. Along a characteristic curve the relation

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)} \tag{4.3}$$

holds. Referring the curve to a suitable parameter t (or denoting the common ratio in (4.3) by dt) we can write the condition defining characteristic curves in the more familiar form of a system of ordinary differential equations

$$\frac{dx}{dt} = a(x,y,z), \qquad \frac{dy}{dt} = b(x,y,z), \qquad \frac{dz}{dt} = c(x,y,z). \tag{4.4}$$

The system is "autonomous" (the independent variable t does not appear explicitly). The choice of the parameter t in (4.4) is artificial. Using any other parameter along the curve amounts to replacing a,b,c by proportional quantities, which does not change the characteristic curve in xyz-space or the P.D.E. (4.2). Assuming that a,b,c are of class C^1 in a region

Ω, we know from the theory of O.D.E.s that through each point of Ω there passes exactly one characteristic curve. There is a 2-parameter family of characteristic curves in xyz-space (but a 3-parameter family of solutions $(x(t),y(t),z(t))$ of (4.4), since replacing the independent variable t by $t+c$ with a constant c changes the solution (x,y,z), but not the characteristic curve, which is its range).

If a surface S: $z=u(x,y)$ is a union of characteristic curves, then S is an integral surface. For then through any point P of S there passes a characteristic curve γ contained in S. The tangent to γ at P necessarily lies in the tangent plane of S at P. Since the tangent to γ has the characteristic direction, the normal to S at P is perpendicular to the characteristic direction, which makes S an integral surface. Conversely we can show that every integral surface S is the union of characteristic curves, or that through every point of S there passes a characteristic curve contained in S. This is a consequence of the following theorem:

Theorem. *Let the point $P=(x_0,y_0,z_0)$ lie on the integral surface S: $z=u(x,y)$. Let γ be the characteristic curve through P. Then γ lies completely on S.*

PROOF. Let γ given by $(x(t),y(t),z(t))$ be the solution of (4.4) for which $(x,y,z)=(x_0,y_0,z_0)$ for $t=t_0$. From γ and S we form the expression

$$U=z(t)-u(x(t),y(t))=U(t). \tag{4.5}$$

Obviously $U(t_0)=0$ since P lies on S. By (4.4)

$$\begin{aligned}\frac{dU}{dt}&=\frac{dz}{dt}-u_x(x(t),y(t))\frac{dx}{dt}-u_y(x(t),y(t))\frac{dy}{dt}\\ &=c(x,y,z)-u_x(x,y)a(x,y,z)-u_y(x,y,z)b(x,y,z).\end{aligned}$$

This can be written as the ordinary differential equation

$$\begin{aligned}\frac{dU}{dt}=c(x,y,U+u(x,y))-u_x(x,y)a(x,y,U+u(x,y))\\ -u_y(x,y)b(x,y,U+u(x,y))\end{aligned} \tag{4.6}$$

for U, where for x,y we have to substitute the functions $x(t),y(t)$ from the description of γ. Now $U\equiv 0$ is a particular solution of (4.6), since $u(x,y)$ satisfies (4.2). By the uniqueness theorem for O.D.E.s, this is the only solution vanishing for $t=t_0$. It follows that the function $U(t)$ defined by (4.5) vanishes identically. But that just means that the whole curve γ lies on S. □

As a consequence of the theorem two integral surfaces that have a point P in common intersect along the whole characteristic curve γ through P. Conversely if the integral surfaces S_1 and S_2 intersect, without touching, along a curve γ, then γ is characteristic. For consider the tangent planes π_1,π_2 to S_1,S_2 at a point P of γ. Each of the planes has to contain the characteristic direction (a,b,c) at P. Since $\pi_1\neq\pi_2$ it follows that the intersection of π_1 and π_2 has the direction (a,b,c). Since the tangent T to γ at P also has to belong to both π_1 and π_2, it follows that T has the direction (a,b,c), and hence that γ is characteristic.

5. The Cauchy Problem for the Quasi-linear Equation

We now have a simple description for the *general solution* u of (4.2): The integral surface $z=u(x,y)$ is the union of characteristic curves. To get a better insight into the structure of the manifold of solutions it is desirable to have a definite method of generating solutions in terms of a prescribed set F of functions, called "data." Ideally we have a mapping $F\to u$ of data F onto solutions u of the P.D.E. The space of solutions is then described by the usually simpler space of data. A good deal of the theory of P.D.E.s is concerned with the "problem" of actually finding the u belonging to a given F. (Here "finding" commonly is equated with "establishing existence.")

A simple way of selecting an individual $u(x,y)$ out of the infinite set of all solutions of (4.2) consists in prescribing a curve Γ in xyz-space which is to be contained in the integral surface $z=u(x,y)$. Let Γ be represented parametrically by

$$x=f(s),\qquad y=g(s),\qquad z=h(s). \tag{5.1}$$

We are asking for a solution $u(x,y)$ of (4.2) such that the relation

$$h(s)=u(f(s),g(s)) \tag{5.2}$$

holds identically in s. Finding the function $u(x,y)$ for given data $f(s)$, $g(s)$, $h(s)$ constitutes the *Cauchy problem* for (4.2). Actually the same curve Γ has many different parametric representations (5.1) for different choices of the parameter s. Introducing a different parameter σ by a substitution $s=\phi(\sigma)$ will not change the solution $u(x,y)$ of the Cauchy problem.

We shall be satisfied here with a *local* solution u of our problem, defined for x,y near values $x_0=f(s_0)$, $y_0=g(s_0)$.

In many instances the variable y will be identified with time and x with position in space. It is then natural to pose the problem of finding a solution $u(x,y)$ from its *initial values* at the time $y=0$:

$$u(x,0)=h(x). \tag{5.3}$$

This *initial-value problem* obviously is the special Cauchy problem in which the curve Γ has the form

$$x=s,\qquad y=0,\qquad z=h(s), \tag{5.4}$$

that is, Γ lies in the xz-plane and is referred to x as parameter. We notice that in the initial-value problem we prescribe a single function $h(x)$, which in turn is determined uniquely by u, whereas in the general Cauchy problem many space curves Γ are bound to lead to the same u. An integral surface contains many curves Γ but only one intersection with the xz-plane.

Let then the functions $f(s)$, $g(s)$, $h(s)$ describing Γ be of class C^1 in a neighborhood of a value s_0. Let

$$P_0=(x_0,y_0,z_0)=(f(s_0),g(s_0),h(s_0)). \tag{5.5}$$

Assume that the coefficients $a(x,y,z)$, $b(x,y,z)$, $c(x,y,z)$ in (4.2) are of class C^1 in x,y,z near P_0. It is clear intuitively that the integral surface $z=u(x,y)$ passing through Γ will have to consist of the characteristic curves passing through the various points of Γ. Accordingly we form for each s near s_0 that solution

$$x=X(s,t), \qquad y=Y(s,t), \qquad z=Z(s,t) \tag{5.6}$$

of the characteristic differential equations (4.4) which reduces to $f(s),g(s),h(s)$ for $t=0$. The functions X,Y,Z then satisfy

$$X_t=a(X,Y,Z), \qquad Y_t=b(X,Y,Z), \qquad Z_t=c(X,Y,Z) \tag{5.7}$$

identically in s,t and also satisfy the initial conditions

$$X(s,0)=f(s), \qquad Y(s,0)=g(s), \qquad Z(s,0)=h(s). \tag{5.8}$$

From the general theorems on existence and on continuous dependence on parameters of solutions of systems of ordinary differential equations it follows that there exists a unique set of functions $X(s,t)$, $Y(s,t)$, $Z(s,t)$ of class C^1 for (s,t) near $(s_0,0)$ which satisfy (5.7), (5.8).

Equations (5.6) represent a surface Σ: $z=u(x,y)$ referred to parameters s,t if we can solve the first two equations for s,t in terms of x,y, say in the form $s=S(x,y)$, $t=T(x,y)$. Then the u defined by

$$z=u(x,y)=Z(S(x,y),T(x,y)) \tag{5.9}$$

will be the explicit representation of Σ. By (5.5), (5.8)

$$x_0=X(s_0,0), \qquad y_0=Y(s_0,0). \tag{5.10}$$

Now the implicit function theorem asserts that we can find solutions $s=S(x,y)$, $t=T(x,y)$ of

$$x=X(S(x,y),T(x,y)), \qquad y=Y(S(x,y),T(x,y)) \tag{5.11}$$

of class C^1 in a neighborhood of (x_0,y_0) and satisfying

$$s_0=S(x_0,y_0), \qquad 0=T(x_0,y_0), \tag{5.12}$$

provided the Jacobian

$$J=\begin{vmatrix} X_s(s_0,0) & Y_s(s_0,0) \\ X_t(s_0,0) & Y_t(s_0,0) \end{vmatrix} \tag{5.13}$$

does not vanish. By (5.7), (5.8) this amounts to the condition

$$J=\begin{vmatrix} f'(s_0) & g'(s_0) \\ a(x_0,y_0,z_0) & b(x_0,y_0,z_0) \end{vmatrix} \neq 0. \tag{5.14}$$

Thus (5.14) guarantees that locally (5.6) represents a surface Σ: $z=u(x,y)$. That Σ is an integral surface is clear in the parametric representation (5.6). For at any point P the quantities X_t, Y_t, Z_t give the direction of the tangent to a curve $s=$const. on Σ, which will have to lie in the tangent plane of Σ at P. Thus (5.7) shows that the tangent plane at any point contains the characteristic direction (a,b,c), and hence that Σ is an integral surface. [One can also verify analytically that the function u represented by (5.9) satisfies the differential equation (4.2) by first expressing u_x, u_y in terms of S_x, S_y, T_x, T_y, and then expressing those four quantities in terms of X_s, X_t, Y_s, Y_t using (5.11).]

This completes the local existence proof for the solution of the Cauchy problem, under the assumption (5.14). Uniqueness follows from the theorem on p. 10: Any integral surface through Γ would have to contain the characteristic curves through the points of Γ, hence would have to contain the surface represented parametrically by (5.6), and hence locally would have to be identical with the surface.

Condition (5.14) is essential for the existence of a C^1-solution $u(x,y)$ of the Cauchy problem. For if $J=0$ we would find from (5.2), (4.2) that at $s=s_0$, $x=f(s_0)$, $y=g(s_0)$ the three relations

$$bf'-ag'=0, \qquad h'=f'u_x+g'u_y, \qquad c=au_x+bu_y \tag{5.15}$$

hold. These imply that

$$bh'-cg'=0, \qquad ah'-cf'=0$$

and hence that f',g',h' are proportional to a,b,c. Hence $J=0$ is incompatible with the existence of a solution unless Γ happens to be characteristic at s_0. Incidentally the Cauchy problem will have infinitely many solutions for a characteristic curve Γ, which are obtained by passing any curve Γ^* satisfying (5.14) through a point P_0 of Γ and solving the Cauchy problem for Γ^*.

In the special case of a *linear* P.D.E. we can write (4.2) in the form

$$a(x,y)u_x+b(x,y)u_y=c(x,y)u+d(x,y). \tag{5.16}$$

Here the system of three characteristic O.D.E.s reduces to the pair

$$\frac{dx}{dt}=a(x,y), \qquad \frac{dy}{dt}=b(x,y) \tag{5.17}$$

or even to the single equivalent equation

$$\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)}. \tag{5.18}$$

Equations (5.17) or (5.18) determine a system of curves in the xy-plane, called *characteristic projections*, (also, more commonly and confusingly, just "characteristics") which are the projections onto the xy-plane of the characteristic curves in xyz-space. The characteristic curve is obtained from its projection $x(t), y(t)$ by finding $z(t)$ from the linear O.D.E.

$$\frac{dz}{dt} = c(x(t), y(t))z + d(x(t), y(t)). \tag{5.19}$$

We indicate how to proceed in the more general case of a quasi-linear equation for a function $u = u(x_1, \dots, x_n)$ of n independent variables. Such an equation has the form

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n, u) u_{x_i} = c(x_1, \dots, x_n, u). \tag{5.20}$$

Here the characteristic curves in $x_1 \dots x_n z$-space are given by the system of O.D.E.s

$$\frac{dx_i}{dt} = a_i(x_1, \dots, x_n, z) \quad \text{for } i = 1, \dots, n \tag{5.21a}$$

$$\frac{dz}{dt} = c(x_1, \dots, x_n, z). \tag{5.21b}$$

In the *Cauchy problem* we want to pass an integral surface $z = u(x_1, \dots, x_n)$ in $\mathbb{R}^{n+1}$ through an $(n-1)$-dimensional manifold Γ given parametrically by

$$x_i = f_i(s_1, \dots, s_{n-1}) \quad \text{for } i = 1, \dots, n \tag{5.22a}$$

$$z = h(s_1, \dots, s_{n-1}). \tag{5.22b}$$

For that purpose we pass through each point of Γ with parameters $s_1, \dots, s_{n-1}$ a characteristic curve (solution of (5.21a, b) reducing to $(f_1, \dots, f_n, h)$ for $t = 0$) represented by

$$x_i = X_i(s_1, \dots, s_{n-1}, t) \quad \text{for } i = 1, \dots, n \tag{5.23a}$$

$$z = Z(s_1, \dots, s_{n-1}, t). \tag{5.23b}$$

These equations form a parametric representation for the desired integral surface $z = u(x_1, \dots, x_n)$, provided relations (5.23a) can be solved for $s_1, \dots, s_{n-1}, t$. This is the case when the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial f_1}{\partial s_1} & \cdots & \dfrac{\partial f_n}{\partial s_1} \\ \vdots & & \vdots \\ \dfrac{\partial f_1}{\partial s_{n-1}} & \cdots & \dfrac{\partial f_n}{\partial s_{n-1}} \\ a_1 & \cdots & a_n \end{vmatrix} \tag{5.24}$$

does not vanish.

6. Examples

(1) (See Section 3.)

$$u_y + c u_x = 0 \qquad (c = \text{const.}) \tag{6.1a}$$

$$u(x, 0) = h(x). \tag{6.1b}$$

The initial curve Γ corresponding to (6.1b) is given by

$$x = s, \qquad y = 0, \qquad z = h(s).$$

The characteristic differential equations are

$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = 1, \qquad \frac{dz}{dt} = 0. \tag{6.2}$$

This leads to the parametric representation

$$x = X(s,t) = s + ct, \qquad y = Y(s,t) = t, \qquad z = Z(s,t) = h(s) \tag{6.3}$$

for the integral surface. Eliminating s, t we find for the solution of the initial-value problem (6.1a, b) the representation

$$z = h(x - cy) \tag{6.4}$$

in agreement with (3.3).

(2) Euler's P.D.E. for a *homogeneous* function $u(x_1, \dots, x_n)$:

$$\sum_{k=1}^{n} x_k u_{x_k} = \alpha u \qquad (\alpha = \text{const.} \neq 0). \tag{6.5}$$

Since equation (6.5) is singular at the origin (the J defined by (5.24) cannot be different from 0) we postulate the initial-value problem

$$u(x_1, \dots, x_{n-1}, 1) = h(x_1, \dots, x_{n-1}) \tag{6.6}$$

corresponding to a curve Γ given by

$$x_i = \begin{cases} s_i & \text{for } i = 1, \dots, n-1 \\ 1 & \text{for } i = n \end{cases} \tag{6.7}$$

$$z = h(s_1, \dots, s_{n-1}).$$

Solving the characteristic differential equations

$$\frac{dx_i}{dt} = x_i \quad \text{for } i = 1, \dots, n \tag{6.8}$$

$$\frac{dz}{dt} = \alpha z$$

leads to

$$x_i = \begin{cases} s_i e^t & \text{for } i = 1, \dots, n-1 \\ e^t & \text{for } i = n \end{cases} \tag{6.9}$$

$$z = e^{\alpha t} h(s_1, \dots, s_{n-1})$$