Stochastic Signals and Systems

ARMA processes

Prediction revisited

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Prediction revisited

Let (y_n) be a *completely regular* w.s.st. process:

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k},$$

where (e_n) is the innovation process of (y_n).

The one-step ahead predictor of y_n is:

$$\widehat{y}_n = \sum_{k=1}^{\infty} h_k e_{n-k}.$$

Try to express e in term of y. Best done *in spectral domain*.

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k}$$

The original systems and its inverse **in the frequency domain**:

$$d\zeta^y(\omega) = H(e^{-i\omega}) d\zeta^e(\omega).$$

$$d\zeta^e(\omega) = H^{-1}(e^{-i\omega}) d\zeta^y(\omega).$$

Thus the one-step ahead predictor ($\widehat{y}_n = y_n - e_n$) is given by :

$$d\widehat{\zeta^y}(\omega) = (H(e^{-i\omega}) - 1) d\zeta^e(\omega).$$

$$d\zeta^{\widehat{y}}(\omega) = (H(e^{-i\omega}) - 1)d\zeta^{e}(\omega)$$

Proposition

Let $y = (y_n)$ be a completely regular w.s.st. process given by

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k}.$$

Then its one step ahead predictor $\widehat{y} = (\widehat{y}_n)$ is obtained via the spectral representation measure

$$d\zeta^{\widehat{y}}(\omega) = (1 - H^{-1}(e^{-i\omega}))d\zeta^{y}(\omega).$$

A shortcoming of the result: *it is formulated in spectral domain*.

Prediction of ARMA processes

Let (y_n) be a w.s.st. ARMA process:

$$A(q^{-1})y = C(q^{-1})e.$$

Assume that $A(z^{-1})$ and $C(z^{-1})$ are *stable*.

Then e is the innovation process of y.

Setting $H(e^{-i\omega}) = \dfrac{C(e^{-i\omega})}{A(e^{-i\omega})}$ we get

$$d\zeta^y(\omega) = H(e^{-i\omega})d\zeta^e(\omega).$$

$$d\zeta^y(\omega) = \frac{C(e^{-i\omega})}{A(e^{-i\omega})} d\zeta^e(\omega)$$

Apply Proposition "$d\widehat{\zeta^y}(\omega) = (1 - H^{-1}(e^{-i\omega}))d\zeta^y(\omega)$."

Then we get for the one-step ahead predictor:

$$d\widehat{\zeta^y}(\omega) = \left(1 - \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\right) d\zeta^y(\omega).$$

Multiply both sides by $C(e^{-i\omega})$:

$$C(e^{-i\omega})d\widehat{\zeta^y}(\omega) = \left(C(e^{-i\omega}) - A(e^{-i\omega})\right) d\zeta^y(\omega).$$

$$C(e^{-i\omega})d\zeta^{\widehat{y}}(\omega) = \left(C(e^{-i\omega}) - A(e^{-i\omega})\right)\, d\zeta^{y}(\omega).$$

Write the result in time domain:

Proposition

Let (y_n) *be a w.s.st. ARMA process*

$$A(q^{-1})y = C(q^{-1})e$$

with **stable** $A(z^{-1})$ *and* $C(z^{-1})$.

Then the one-step ahead prediction $\widehat{y}$ *is given by*

$$C(q^{-1})\widehat{y} = \left(C(q^{-1}) - A(q^{-1})\right)\, y.$$

$$C(q^{-1})\widehat{y} = \left(C(q^{-1}) - A(q^{-1})\right)\, y$$

Since $a_0 = c_0 = 1$

$$\Longrightarrow \quad C(q^{-1}) - A(q^{-1}) = \sum_{k=1} (c_k - a_k) q^{-k}.$$

$$\widehat{y}_n + c_1 \widehat{y}_{n-1} + \ldots c_r \widehat{y}_{n-r} = (a_1 - c_1) y_{n-1} + (a_2 - c_2) y_{n-2} + \ldots (a_p - c_p) y_{n-p}.$$

Thus we do get a *genuine* one-step ahead predictor.