Stochastic Signals and Systems

ARMA processes
Prediction revisited

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Prediction revisited
Let \( (y_n) \) be a completely regular w.s.st. process:

\[
y_n = \sum_{k=0}^{\infty} h_k e_{n-k},
\]

where \( (e_n) \) is the innovation process of \( (y_n) \).

The one-step ahead predictor of \( y_n \) is:

\[
\hat{y}_n = \sum_{k=1}^{\infty} h_k e_{n-k}.
\]

Try to express \( e \) in term of \( y \). Best done in *spectral domain.*
\[ y_n = \sum_{k=0}^{\infty} h_k e_{n-k} \]

The original systems and its inverse in the frequency domain:

\[ d\zeta^y(\omega) = H(e^{-i\omega})d\zeta^e(\omega). \]

\[ d\zeta^e(\omega) = H^{-1}(e^{-i\omega})d\zeta^y(\omega). \]

Thus the one-step ahead predictor \((\hat{y}_n = y_n - e_n)\) is given by:

\[ d\zeta^{\hat{y}}(\omega) = (H(e^{-i\omega}) - 1)d\zeta^e(\omega). \]
Proposition

Let \( y = (y_n) \) be a completely regular w.s.st. process given by

\[
y_n = \sum_{k=0}^{\infty} h_k e_{n-k}.
\]

Then its one step ahead predictor \( \hat{y} = (\hat{y}_n) \) is obtained via the spectral representation measure

\[
d\zeta^{\hat{y}}(\omega) = (1 - H^{-1}(e^{-i\omega}))d\zeta^{y}(\omega).
\]

A shortcoming of the result: it is formulated in spectral domain.
Prediction of ARMA processes

Let \((y_n)\) be a w.s.st. ARMA process:

\[
A(q^{-1})y = C(q^{-1})e.
\]

Assume that \(A(z^{-1})\) and \(C(z^{-1})\) are stable.

Then \(e\) is the innovation process of \(y\).

Setting \(H(e^{-i\omega}) = \frac{C(e^{-i\omega})}{A(e^{-i\omega})}\) we get

\[
d\zeta^y(\omega) = H(e^{-i\omega})d\zeta^e(\omega).
\]
\[ d\zeta^y(\omega) = \frac{C(e^{-i\omega})}{A(e^{-i\omega})} d\zeta^e(\omega) \]

Apply Proposition "\(d\hat{\zeta}^y(\omega) = (1 - H^{-1}(e^{-i\omega}))d\zeta^y(\omega)\)."

Then we get for the one-step ahead predictor:

\[ d\hat{\zeta}^y(\omega) = \left(1 - \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\right) d\zeta^y(\omega).\]

Multiply both sides by \(C(e^{-i\omega})\):

\[ C(e^{-i\omega})d\hat{\zeta}^y(\omega) = (C(e^{-i\omega}) - A(e^{-i\omega})) d\zeta^y(\omega).\]
\[ C(e^{-i\omega})d\zeta \hat{y}(\omega) = (C(e^{-i\omega}) - A(e^{-i\omega})) \ d\zeta y(\omega). \]

Write the result in time domain:

**Proposition**

Let \((y_n)\) be a w.s.st. ARMA process

\[ A(q^{-1})y = C(q^{-1})e \]

with stable \(A(z^{-1})\) and \(C(z^{-1})\).

Then the one-step ahead prediction \(\hat{y}\) is given by

\[ C(q^{-1})\hat{y} = (C(q^{-1}) - A(q^{-1})) \ y. \]
\[ C(q^{-1})\hat{y} = (C(q^{-1}) - A(q^{-1})) \, y \]

Since \( a_0 = c_0 = 1 \)

\[ \implies C(q^{-1}) - A(q^{-1}) = \sum_{k=1}^{\infty} (c_k - a_k)q^{-k}. \]

\[ \hat{y}_n + c_1 \hat{y}_{n-1} + \ldots + c_r \hat{y}_{n-r} = (a_1 - c_1)y_{n-1} + (a_2 - c_2)y_{n-2} + \ldots + (a_p - c_p)y_{n-p}. \]

Thus we do get a genuine one-step ahead predictor.