Stochastic Signals and Systems

Lecture 8.
Unstable AR and MA processes

27 November 2020

REMINDER

MA processes, I.

Let $e = (e_n)$ be a w.s.st. orthogonal process, and define with some c_k - s

$$.y_n = \sum_{k=0}^{m} c_k e_{n-k}. \tag{1}$$

Definition The w.s.st. process $y = (y_n)$ is called a moving average or MA process, or more precisely a MA(m) process.

A compact notation: define $C(q^{-1}) = \sum_{k=0}^{m} c_k q^{-k}$, then we can write

$$.y = C(q^{-1})e. \tag{2}$$

Briefly: y is obtained by passing e through a finite impulse response (FIR) filter.

A basic exercise

Exercise. Show that the w.s.st. processes:

$$
\begin{aligned}
y_n &= c_0 e_n + c_1 e_{n-1} \\
z_n &= c_1 e_n + c_0 e_{n-1}.
\end{aligned}
$$

have the same auto-covariance functions.

Hint: Show that the two processes have the same spectral density.

MA processes, II.

Proposition

Assume that $C(z^{-1})$ is a stable polynomial. Then $e = (e_n)$ is the innovation process of $y = (y_n)$.

Proof (Outline). Obviously, $y_n \in H_n^e$. To prove the converse consider the equality

$$d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})}\, d\zeta^y(\omega) = \left(\sum_{k=0}^{\infty} h_k e^{-ik\omega}\right) d\zeta^y(\omega).$$

Since $C(z^{-1})$ is a stable polynomial we can write

$$\frac{1}{C(e^{-i\omega})} = \sum_{k=0}^{\infty} h_k e^{-ik\omega}$$

ith uniform convergence on $\{z : |z| = 1\}$.

MA processes, III.

But: since the spectral density of $y = (y_n)$ is bounded (why ?) we have also convergence in $L_2^c(dF^y)$! Thus

$$e_n = \int_0^{2\pi} e^{in\omega} \frac{1}{C(e^{-i\omega})} \, d\zeta^y(\omega) = \int_0^{2\pi} e^{in\omega} \left(\sum_{k=0}^{\infty} h_k e^{-ik\omega} \right) d\zeta^y(\omega).$$

implies

$$e_n = \sum_{k=0}^{\infty} h_k \int_0^{2\pi} e^{in\omega} e^{-ik\omega} \, d\zeta^y(\omega) = \sum_{k=0}^{\infty} h_k \, y_{n-k}.$$

Unstable MA processes, I.

Consider now an MA process s.t. $C(z^{-1})$ not necessarily stable.

An innocent looking example is to take $|c'| > 1$ and consider

$$y_n = e'_n - c' e'_{n-1}$$

Challenge: find representation of (y_n) in terms of its innovation process.

More generally: consider an MA process (y_n) given by

$$y = C'(q^{-1})e' \tag{3}$$

where $C'(q^{-1})$ is a polynomial of q^{-1} and (e'_n) is a w.s.st. orthogonal process.

Unstable MA processes, II.

Assuming $\sigma^2(e') = 1$ the spectral density of (y_n) is given by

$$f(\omega) = |C'(e^{-i\omega})|^2.$$

This can be obtained by restricting the complex-function

$$g(z) = C'(z^{-1})C'(z)$$

to $z = e^{-i\omega}$.

Let us now assume that $C'(z^{-1})$ has an unstable root, say γ'.

Then factorizing $C'(z^{-1})$ we will have a factor of the form

$$c'(z^{-1}) = 1 - \gamma' z^{-1} \qquad \text{with} \qquad |\gamma'| > 1.$$

Unstable MA processes, III.

Repeat: factorizing $C'(z^{-1})$ we will have a factor of the form:

$$c'(z^{-1}) = 1 - \gamma' z^{-1} \qquad \text{with} \qquad |\gamma'| > 1.$$

The effect of this factor in $g(z) = C'(z^{-1})C'(z)$ is

$$c'(z^{-1})c'(z) = (1 - \gamma' z^{-1})(1 - \gamma' z) = (z - \gamma')(z^{-1} - \gamma'). \tag{4}$$

Let us now swap the role of γ' and $1/\gamma'$.

Write the second term on the r.h.s. as

$$z^{-1} - \gamma' = (\frac{1}{\gamma'} z^{-1} - 1) \cdot \gamma' = -(1 - \frac{1}{\gamma'} z^{-1}) \cdot \gamma' =: c(z^{-1}) \tag{5}$$

Then it is readily seen that $c(z^{-1})$ is stable, having a single root $\frac{1}{\gamma'}$.

Unstable MA processes, IV.

Repeat: it is readily seen that

$$c(z^{-1}) = -(1 - \frac{1}{\gamma'} z^{-1}) \cdot \gamma'$$

is stable, having a single root $\dfrac{1}{\gamma'}$. Moreover

$$c'(z^{-1})c'(z) = (z - \gamma')(z^{-1} - \gamma') = c(z)c(z^{-1}) = c(z^{-1})c(z).$$

Replacing all factors of $C'(z^{-1})$ by stable ones we come to the following conclusion.

Proposition

Let $C'(z^{-1})$ be a polynomial of z^{-1} such that $C'(z^{-1}) \neq 0$ for $|z| = 1$.

Then $\exists$ a stable polynomial $C(z^{-1})$ with $\deg C = \deg C'$ such that

$$C'(z^{-1})C'(z) = C(z^{-1})C(z). \tag{6}$$

The stable factor is unique if the leading coefficient is fixed to 1. .

Spectral factorization, I.

The above proposition implies: the spectral density of a MA process can be written as

$$f(\omega) = C(e^{-i\omega})C(e^{i\omega}) = |C(e^{-i\omega})|^2.$$

where $C(z^{-1})$ is a stable polynomial. This is called spectral factorization, and $C(e^{-i\omega})$ is called a stable spectral factor.

Define a new process e by its spectral representation process as

$$d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})} d\zeta^y(\omega). \tag{7}$$

Exercise. Show that the r.h.s. is well defined.

Eq. (7) can be written in the time domain as

$$C(q^{-1})e = y.$$

Spectral factorization, II.

Write the spectral representation process of e as

$$d\zeta^e(\omega) = \frac{1}{C(e^{-i\omega})}\, d\zeta^y(\omega) = \frac{C'(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^{e'}(\omega). \tag{8}$$

Exercise. Show that e is a w.s.st. orthogonal process by showing that for the spectral density of e we have

$$f^e(\omega) = \left| \frac{C'(e^{-i\omega})}{C(e^{-i\omega})} \right|^2 = 1.$$

The definition of e, given in (8), implies

$$y = C(q^{-1})e,$$

and since $C(z^{-1})$ stable, e is the innovation process of y. To summarize:

Spectral factorization, III.

Proposition

Let $y = (y_n)$ be an MA process given in (3). Assume that $C'(z^{-1}) \neq 0$ for $|z| = 1$. Let $C(e^{-i\omega})$ be the stable spectral factor of $f^y(\omega)$, and define $e = (e_n)$ by (7). Then e is a w.s.st. orthogonal process,

$$y = C(q^{-1})e,$$

and e is the innovation process of y.

AR(p) processes re-visited, I.

Let us now consider a possibly unstable AR(p) process $y = (y_n)$ defined by

$$A'(q^{-1})\, y = e' \quad \text{with} \quad \deg A = p > 1, \tag{9}$$

where (e'_n) is a w.s.st. orthogonal process, $\sigma^2(e') = 1$.

Question: what is the innovation process of y ?

A simple (but less innocent looking) example is to take $|a'| > 1$ and consider

$$y_n + a' e_{n-1} = e'_n.$$

We can not iterate and express y_n in terms of the past of values of e' !

AR(p) processes re-visited, II.

Assuming $A'(e^{-i\omega}) \neq 0$ for all ω the spectral density of (y_n) is given by

$$f^y(\omega) = \frac{1}{|A'(e^{-i\omega})|^2}.$$

Proceed with spectral factorization: by Proposition 2, there $\exists$ a stable polynomial $A(z^{-1})$ with $\deg A = \deg A'$ such that

$$A'(z^{-1})\, A'(z) = A(z^{-1})\, A(z). \tag{10}$$

It follows that for all ω

$$A'(e^{-i\omega})\, A'(e^{i\omega}) = A(e^{-i\omega})\, A(e^{i\omega})$$

and thus

$$f^y(\omega) = \frac{1}{|A'(e^{-i\omega})|^2} = \frac{1}{|A(e^{-i\omega})|^2}.$$

AR(p) processes re-visited, III.

Define a new process e by its spectral representation process as

$$d\zeta^e(\omega) = A(e^{-i\omega})\, d\zeta^y(\omega). \tag{11}$$

Exercise. Show that, assuming $A'(e^{-i\omega}) \neq 0$ for all ω, the r.h.s. is well-defined, i.e. $A(e^{-i\omega})$ is in $L_2^c(dF^y)$.

Elaborating the definition of $d\zeta^e(\omega)$ we get:

$$d\zeta^e(\omega) = A(e^{-i\omega})\, d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{A'(e^{-i\omega})}\, d\zeta^{e'}(\omega). \tag{12}$$

Exercise. Show that e is a w.s.st. orthogonal process by showing that for the spectral density of e we have

$$f^e(\omega) = \left| \frac{A(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = 1.$$

AR(p) processes re-visited, IV.

Now in view of (12) we have

$$A(q^{-1})\, y = e, \tag{13}$$

and since $A(z^{-1})$ stable, e is the innovation process of y. To summarize:

Proposition

Let $y = (y_n)$ be an AR(p)-process given in (9). Assume that $A'(z^{-1}) \neq 0$ for $|z| = 1$. Let $A(e^{-i\omega})$ be the stable spectral factor of $f^y(\omega)$, and define $e = (e_n)$ by (13). Then e is the innovation process of y.

Exercise. Let y be an AR(1) process defined by $y_n = a' y_{n-1} + e'_n$ with $a' > 1$, and (e'_n) being a w.s.st. orthogonal process. Find an expression for the innovation process of y !

ARMA processes, I.

The combination of AR and MA processes is called an ARMA process.

Definition

A w.s.st. process $y = (y_n)$ is called an ARMA process if it satisfies the dynamics

$$A(q^{-1})\, y = C(q^{-1})\, e, \tag{14}$$

where (e_n) is a w.s.st. orthogonal process, and $A(q^{-1})$ and $C(q^{-1})$ are polynomials of the backward shift operator q^{-1}. The degrees $p = \deg A(q^{-1})$ and $r = \deg C(q^{-1})$ are called the orders of the ARMA process. In emphasizing the orders we call $y = (y_n)$ an ARMA(p, r) process.

ARMA processes, II.

We use the following notations in defining an ARMA(p, r) process:

$$A(q^{-1}) = \sum_{k=0}^{p} a_k q^{-k}, \qquad C(q^{-1}) = \sum_{k=0}^{r} c_k q^{-k},$$

assuming that $a_0 = c_0 = 1$, and $a_p \neq 0$ and $c_r \neq 0$.

ARMA processes, III.

Straightforward extensions of Propositions on the existence of an AR process is the following:

Proposition

Consider the ARMA dynamics (14). *Assume that* $A(z^{-1}) \neq 0$ *for* $|z| = 1$. *Then there is a unique w.s.st. process* $y = (y_n)$ *satisfying (14). The process* (y_n) *has a spectral density equal to*

$$f^y(\omega) = \sigma^2(e)\,\frac{|C(e^{-i\omega})|^2}{|A(e^{-i\omega})|^2}.$$

Proof: Assume that a w.s.st. solution (y_n) does exist. Let the spectral representation processes of (e_n) and (y_n) be denoted by $d\zeta^e(\omega)$ and $d\zeta^y(\omega)$, respectively. Then

$$A(e^{-i\omega})\,d\zeta^y(\omega) = C(e^{-i\omega})\,d\zeta^e(\omega).$$

ARMA processes, IV.

From this we get,

$$d\zeta^y(\omega) = \frac{1}{A(e^{-i\omega})} \cdot C(e^{-i\omega})\, d\zeta^e(\omega) = C(e^{-i\omega}) \cdot \frac{1}{A(e^{-i\omega})}\, d\zeta^e(\omega). \quad (15)$$

Now, if $A(e^{-i\omega}) \neq 0$ for all ω, then the spectral representation measure

$$d\zeta^v(\omega) = \frac{1}{A(e^{-i\omega})}\, d\zeta^e(\omega) \quad (16)$$

is well-defined, and obviously so is the far r.h.s. of (15).

Note that the process $v = (v_n)$ with spectral representation process $d\zeta^v(\omega)$ satisfies the AR dynamics $A(q^{-1})\, v = e$.

ARMA processes, V.

Claim. Let $d\zeta(\omega)$ be a random measure with orthogonal increments. Let $G(e^{-i\omega})$ and $H(e^{-i\omega})$ frequency response functions such that $d\zeta'(\omega) = G(e^{-i\omega}) \cdot \left(\; H(e^{-i\omega}) \, d\zeta(\omega) \; \right)$ is well-defined. Prove that in this case $H(e^{-i\omega}) \cdot \left(G(e^{-i\omega}) \, d\zeta(\omega) \; \right)$ is also well-defined and

$$H(e^{-i\omega}) \cdot \left(G(e^{-i\omega}) \, d\zeta(\omega) \; \right) = G(e^{-i\omega}) \cdot \left(\; H(e^{-i\omega}) \, d\zeta(\omega) \; \right). \tag{17}$$

Exercise. Prove the above claim.

It follows that $d\zeta^y(\omega)$, is well-defined via (15). Uniqueness is thus proved.

ARMA processes, VI.

Exercise. Complete the proof of Proposition 5 by showing that the process $y = (y_n)$ defined by

$$y_n := \int_0^{2\pi} e^{in\omega}\, C(e^{-i\omega}) \cdot \frac{1}{A(e^{-i\omega})}\, d\zeta^e(\omega) = \int_0^{2\pi} e^{in\omega}\, C(e^{-i\omega})\, d\zeta^v(\omega)$$

does satisfy the ARMA dynamics (14).

Exercise. Derive the expression for the spectral density of $y = (y_n)$ given in the proposition.

Remark. A novel feature of the ARMA dynamics compared to AR or MA dynamics is that pole-zero cancellation may occur: if $A(z^{-1})$ and $C(z^{-1})$ have a common factor it will be cancelled in $C(z^{-1})/A(z^{-1})$!

A common remedy: assume that $A(z^{-1})$ and $C(z^{-1})$ have no common factor.

Stability and inverse stability, I.

Let us consider an ARMA(p, r) process $y = (y_n)$ defined by

$$A(q^{-1})\, y = C(q^{-1})\, e \quad \text{with} \quad \deg A = p, \;\; \deg C = r. \tag{18}$$

Here $e = (e_n)$ is a w.s.st. orthogonal process. We can ask ourselves: under what conditions e is the innovation process of y.

Proposition

Assume that both $A(z^{-1})$ and $C(z^{-1})$ are stable, i.e. $A(z^{-1}) \neq 0$ and $C(z^{-1}) \neq 0$ for $|z| \geq 1$. Then $e = (e_n)$ is the innovation process of $y = (y_n)$.

The idea of the proof: expand both $\dfrac{C(e^{-i\omega})}{A(e^{-i\omega})}$ and $\dfrac{A(e^{-i\omega})}{C(e^{-i\omega})}$

into a power series of $e^{-i\omega}$, to infer both $H_n^y \subset H_n^e$ and $H_n^e \subset H_n^y \quad \forall n$.

Stability and inverse stability, II.

A simple extension of the lemma on the expansion of $1/A(e^{-i\omega})$:

Lemma

If both $A(z^{-1})$ and $C(z^{-1})$ are stable, $a_0 = c_0 = 1$, then

$$\frac{C(e^{-i\omega})}{A(e^{-i\omega})} = \sum_{k=0}^{\infty} h_k e^{-ik\omega}, \qquad \text{and} \qquad \frac{A(e^{-i\omega})}{C(e^{-i\omega})} = \sum_{k=0}^{\infty} g_k e^{-ik\omega},$$

with $h_0 = g_0 = 1$, where convergence on the r.h.s. is uniform in ω.

Exercise. Derive this Lemma directly from the previous one.

(*Hint*: Directly: meaning without redoing the proof.)

It follows that the r.h.s. of $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}$ converges in $L_2^c(dF^e)$, hence the random measure $\frac{C(e^{-i\omega})}{A(e^{-i\omega})} d\zeta^e(\omega)$ is well-defined. Similarly, the r.h.s. of $\frac{A(e^{-i\omega})}{C(e^{-i\omega})}$ converges in $L_2^c(dF^y)$, hence the random measure $\frac{A(e^{-i\omega})}{C(e^{-i\omega})} d\zeta^y(\omega)$ is well-defined.

Proof of Proposition 6, I.

Let us now return to the ARMA(p, r) process (y_n) defined by

$$A(q^{-1})\, y = C(q^{-1})\, e \tag{19}$$

where $A(z^{-1})$ and $C(z^{-1})$ are stable polynomials of z^{-1}.

Proof: To prove $y_n \in H_n^e$ consider the familiar expression

$$d\zeta^y(\omega) = \frac{C(e^{-i\omega})}{A(e^{-i\omega})}\, d\zeta^e(\omega) = \left(\sum_{k=0}^{\infty} h_k e^{-ik\omega}\right) d\zeta^e(\omega),$$

The infinite sum on the r.h.s. converges in $L_2^c(dF^e)$, see the Lemma above.

Hence, following the with arguments for stable AR(p) processes, we get:

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k},$$

where the r.h.s. converges in $L_2(\Omega, \mathcal{F}, \mathcal{P})$. In particular, $\sum_{k=0}^{\infty} h_k^2 < \infty$.

Proof of Proposition 6, II.

To prove $e_n \in H_n^y$ we proceed symmetrically: write

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \, d\zeta^y(\omega) = \left(\sum_{k=0}^{\infty} g_k e^{-ik\omega} \right) d\zeta^y(\omega),$$

The infinite sum on the r.h.s. converges in $L_2^c(dF^y)$ (!), see Lemma above.

Hence, following the arguments for stable AR(p) processes, we get:

$$e_n = \sum_{k=0}^{\infty} g_k y_{n-k},$$

where the r.h.s. converges in $L_2(\Omega, \mathcal{F}, \mathcal{P})$. Q.e.d.

Unstable ARMA processes, I.

The analysis of unstable MA or AR processes can be extended.
Let $y = (y_n)$ be a w.s.st. ARMA process defined by

$$A'(q^{-1})\, y = C'(q^{-1})\, e' \tag{20}$$

where the polynomials $A'(z^{-1})$ and $C'(z^{-1})$ are not necessarily stable.

The process e' is a w.s.st. orthogonal process with $\sigma^2(e') = 1$.
Assume $A'(z^{-1}) \neq 0$ and $C'(z^{-1}) \neq 0$ for $|z| = 1$. The spectral density of y:

$$f(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 .$$

Let $A(e^{-i\omega})$ and $C(e^{-i\omega})$ be the stable spectral factors of the denominator

and the numerator, resp.

Unstable ARMA processes, II.

Then

$$f^y(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = \left| \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \right|^2 .$$

The rational function $\frac{C(e^{-i\omega})}{A(e^{-i\omega})}$ is called a <u>stable spectral factor</u> of f^y.

Now define the w.s.st. process e by

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^y(\omega),$$

Equivalently, define the process e by the inverse dynamics

$$C(q^{-1})\, e = A(q^{-1})\, y \tag{21}$$

Let us spell out the definition of e:

$$d\zeta^e(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^y(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})}\, d\zeta^{e'}(\omega).$$

Unstable ARMA processes, III.

$$d\zeta^{e}(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})}\, d\zeta^{y}(\omega) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})}\, d\zeta^{e'}(\omega).$$

Note that the transfer function

$$G(e^{-i\omega}) = \frac{A(e^{-i\omega})}{C(e^{-i\omega})} \cdot \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})}$$

is such that

$$G(e^{-i\omega})G(e^{i\omega}) = |G(e^{-i\omega})|^{2} = 1 \qquad \forall\omega. \tag{22}$$

We say that the transfer function $G(e^{-i\omega})$ is all-pass: all frequencies are passed through the filter G with unchanged energy.

It is readily seen that the process $e = (e_n)$ is a w.s.st. orthogonal process:

Unstable ARMA processes, IV.

Exercise. Let $G(.)$ be an all-pass transfer function, and let e' be a w.s.st. orthogonal process. Then the process e defined by

$$d\zeta^e(\omega) = G(e^{-i\omega}) d\zeta^{e'}(\omega)$$

is also a w.s.st. orthogonal process.

Exercise. **HW**. Let $\gamma' \in \mathbb{C}$ arbitrary, $|\gamma'| \neq 1$.

Show that the transfer function

$$G(e^{-i\omega}) = \frac{1 - \gamma' e^{-i\omega}}{\gamma' - e^{-i\omega}}$$

is all-pass.

Unstable ARMA processes. Summary.

Proposition

Let $y = (y_n)$ be a w.s.st. ARMA process defined by

$$A'(q^{-1})\, y = C'(q^{-1})\, e' \tag{23}$$

where e' is a w.s.st. orthogonal process with $\sigma^2(e') = 1$. Assume $A'(z^{-1}) \neq 0$ and $C'(z^{-1}) \neq 0$ for $|z| = 1$. Let $C(e^{-i\omega})/A(e^{-i\omega})$ denote the stable spectral factor of the spectral density of y, denoted by $f(.)$:

$$f(\omega) = \left| \frac{C'(e^{-i\omega})}{A'(e^{-i\omega})} \right|^2 = \left| \frac{C(e^{-i\omega})}{A(e^{-i\omega})} \right|^2 .$$

Then the innovation process of y is obtained from the equation below:

$$C(q^{-1})\, e = A(q^{-1})\, y, \tag{24}$$

END of Lecture 8