Lecture 8.

Unstable AR and MA processes

27 November 2020
REMINDER
MA processes, I.

Let \( e = (e_n) \) be a w.s.st. orthogonal process, and define with some \( c_k \) - s

\[ y_n = \sum_{k=0}^{m} c_k e_{n-k}. \]  

(1)

**Definition** The w.s.st. process \( y = (y_n) \) is called a [moving average](https://en.wikipedia.org/wiki/Moving_average) or MA process, or more precisely a MA(\(m\)) process.

A compact notation: define \( C(q^{-1}) = \sum_{k=0}^{m} c_k q^{-k} \), then we can write

\[ y = C(q^{-1})e. \]  

(2)

Briefly: \( y \) is obtained by passing \( e \) through a finite impulse response (FIR) filter.
A basic exercise

Exercise. Show that the w.s.st. processes:

\[ y_n = c_0 e_n + c_1 e_{n-1} \]
\[ z_n = c_1 e_n + c_0 e_{n-1}. \]

have the same auto-covariance functions.

Hint: Show that the two processes have the same spectral density.
MA processes, II.

Proposition

Assume that \( C(z^{-1}) \) is a stable polynomial. Then \( e = (e_n) \) is the innovation process of \( y = (y_n) \).

Proof (Outline). Obviously, \( y_n \in H^e_n \). To prove the converse consider the equality

\[
d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})} \, d\zeta^y(\omega) = \left( \sum_{k=0}^{\infty} h_k e^{-ik\omega} \right) \, d\zeta^y(\omega).
\]

Since \( C(z^{-1}) \) is a stable polynomial we can write

\[
\frac{1}{C(e^{-i\omega})} = \sum_{k=0}^{\infty} h_k e^{-ik\omega}
\]

with uniform convergence on \( \{z : |z| = 1\} \).
MA processes, III.

But: since the spectral density of \( y = (y_n) \) is bounded (why?) we have also convergence in \( L_2^c(dF^y) \) ! Thus

\[
e_n = \int_0^{2\pi} e^{i\omega} \frac{1}{C(e^{-i\omega})} d\zeta^y(\omega) = \int_0^{2\pi} e^{i\omega} \left( \sum_{k=0}^{\infty} h_k e^{-ik\omega} \right) d\zeta^y(\omega).
\]

implies

\[
e_n = \sum_{k=0}^{\infty} h_k \int_0^{2\pi} e^{i\omega} e^{-ik\omega} d\zeta^y(\omega) = \sum_{k=0}^{\infty} h_k y_{n-k}.
\]
Unstable MA processes, I.

Consider now an MA process s.t. $C(z^{-1})$ not necessarily stable. An innocent looking example is to take $|c'| > 1$ and consider

$$y_n = e'_n - c' e'_{n-1}$$

Challenge: find representation of $(y_n)$ in terms of its innovation process.

More generally: consider an MA process $(y_n)$ given by

$$y = C'(q^{-1})e'$$

(3)

where $C'(q^{-1})$ is a polynomial of $q^{-1}$ and $(e'_n)$ is a w.s.st. orthogonal process.
Unstable MA processes, II.

Assuming $\sigma^2(e') = 1$ the spectral density of $(y_n)$ is given by

$$f(\omega) = |C'(e^{-i\omega})|^2.$$  

This can be obtained by restricting the complex-function

$$g(z) = C'(z^{-1})C'(z)$$

to $z = e^{-i\omega}$.

Let us now assume that $C'(z^{-1})$ has an unstable root, say $\gamma'$. Then factorizing $C'(z^{-1})$ we will have a factor of the form

$$c'(z^{-1}) = 1 - \gamma'z^{-1} \quad \text{with} \quad |\gamma'| > 1.$$
Unstable MA processes, III.

Repeat: factorizing \( C'(z^{-1}) \) we will have a factor of the form:

\[
c'(z^{-1}) = 1 - \gamma' z^{-1} \quad \text{with} \quad |\gamma'| > 1.
\]

The effect of this factor in \( g(z) = C'(z^{-1})C'(z) \) is

\[
c'(z^{-1})c'(z) = (1 - \gamma' z^{-1})(1 - \gamma' z) = (z - \gamma')(z^{-1} - \gamma'). \quad (4)
\]

Let us now swap the role of \( \gamma' \) and \( 1/\gamma' \).

Write the second term on the r.h.s. as

\[
z^{-1} - \gamma' = \left(\frac{1}{\gamma'} z^{-1} - 1\right) \cdot \gamma' = -(1 - \frac{1}{\gamma'} z^{-1}) \cdot \gamma' =: c(z^{-1}) \quad (5)
\]

Then it is readily seen that \( c(z^{-1}) \) is stable, having a single root \( \frac{1}{\gamma'} \).
Unstable MA processes, IV.

Repeat: it is readily seen that

\[ c(z^{-1}) = -(1 - \frac{1}{\gamma'} z^{-1}) \cdot \gamma' \]

is stable, having a single root \( \frac{1}{\gamma'} \). Moreover

\[ c'(z^{-1})c'(z) = (z - \gamma')(z^{-1} - \gamma') = c(z)c(z^{-1}) = c(z^{-1})c(z). \]

Replacing all factors of \( C'(z^{-1}) \) by stable ones we come to the following conclusion.

**Proposition**

Let \( C'(z^{-1}) \) be a polynomial of \( z^{-1} \) such that \( C'(z^{-1}) \neq 0 \) for \( |z| = 1 \).

Then \( \exists \) a stable polynomial \( C(z^{-1}) \) with \( \deg C = \deg C' \) such that

\[ C'(z^{-1})C'(z) = C(z^{-1})C(z). \] (6)

The stable factor is unique if the leading coefficient is fixed to 1.
Spectral factorization, I.

The above proposition implies: the spectral density of a MA process can be written as

\[ f(\omega) = C(e^{-i\omega})C(e^{i\omega}) = |C(e^{-i\omega})|^2. \]

where \( C(z^{-1}) \) is a stable polynomial. This is called spectral factorization, and \( C(e^{-i\omega}) \) is called a stable spectral factor.

Define a new process \( e \) by its spectral representation process as

\[ d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})}d\zeta^y(\omega). \]  

(7)

Exercise. Show that the r.h.s. is well defined.

Eq. (7) can be written in the time domain as

\[ C(q^{-1})e = y. \]
Spectral factorization, II.

Write the spectral representation process of $e$ as

$$dζ^e(ω) = \frac{1}{C(e^{-iω})} dζ^y(ω) = \frac{C'(e^{-iω})}{C(e^{-iω})} dζ^e'(ω). \quad (8)$$

**Exercise.** Show that $e$ is a w.s.st. orthogonal process by showing that for the spectral density of $e$ we have

$$f^e(ω) = \left| \frac{C'(e^{-iω})}{C(e^{-iω})} \right|^2 = 1.$$  

The definition of $e$, given in (8), implies

$$y = C(q^{-1})e,$$

and since $C(z^{-1})$ stable, $e$ is the innovation process of $y$. To summarize:
Spectral factorization, III.

Proposition

Let $y = (y_n)$ be an MA process given in (3). Assume that $C'(z^{-1}) \neq 0$ for $|z| = 1$. Let $C(e^{-i\omega})$ be the stable spectral factor of $f^y(\omega)$, and define $e = (e_n)$ by (7). Then $e$ is a w.s.st. orthogonal process,

$$y = C(q^{-1})e,$$

and $e$ is the innovation process of $y$. 
AR(p) processes re-visited, I.

Let us now consider a possibly unstable AR(p) process \( y = (y_n) \) defined by

\[
A'(q^{-1}) y = e' \quad \text{with} \quad \deg A = p > 1,
\]

where \((e'_n)\) is a w.s.st. orthogonal process, \(\sigma^2(e') = 1\).

**Question:** what is the innovation process of \( y \)?

A simple (but less innocent looking) example is to take \(|a'| > 1\) and consider

\[
y_n + a' e_{n-1} = e'_n.
\]

We can not iterate and express \( y_n \) in terms of the past of values of \( e' \)!
AR(p) processes re-visited, II.

Assuming $A'(e^{-i\omega}) \neq 0$ for all $\omega$ the spectral density of $(y_n)$ is given by

$$f_y(\omega) = \frac{1}{|A'(e^{-i\omega})|^2}.$$  

Proceed with spectral factorization: by Proposition 2, there $\exists$ a stable polynomial $A(z^{-1})$ with $\deg A = \deg A'$ such that

$$A'(z^{-1}) A'(z) = A(z^{-1}) A(z). \quad (10)$$

It follows that for all $\omega$

$$A'(e^{-i\omega}) A'(e^{i\omega}) = A(e^{-i\omega}) A(e^{i\omega})$$

and thus

$$f_y(\omega) = \frac{1}{|A'(e^{-i\omega})|^2} = \frac{1}{|A(e^{-i\omega})|^2}.$$
AR(p) processes re-visited, III.

Define a new process \( e \) by its spectral representation process as

\[
d\zeta^e(\omega) = A(e^{-i\omega}) d\zeta^y(\omega).
\] (11)

**Exercise.** Show that, assuming \( A'(e^{-i\omega}) \neq 0 \) for all \( \omega \), the r.h.s. is well-defined, i.e. \( A(e^{-i\omega}) \) is in \( L^2_y(dF_y) \).

Elaborating the definition of \( d\zeta^e(\omega) \) we get:

\[
d\zeta^e(\omega) = A(e^{-i\omega}) d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{A'(e^{-i\omega})} d\zeta^e'(\omega).
\] (12)

**Exercise.** Show that \( e \) is a w.s.st. orthogonal process by showing that for the spectral density of \( e \) we have

\[
f^e(\omega) = \left| \frac{A(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = 1.
\]
AR(p) processes re-visited, IV.

Now in view of (12) we have

$$A(q^{-1})y = e,$$  \hspace{1cm} (13)

and since $A(z^{-1})$ stable, $e$ is the innovation process of $y$. To summarize:

**Proposition**

Let $y = (y_n)$ be an AR(p)-process given in (9). Assume that $A'(z^{-1}) \neq 0$ for $|z| = 1$. Let $A(e^{-i\omega})$ be the stable spectral factor of $f^y(\omega)$, and define $e = (e_n)$ by (13). Then $e$ is the innovation process of $y$.

**Exercise.** Let $y$ be an AR(1) process defined by $y_n = a'y_{n-1} + e'_n$ with $a' > 1$, and $(e'_n)$ being a w.s.st. orthogonal process. Find an expression for the innovation process of $y$!
The combination of AR and MA processes is called an ARMA process.

**Definition**

A w.s.st. process \( y = (y_n) \) is called an **ARMA** process if it satisfies the dynamics

\[
A(q^{-1}) y = C(q^{-1}) e, \tag{14}
\]

where \((e_n)\) is a w.s.st. orthogonal process, and \(A(q^{-1})\) and \(C(q^{-1})\) are polynomials of the backward shift operator \(q^{-1}\). The degrees \(p = \text{deg } A(q^{-1})\) and \(r = \text{deg } C(q^{-1})\) are called the orders of the ARMA process. In emphasizing the orders we call \( y = (y_n) \) an ARMA\((p, r)\) process.
ARMA processes, II.

We use the following notations in defining an ARMA($p$, $r$) process:

\[ A(q^{-1}) = \sum_{k=0}^{p} a_k q^{-k}, \quad C(q^{-1}) = \sum_{k=0}^{r} c_k q^{-k}, \]

assuming that $a_0 = c_0 = 1$, and $a_p \neq 0$ and $c_r \neq 0$. 
ARMA processes, III.

Straightforward extensions of Propositions on the existence of an AR process is the following:

**Proposition**

Consider the ARMA dynamics (14). Assume that $A(z^{-1}) \neq 0$ for $|z| = 1$. Then there is a unique w.s.st. process $y = (y_n)$ satisfying (14). The process $(y_n)$ has a spectral density equal to

$$f_y(\omega) = \sigma^2(e) \frac{|C(e^{-i\omega})|^2}{|A(e^{-i\omega})|^2}.$$

**Proof:** Assume that a w.s.st. solution $(y_n)$ does exist. Let the spectral representation processes of $(e_n)$ and $(y_n)$ be denoted by $d\zeta^e(\omega)$ and $d\zeta^y(\omega)$, respectively. Then

$$A(e^{-i\omega}) d\zeta^y(\omega) = C(e^{-i\omega}) d\zeta^e(\omega).$$
ARMA processes, IV.

From this we get,

\[
dζ^y(ω) = \frac{1}{A(e^{-iω})} \cdot C(e^{-iω}) \ dζ^e(ω) = C(e^{-iω}) \cdot \frac{1}{A(e^{-iω})} \ dζ^e(ω). \quad (15)
\]

Now, if \( A(e^{-iω}) \neq 0 \) for all \( ω \), then the spectral representation measure

\[
dζ^v(ω) = \frac{1}{A(e^{-iω})} \ dζ^e(ω) \quad (16)
\]

is well-defined, and obviously so is the far r.h.s. of (15).

Note that the process \( v = (v_n) \) with spectral representation process \( dζ^v(ω) \) satisfies the AR dynamics \( A(q^{-1}) v = e \).
Claim. Let \( d\zeta(\omega) \) be a random measure with orthogonal increments. Let \( G(e^{-i\omega}) \) and \( H(e^{-i\omega}) \) frequency response functions such that
\[
d\zeta'(\omega) = G(e^{-i\omega}) \cdot (H(e^{-i\omega}) \, d\zeta(\omega))
\]
is well-defined. Prove that in this case
\[
H(e^{-i\omega}) \cdot (G(e^{-i\omega}) \, d\zeta(\omega))
\]
is also well-defined and
\[
H(e^{-i\omega}) \cdot (G(e^{-i\omega}) \, d\zeta(\omega)) = G(e^{-i\omega}) \cdot (H(e^{-i\omega}) \, d\zeta(\omega)). \quad (17)
\]
Exercise. Prove the above claim.
It follows that \( d\zeta'(\omega) \), is well-defined via (15). Uniqueness is thus proved.
Exercise. Complete the proof of Proposition 5 by showing that the process $y = (y_n)$ defined by

$$y_n := \int_0^{2\pi} e^{i\omega} C(e^{-i\omega}) \cdot \frac{1}{A(e^{-i\omega})} d\zeta^e(\omega) = \int_0^{2\pi} e^{i\omega} C(e^{-i\omega}) d\zeta^v(\omega)$$

does satisfy the ARMA dynamics (14).

Exercise. Derive the expression for the spectral density of $y = (y_n)$ given in the proposition.

Remark. A novel feature of the ARMA dynamics compared to AR or MA dynamics is that pole-zero cancellation may occur: if $A(z^{-1})$ and $C(z^{-1})$ have a common factor it will be cancelled in $C(z^{-1})/A(z^{-1})$!

A common remedy: assume that $A(z^{-1})$ and $C(z^{-1})$ have no common factor.
Stability and inverse stability, I.

Let us consider an ARMA($p$, $r$) process $y = (y_n)$ defined by

$$A(q^{-1}) y = C(q^{-1}) e \quad \text{with} \quad \deg A = p, \quad \deg C = r. \quad (18)$$

Here $e = (e_n)$ is a w.s.st. orthogonal process. We can ask ourselves:
under what conditions $e$ is the innovation process of $y$.

**Proposition**

Assume that both $A(z^{-1})$ and $C(z^{-1})$ are stable, i.e. $A(z^{-1}) \neq 0$ and $C(z^{-1}) \neq 0$ for $|z| \geq 1$. Then $e = (e_n)$ is the innovation process of $y = (y_n)$.

The idea of the proof: expand both $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}$ and $\frac{A(e^{-i\omega})}{C(e^{-i\omega})}$
into a power series of $e^{-i\omega}$, to infer both $H^y_n \subset H^e_n$ and $H^e_n \subset H^y_n \ \forall n.$
Stability and inverse stability, II.

A simple extension of the lemma on the expansion of $1/A(e^{-i\omega})$:

**Lemma**

If both $A(z^{-1})$ and $C(z^{-1})$ are stable, $a_0 = c_0 = 1$, then

$$\frac{C(e^{-i\omega})}{A(e^{-i\omega})} = \sum_{k=0}^{\infty} h_k e^{-ik\omega}, \text{ and } \frac{A(e^{-i\omega})}{C(e^{-i\omega})} = \sum_{k=0}^{\infty} g_k e^{-ik\omega},$$

with $h_0 = g_0 = 1$, where convergence on the r.h.s. is uniform in $\omega$.

**Exercise.** Derive this Lemma directly from the previous one.

*(Hint: Directly: meaning without redoing the proof.)*

It follows that the r.h.s. of $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}$ converges in $L^c_2(dF^e)$, hence the random measure $\frac{C(e^{-i\omega})}{A(e^{-i\omega})} d\zeta^e(\omega)$ is well-defined. Similarly, the r.h.s. of $\frac{A(e^{-i\omega})}{C(e^{-i\omega})}$ converges in $L^c_2(dF^y)$, hence the random measure $\frac{A(e^{-i\omega})}{C(e^{-i\omega})} d\zeta^y(\omega)$ is well-defined.
Proof of Proposition 6, I.

Let us now return to the ARMA($p, r$) process ($y_n$) defined by

$$A(q^{-1}) y = C(q^{-1}) e$$

where $A(z^{-1})$ and $C(z^{-1})$ are stable polynomials of $z^{-1}$.

**Proof:** To prove $y_n \in H_n^e$ consider the familiar expression

$$d\zeta^y(\omega) = \frac{C(e^{-i\omega})}{A(e^{-i\omega})} d\zeta^e(\omega) = \left(\sum_{k=0}^{\infty} h_k e^{-ik\omega}\right) d\zeta^e(\omega),$$

The infinite sum on the r.h.s. converges in $L_2^c (dF^e)$, see the Lemma above.

Hence, following the with arguments for stable AR($p$) processes, we get:

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k},$$

where the r.h.s. converges in $L_2(\Omega, F, P)$. In particular, $\sum_{k=0}^{\infty} h_k^2 < \infty$. 

© L. Gerencsér, Zs. Vágó and B. Gerencsér
Proof of Proposition 6, II.

To prove $e_n \in H^x_n$ we proceed symmetrically: write

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} d\zeta^y(\omega) = \left( \sum_{k=0}^{\infty} g_k e^{-ik\omega} \right) d\zeta^y(\omega),$$

The infinite sum on the r.h.s. converges in $L^2_c(dF^y)$ (!), see Lemma above.

Hence, following the arguments for stable AR$(p)$ processes, we get:

$$e_n = \sum_{k=0}^{\infty} g_k y_{n-k},$$

where the r.h.s. converges in $L_2(\Omega, \mathcal{F}, \mathcal{P})$. Q.e.d.
Unstable ARMA processes, I.

The analysis of unstable MA or AR processes can be extended. Let \( y = (y_n) \) be a w.s.st. ARMA process defined by

\[
A'(q^{-1}) y = C'(q^{-1}) e'
\]

(20)

where the polynomials \( A'(z^{-1}) \) and \( C'(z^{-1}) \) are not necessarily stable.

The process \( e' \) is a w.s.st. orthogonal process with \( \sigma^2(e') = 1 \).

Assume \( A'(z^{-1}) \neq 0 \) and \( C'(z^{-1}) \neq 0 \) for \( |z| = 1 \). The spectral density of \( y \):

\[
f(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 .
\]

Let \( A(e^{-i\omega}) \) and \( C(e^{-i\omega}) \) be the stable spectral factors of the denominator

and the numerator, resp.
Unstable ARMA processes, II.

Then

\[ f_y(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = \left| \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \right|^2. \]

The rational function \( \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \) is called a stable spectral factor of \( f_y \).

Now define the w.s.st. process \( e \) by

\[ d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} d\zeta^y(\omega), \]

Equivalently, define the process \( e \) by the inverse dynamics

\[ C(q^{-1})e = A(q^{-1})y \]

Let us spell out the definition of \( e \):

\[ d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} d\zeta^e'(\omega). \]
Unstable ARMA processes, III.

\[ d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} d\zeta'^e(\omega). \]

Note that the transfer function

\[ G(e^{-i\omega}) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \]

is such that

\[ G(e^{-i\omega})G(e^{i\omega}) = |G(e^{-i\omega})|^2 = 1 \quad \forall \omega. \]  

(22)

We say that the transfer function \( G(e^{-i\omega}) \) is all-pass: all frequencies are passed through the filter \( G \) with unchanged energy.

It is readily seen that the process \( e = (e_n) \) is a w.s.st. orthogonal process:
Unstable ARMA processes, IV.

**Exercise.** Let $G(.)$ be an all-pass transfer function, and let $e'$ be a w.s.st. orthogonal process. Then the process $e$ defined by

$$dζ^e(ω) = G(e^{-iω})dζ^{e'}(ω)$$

is also a w.s.st. orthogonal process.

**Exercise.** **HW.** Let $γ' \in \mathbb{C}$ arbitrary, $|γ'| ≠ 1$.

Show that the transfer function

$$G(e^{-iω}) = \frac{1 - γ'e^{-iω}}{γ' - e^{-iω}}$$

is all-pass.
Unstable ARMA processes. Summary.

Proposition

Let \( y = (y_n) \) be a w.s.st. ARMA process defined by

\[
A'(q^{-1}) y = C'(q^{-1}) e'
\]

(23)

where \( e' \) is a w.s.st. orthogonal process with \( \sigma^2(e') = 1 \). Assume \( A'(z^{-1}) \neq 0 \) and \( C'(z^{-1}) \neq 0 \) for \( |z| = 1 \). Let \( C(e^{-i\omega})/A(e^{-i\omega}) \) denote the stable spectral factor of the spectral density of \( y \), denoted by \( f(\cdot) \):

\[
f(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = \left| \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \right|^2.
\]

Then the innovation process of \( y \) is obtained from the equation below:

\[
C(q^{-1}) e = A(q^{-1}) y,
\]

(24)
END of Lecture 8