# Stochastic Signals and Systems

Lecture 9 .

**Unstable AR and MA processes**

**Multivariate systems**

3 December 2020

# REMINDER

# Unstable MA processes, I.

Consider now an MA process s.t. $C(z^{-1})$ not necessarily stable.

$$y = C'(q^{-1})e' \tag{1}$$

where $C'(q^{-1})$ is a polynomial of $q^{-1}$ and $(e'_n)$ is a w.s.st. orthogonal process.

Assuming $\sigma^2(e') = 1$ the spectral density of $(y_n)$ is given by

$$f(\omega) = |C'(e^{-i\omega})|^2.$$

This can be obtained by restricting the complex-function

$$g(z) = C'(z^{-1})C'(z)$$

to $z = e^{-i\omega}$.

# Unstable MA processes, II.

**Proposition**

*Let* $C'(z^{-1})$ *be a polynomial of* $z^{-1}$ *such that* $C'(z^{-1}) \neq 0$ *for* $|z| = 1$.

*Then* $\exists$ *a stable polynomial* $C(z^{-1})$ *with* $\deg C = \deg C'$ *such that*

$$C'(z^{-1})C'(z) = C(z^{-1})C(z). \tag{2}$$

*The stable factor is unique if the leading coefficient is fixed to 1. .*

# Spectral factorization, I.

The spectral density of a MA process can be written as

$$f(\omega) = C(e^{-i\omega})C(e^{i\omega}) = |C(e^{-i\omega})|^2.$$

where $C(z^{-1})$ is a stable polynomial. This is called spectral factorization, and $C(e^{-i\omega})$ is called a stable spectral factor.

Define a new process $e$ by its spectral representation process as

$$d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})} d\zeta^y(\omega). \tag{3}$$

Write the spectral representation process of $e$ as

$$d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})}\, d\zeta^y(\omega) = \frac{C'(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^{e'}(\omega). \tag{4}$$

The definition of $e$, given in (4), implies

$$y = C(q^{-1})e,$$

and since $C(z^{-1})$ stable, $e$ is the innovation process of $y$.

# Spectral factorization, II.

**Proposition**

*Let $y = (y_n)$ be an MA process given by $y = C'(q^{-1})e'$ (1). Assume that $C'(z^{-1}) \neq 0$ for $|z| = 1$. Let $C(e^{-i\omega})$ be the stable spectral factor of $f^y(\omega)$, and define $e = (e_n)$ by (3). Then $e$ is a w.s.st. orthogonal process,*

$$y = C(q^{-1})e,$$

*and $e$ is the innovation process of $y$.*

# ARMA processes, I.

The combination of AR and MA processes is called an ARMA process.

## Definition

*A w.s.st. process $y = (y_n)$ is called an ARMA process if it satisfies the dynamics*

$$A(q^{-1})\, y = C(q^{-1})\, e, \tag{5}$$

*where $(e_n)$ is a w.s.st. orthogonal process, and $A(q^{-1})$ and $C(q^{-1})$ are polynomials of the backward shift operator $q^{-1}$.*

We use the following notations in defining an ARMA$(p, r)$ process:

$$A(q^{-1}) = \sum_{k=0}^{p} a_k q^{-k}, \qquad C(q^{-1}) = \sum_{k=0}^{r} c_k q^{-k},$$

assuming that $a_0 = c_0 = 1$, and $a_p \neq 0$ and $c_r \neq 0$.

# ARMA processes, II.

**Proposition**

*Consider the ARMA dynamics (5). Assume that $A(z^{-1}) \neq 0$ for $|z| = 1$. Then there is a unique w.s.st. process $y = (y_n)$ satisfying (5). The process $(y_n)$ has a spectral density equal to*

$$f^y(\omega) = \sigma^2(e) \frac{|C(e^{-i\omega})|^2}{|A(e^{-i\omega})|^2}.$$

**Remark.** We assume that $A(z^{-1})$ and $C(z^{-1})$ have no common factor.

# Stability and inverse stability, I.

Let us consider an ARMA$(p,r)$ process $y=(y_n)$ defined by

$$A(q^{-1})\,y = C(q^{-1})\,e \quad \text{with} \quad \deg A = p, \;\; \deg C = r. \tag{6}$$

Here $e=(e_n)$ is a w.s.st. orthogonal process. We can ask ourselves: under what conditions $e$ is the innovation process of $y$.

**Proposition**

*Assume that both $A(z^{-1})$ and $C(z^{-1})$ are stable, i.e. $A(z^{-1}) \neq 0$ and $C(z^{-1}) \neq 0$ for $|z| \geq 1$. Then $e=(e_n)$ is the innovation process of $y=(y_n)$.*

The idea of the proof: expand both $\dfrac{C(e^{-i\omega})}{A(e^{-i\omega})}$ and $\dfrac{A(e^{-i\omega})}{C(e^{-i\omega})}$

into a power series of $e^{-i\omega}$, to infer both $H_n^y \subset H_n^e$ and $H_n^e \subset H_n^y \quad \forall n$.

# Stability and inverse stability, II.

A simple extension of the lemma on the expansion of $1/A(e^{-i\omega})$:

**Lemma**

*If both $A(z^{-1})$ and $C(z^{-1})$ are stable, $a_0 = c_0 = 1$, then*

$$\frac{C(e^{-i\omega})}{A(e^{-i\omega})} = \sum_{k=0}^{\infty} h_k e^{-ik\omega}, \qquad \text{and} \qquad \frac{A(e^{-i\omega})}{C(e^{-i\omega})} = \sum_{k=0}^{\infty} g_k e^{-ik\omega},$$

*with $h_0 = g_0 = 1$, where convergence on the r.h.s. is uniform in $\omega$.*

It follows that the r.h.s. of $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}$ converges in $L_2^c(dF^e)$, hence the random measure $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}\, d\zeta^e(\omega)$ is well-defined. Similarly, the r.h.s. of $\frac{A(e^{-i\omega})}{C(e^{-i\omega})}$ converges in $L_2^c(dF^y)$, hence the random measure $\frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^y(\omega)$ is well-defined.

# End of REMINDER

# Unstable ARMA processes, I.

The analysis of unstable MA or AR processes can be extended.

Let $y = (y_n)$ be a w.s.st. ARMA process defined by

$$A'(q^{-1})\, y = C'(q^{-1})\, e' \tag{7}$$

where the polynomials $A'(z^{-1})$ and $C'(z^{-1})$ are not necessarily stable.

The process $e'$ is a w.s.st. orthogonal process with $\sigma^2(e') = 1$.

Assume $A'(z^{-1}) \neq 0$ and $C'(z^{-1}) \neq 0$ for $|z| = 1$. The spectral density of $y$:

$$f(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 .$$

Let $A(e^{-i\omega})$ and $C(e^{-i\omega})$ be the stable spectral factors of

the denominator and the numerator, resp.

# Unstable ARMA processes, II.

Then

$$f^y(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = \left| \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \right|^2.$$

The rational function $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}$ is called a stable spectral factor of $f^y$.

Now define the w.s.st. process $e$ by

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^y(\omega),$$

Equivalently, define the process $e$ by the inverse dynamics

$$C(q^{-1})\, e = A(q^{-1})\, y \tag{8}$$

Let us spell out the definition of $e$:

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})}\, d\zeta^{e'}(\omega).$$

# Unstable ARMA processes, III.

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})}\, d\zeta^{e'}(\omega).$$

Note that the transfer function

$$G(e^{-i\omega}) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})}$$

is such that

$$G(e^{-i\omega})G(e^{i\omega}) = |G(e^{-i\omega})|^2 = 1 \qquad \forall \omega. \tag{9}$$

We say that the transfer function $G(e^{-i\omega})$ is all-pass: all frequencies are passed through the filter $G$ with unchanged energy.

It is readily seen that the process $e = (e_n)$ is a w.s.st. orthogonal process:

# Unstable ARMA processes. Summary.

## Proposition

*Let $y = (y_n)$ be a w.s.st. ARMA process defined by*

$$A'(q^{-1})\, y = C'(q^{-1})\, e' \tag{10}$$

*where $e'$ is a w.s.st. orthogonal process with $\sigma^2(e') = 1$.*

*Assume $A'(z^{-1}) \neq 0$ and $C'(z^{-1}) \neq 0$ for $|z| = 1$. Let $C(e^{-i\omega})/A(e^{-i\omega})$ denote the stable spectral factor of the spectral density of $y$:*

$$f(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = \left| \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \right|^2 .$$

*Then the innovation process of $y$ is obtained from the equation below:*

$$C(q^{-1})\, e = A(q^{-1})\, y, \tag{11}$$

# VECTOR VALUED PROCESSES

# Vector valued w.s.st. processes, I.

Motivation: simultaneous price movements of several commodities.

Interaction between individual prices: get better predictions.

**Definition**

*The $\mathbb{R}^s$-valued stochastic process $y = (y_n)$, $-\infty < n < +\infty$ is called wide sense stationary if $\mathrm{E}|y_n|^2 < \infty$ for all $n$, $\mathrm{E}y_n = 0$ for all $n$, and the covariance matrix*

$$R(\tau) = R^y(\tau) = \mathrm{E}(y_{n+\tau} y_n^\top)$$

*is independent of $n$.*

The condition $\mathrm{E}y_n = 0$ can be replaced by the condition that $\mathrm{E}y_n = m$ with some fixed vector $m \in \mathbb{R}^s$ for all $n$.

The matrix-valued function $R(.)$ is called the *auto-covariance function* of $(y_n)$. Obviously, we have

$$R(-\tau) = R(\tau)^\top.$$

# Vector valued w.s.st. processes, II.

The definition extends to $\mathbb{C}^s$-valued (complex-valued) processes requiring

$$R(\tau) = \mathrm{E}(y_{n+\tau}\overline{y_n}^\top)$$

to be independent of $n$. Obviously, $R(-\tau) = \overline{R}(\tau)^\top$.

An eminent role is played by vector-valued w.s.st. orthogonal processes.

**Definition**

*A $\mathbb{R}^s$-valued w.s.st. stochastic process $(e_n)$ is called a w.s.st orthogonal process if*

$$\mathrm{E}e_{n+\tau}e_n^\top = 0 \quad \text{for} \quad \tau \neq 0 \quad \text{and} \quad \mathrm{E}e_n e_n^\top = \Sigma \quad \text{for all} \quad n,$$

*where $\Sigma$ is a fixed, $s \times s$ symmetric positive semi-definite matrix.*

Note that $\Sigma$ is not assumed to be the identity $I$, it may be even singular.

# Results for vector-valued w.s.st. processes

Let $(y_n)$ be a $\mathbb{R}^s$-valued w.s.st. process. Define a new process:

$$z_n = \sum_{k=1}^{p} a_k^T y_{n-k}, \quad \text{with} \quad a_1, \ldots a_p \in \mathbb{R}^s.$$

**Exercise.** (HW) Show that $(z_n)$ is an $\mathbb{R}$-valued w.s.st. process, and

$$\mathrm{E} z_n^2 = \sum_{k=1}^{p} \sum_{j=1}^{p} a_k^T R(k-j) a_j \geq 0.$$

**Corollary.** The *block matrix* $R$, which is defined by the *blocks*

$$R_{k,j} = R(k-j) \in \mathbb{R}^{s \times s}, \qquad k, j = 1, \ldots, p \tag{12}$$

is positive semi-definite. The size of $R$ is $(ps) \times (ps)$.

# Prediction and innovation, I.

Let $(y_n)$ be a $\mathbb{R}^s$-valued w.s.st. process.

Any *component of $y_n$* or a linear combination of components of $y_n$

will be predicted by taking a set of vectors $a_1, \ldots a_p$ in $\mathbb{R}^s$, and defining

$$z_n = \sum_{k=1}^{p} a_k^\top y_{n-k}.$$

**Exercise.** Show that $(z_n)$ is an $\mathbb{R}$-valued w.s.st. process and we have

$$\mathrm{E} z_n^2 = \sum_{k=1}^{p} \sum_{l=1}^{p} a_k^\top R(k-l) a_l. \tag{13}$$

Thus the block matrix $R = (R_{k,l})$ defined by the blocks

$$R_{k,l} = R(k-l), \qquad k, l = 1, \ldots, p \tag{14}$$

is positive semi-definite. The size of $R$ is $ps \times ps$.

# Block Toeplitz matrices

Recall: $R_{k,j} = R(k-j) = \mathrm{E}\left(y(n-j)y^T(n-k)\right), \quad k,j = 1,\ldots,p$, and

$$R = \begin{pmatrix} R(0) & R(1) & \ldots & \ldots & R(p-1) \\ R(-1) & R(0) & R(1) & \ldots & R(p-2) \\ R(-2) & R(-1) & R(0) & \ldots & R(p-3) \\ \vdots & \vdots & \ddots & \ddots & \\ R(1-p) & R(2-p) & \ldots & \ldots & R(0) \end{pmatrix}$$

**Definition**

*A $ps \times ps$ matrix $R$ consisting of $m \times m$ blocks is called a*

***block-Toeplitz matrix****.*

**Note.** A block-Toeplitz matrix is typically not Toeplitz in the usual sense.

# The history of $(y_n)$

Let $(y_n)$ be an $\mathbb{R}^s$-valued w.s.st. process.

Consider first the linear space of $\mathbb{R}$-valued (!) random variables:

$$\mathcal{L}^y_{n-1} = \left\{ \sum_{k=1}^{p} a_k^T y_{n-k}, \text{ for some } p, \text{ and } a_k \in \mathbb{R}^s \right\} \subset L_2(\Omega, \mathcal{F}, P).$$

We define $H^y_{n-1}$ as the closure of $\mathcal{L}^y_{n-1}$ in $L_2(\Omega, \mathcal{F}, P)$.

**Remark.** $H^y_{n-1}$ is a Hilbert space of *real-valued* r.v.-s.

# Projection in $L_2^s(\Omega, \mathcal{F}, P)$

We define $L_2^s(\Omega, \mathcal{F}, P) = \left\{ x \in \mathbb{R}^s \text{ is a r.v. } : \ \mathrm{E}|x|^2 < \infty \right\}.$

The projection of $x \in L_2^s(\Omega, \mathcal{F}, P)$ onto $H_{n-1}^y$ is defined *componentwise*:

$$\widehat{x} = (x | H_{n-1}^y) = \left( (x_1 | H_{n-1}^y), \ldots, (x_s | H_{n-1}^y) \right)^T.$$

For the error vector $\tilde{x} = x - \widehat{x}$ we have

$$\tilde{x} \perp H_{n-1}^y \quad \text{i.e.} \quad \tilde{x}_k \perp H_{n-1}^y \quad \forall\, k.$$

# The innovation process

Define innovation process of $y = (y_n)$, denoted by $e = (e_n)$, via

$$e_n = y_n - (y_n | H^y_{n-1}).$$

A vector valued process $(y_n)$ is singular, if its innovation process identically $0$, or equivalently if

$$H^y_n = H^y_{n-1} \text{ for all } n. \tag{15}$$

A novel phenomenon is that $\Lambda = \mathrm{E} e_n e_n^T$ may be singular, although $\Lambda \neq 0$.

# Spectral theory, I.

Question: can we extend Herglotz's theorem on the representation of the auto-covariance sequence to vector valued w.s.st. processes ?

Assume first that the auto-covariance function of $(y_n)$ satisfies

$$\sum_{\tau=-\infty}^{\infty} \|R(\tau)\|^2 < \infty, \tag{16}$$

where $\|R\|$ denotes the operator norm of the matrix $R$, i.e.

$$\|R\| = \max_{x \neq 0} |Rx|/|x|.$$

# Spectral theory, II.

**Theorem**

*Let $y = (y_n)$ be an $\mathbb{R}^s$-valued w.s.st. process, and let $R(.)$ be its auto-covariance function. Assume that $R(.)$ satisfies (16). Then we have*

$$R(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tau\omega} f(\omega) d\omega, \tag{17}$$

*where $f(.)$ is a symmetric, positive semi-definite function in $L_2^{s\times s}(d\omega)$.*

The function $f(.)$ is called the spectral density of $y = (y_n)$.

**Exercise.** **HW** Show that for an $\mathbb{R}^s$-valued orthogonal wide sense stationary process $(e_n)$ with covariance matrix $\Lambda = \mathbb{E} e_n e_n^\top$ we have

$$f(\omega) = \Lambda \quad \forall \omega.$$

# Spectral theory, III.

**Proof** (Outline.) Let $\alpha \in \mathbb{R}^s$ and consider the scalar process

$$z_{\alpha,n} = \alpha^\top y_n.$$

The auto-covariance function of $z_{\alpha,n}$ is

$$r^\alpha(\tau) = \alpha^\top R(\tau)\alpha.$$

It is obvious by (16) that $\sum_{\tau=-\infty}^{\infty} \left(r^\alpha(\tau)\right)^2 < +\infty$. By the special case of the scalar Herglotz's theorem we have

$$r^\alpha(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega\tau} f^\alpha(\omega) d\omega,$$

where $f^\alpha(\omega) \geq 0$ is the spectral density corresponding to $r^\alpha(\tau)$.

# Spectral theory, IV.

We know how to get $f^{\alpha}(.)$ from $r^{\alpha}(.)$ explicitly:

$$f^{\alpha}(\omega) = \sum_{\tau=-\infty}^{\infty} r^{\alpha}(\tau)\, e^{-i\tau\omega}.$$

Here convergence on the r.h.s. is meant in $L_2^c(d\omega) = L_2^c([0, 2\pi], d\omega)$ !

Substituting $r^{\alpha}(\tau) = \alpha^{\top} R(\tau)\alpha$ we get

$$f^{\alpha}(\omega) = \sum_{\tau=-\infty}^{\infty} \alpha^{\top} R(\tau)\alpha\, e^{-i\tau\omega}\,. \tag{18}$$

Taking finite truncations of the r.h.s. we get that

$$\sum_{\tau=-N}^{N} \alpha^{\top} R(\tau)\alpha e^{-i\tau\omega} = \alpha^{\top}\Big(\sum_{\tau=-N}^{N} R(\tau)e^{-i\tau\omega}\Big)\alpha \tag{19}$$

converges in $L_2(d\omega)$ for <u>any $\alpha$.</u>

# Spectral theory, V.

From here we would like to conclude that

$$f_N(\omega) = \sum_{\tau=-N}^{N} R(\tau) e^{-i\tau\omega}$$

itself converges in $L_2^{s \times s}(d\omega)$. To see this we need what follows:

**Exercise.** Prove that a quadratic form $\alpha^\top F \alpha$, with $F$ symmetric, determines the bilinear form corresponding to $F$ uniquely as

$$\beta^\top F \gamma = \frac{1}{4}\left((\beta+\gamma)^\top F(\beta+\gamma) - (\beta-\gamma)^\top F(\beta-\gamma)\right). \qquad (20)$$

 

# Spectral theory, VI.

Take $F = f_N$, and any pair of unit vectors in $\mathbb{R}^s$, say, $\beta = e_k, \gamma = e_l$.

We conclude that infinite series below converges in $L_2(d\omega)$ $\forall k, l$:

$$e_k^\top \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-i\tau\omega} e_l = \sum_{\tau=-\infty}^{\infty} R_{k,l}(\tau)\, e^{-i\tau\omega} =: f_{k,l}(\omega).$$

Thus the infinite series below converges in $L_2^{s\times s}(d\omega)$:

$$\sum_{\tau=-\infty}^{\infty} R(\tau) e^{-i\tau\omega} =: f(\omega)$$

It is readily seen that

$$f^\alpha(\omega) = \alpha^\top f(\omega)\alpha \quad \forall \alpha \in \mathbb{R}^s.$$

Obviously $f(\omega)$ is symmetric, and $f^\alpha(\omega) \geq 0$

that $f(\omega)$ is positive semi-definite. This concludes the proof.

# Spectral theory: the general case

**Theorem**

*Let $y = (y_n)$ be an $\mathbb{R}^s$-valued w.s.st. process, and let $R(.)$ be its auto-covariance function. Then we have*

$$R(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tau\omega} dF(\omega),$$

*where $F(.)$ is a matrix-valued function such that $F(0) = 0$ and $F(2\pi)$ is finite, and its increments of $F(\cdot)$ are symmetric positive semi-definite matrices.*

The matrix-valued $F(\omega)$ is the spectral distribution function of $(y_n)$