Stochastic Signals and Systems

Lecture 5.
Spectral theory, III.

REMINDER:

Introducing spectral representation
Interpreting $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, 1.

Let us return to the interpretation of the FT of a w.s.st. process $(y_n)$:

$$\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}.$$ 

Consider a special case: $y_n = \xi e^{in\omega_0}$ with $\mathbb{E} \xi = 0$, and interpret

$$\sum_{n=-\infty}^{\infty} \xi e^{in(\omega_0-\omega)}.$$ 

Recall the formal Fourier series of the Dirac-delta function $\delta(\omega - \omega_0)$:

$$\int_0^{2\pi} \delta(\omega - \omega_0) e^{in\omega} d\omega = e^{in\omega_0}$$

formally implies

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\omega_0} e^{-in\omega} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\omega_0-\omega)} = \delta(\omega - \omega_0).$$
Interpreting \( \sum_{n=-\infty}^{\infty} y_n e^{-in\omega} \), II.

Summarizing we get:

\[
\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \xi e^{in(\omega_0 - \omega)} = \xi \delta(\omega - \omega_0).
\]

Consider now another special case: \( y_n = \sum_{k=1}^{m} \xi(\omega_k) e^{in\omega_k} \) where

\[
E\xi(\omega_k) = 0 \quad \text{and} \quad E\xi(\omega_k) \overline{\xi(\omega_l)} = 0 \quad \text{for} \quad k \neq l.
\]

Extending the above formal argument we get (formally):

\[
\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} y_n e^{-in\omega} = \sum_{k=1}^{m} \xi(\omega_k) \delta(\omega - \omega_k).
\]

Note: the Dirac delta function \( \delta(\omega - \omega_k) \) is a generalized function, but its integral is an ordinary function: the unit step function \( H(\omega - \omega_k) \).
The integral of \( \sum_{n=-\infty}^{\infty} y_n e^{-in\omega} \), I.

We are motivated to consider the integrated (and truncated) process

\[
\zeta_N(\omega') = \int_{0}^{\omega'} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega.
\]

Rewrite the r.h.s. as

\[
\sum_{n=-N}^{N} y_n \int_{0}^{\omega'} e^{-in\omega} \, d\omega = \sum_{n=-N}^{N} y_n c_n
\]

Here

\[
c_n = \int_{0}^{\omega'} e^{-in\omega} \, d\omega = \int_{0}^{2\pi} \chi[0,\omega')(\omega) e^{-in\omega} \, d\omega,
\]

where \( \chi[0,\omega')(\omega) \) is the characteristic function of the interval \([0, \omega')\).
The integral of $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, II.

Repeat: we have

$$c_n = \int_{\omega'}^{\omega'} e^{-in\omega} \, d\omega = \int_0^{2\pi} \chi[0, \omega')(\omega) e^{-in\omega} \, d\omega.$$ 

Thus $\frac{1}{\sqrt{2\pi}} c_n$ is the Fourier coefficients of $\chi[0, \omega')(\cdot)$. Hence setting

$$C_N(e^{i\omega}) = \sum_{n=-N}^{N} c_n e^{in\omega} = 2\pi \sum_{n=-N}^{N} \frac{c_n}{(2\pi)^{1/2}} \frac{e^{in\omega}}{(2\pi)^{1/2}},$$

and letting $N \to \infty$ we get, with convergence in $L_2[0, 2\pi)$,

$$\lim_{N \to \infty} C_N(e^{i\omega}) = 2\pi \chi[0, \omega')(\omega).$$
The variance of \( \int \sum_{n=\infty}^{\infty} y_n e^{-in\omega}, \) is:

\[
\zeta_N(\omega') = \int_0^{\omega'} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega = \sum_{n=-N}^{N} y_n c_n
\]

\[
C_N(e^{i\omega}) = \sum_{n=-N}^{N} c_n e^{in\omega} \to 2\pi \chi_{[0,\omega')}(\omega).
\]

Let us now compute \( \mathbb{E} |\zeta_N(\omega')|^2 \). We have by Herglotz’s theorem

\[
\mathbb{E} |\zeta_N(\omega')|^2 = \mathbb{E} \left| \sum_{n=-N}^{N} y_n c_n \right|^2 = \frac{1}{2\pi} \int_0^{2\pi} |C_N(e^{i\omega})|^2 \, f(\omega) \, d\omega.
\]

But \( f(\omega) \in L_2[0,2\pi] \), (why?), and the scalar product in \( L_2[0,2\pi] \) continuous in its variables, hence we conclude:

\[
\lim_{N \to \infty} \mathbb{E} |\zeta_N(\omega')|^2 = 2\pi \int_0^{2\pi} \chi_{[0,\omega')}(\omega) f(\omega) d\omega = 2\pi F(\omega'). \tag{1}
\]
The variance of \( \int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega} \), II.

By similar arguments we get for any pair of integers \( 0 < M, N \):

\[
\mathbb{E} |\zeta_M(\omega') - \zeta_N(\omega')|^2 = \frac{1}{2\pi} \int_0^{2\pi} |C_M(e^{i\omega}) - C_N(e^{i\omega})|^2 f(\omega) \, d\omega.
\]

Recall that \( C_N(e^{i\omega}) \) converges in \( L_2[0, 2\pi) \) (to \( \chi_{[0,\omega']}(\omega) \)), hence it is a Cauchy sequence in \( L_2[0, 2\pi) \). Assuming that \( f(.) \) is bounded infer that \( C_N(e^{i\omega}) \) is a Cauchy sequence also in \( L_2[0, 2\pi, dF) \)!

But then \( \zeta_N(\omega') \) itself is a Cauchy sequence in \( L_2(\Omega, \mathcal{F}, \mathcal{P}) \), and thus

\[
\lim_{N \to \infty} \zeta_N(\omega') =: \zeta_N(\omega') \quad \text{in} \quad L_2(\Omega, \mathcal{F}, \mathcal{P}). \quad (2)
\]
Orthogonality of \[ \int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega}, \]

By similar arguments, we get for \(0 \leq a < b < 2\pi\):

\[ \zeta_N(b) - \zeta_N(a) = \int_{a}^{b} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega = \sum_{n=-N}^{N} y_n g_n. \]

\[ G_N(e^{i\omega}) = \sum_{n=-N}^{N} g_n e^{in\omega} \to 2\pi \chi_{[a,b]}(\omega). \]

Taking another interval \([c, d]\), we can write

\[ \zeta_N(d) - \zeta_N(c) = \int_{c}^{d} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega = \sum_{n=-N}^{N} y_n h_n \]

\[ H_N(e^{i\omega}) = \sum_{n=-N}^{N} h_n e^{in\omega} \to 2\pi \chi_{[c,d]}(\omega). \]
Orthogonality of \[ \int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega}, \quad \text{II.} \]

Take now 2 non-overlapping intervals \([a, b]\) and \([c, d]\).

Applying Herglotz’s theorem to FIR filtered processes, we get

\[
\mathbb{E} (\zeta_N(b) - \zeta_N(a)) (\zeta_N(d) - \zeta_N(c)) = \frac{1}{2\pi} \int_0^{2\pi} G_N(e^{i\omega}) H_N(e^{-i\omega}) dF(\omega) .
\]

Taking limit for \(N\), and noting that \(\chi_{[a,b]}(\omega) \cdot \chi_{[c,d]}(\omega) = 0\), we get

\[
\mathbb{E} \zeta(a)(\zeta(a) - \zeta(b))(\zeta(d) - \zeta(c)) = 0.
\]

I.e. the process \(\zeta(.)\) has orthogonal increments.
A spectral representation measure

**Theorem**

Let $y = (y_n)$ be a w.s.s.t. process with auto-covariance function such that

$$\sum_{\tau=0}^{\infty} r^2(\tau) < \infty \quad \text{and} \quad f(\omega) := \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i\tau \omega} \leq K$$

for all $\omega \in [0, 2\pi)$. Then, with convergence meant in $L_2^c(\Omega, \mathcal{F}, P)$:

$$\lim_{N \to \infty} \int_{0}^{\omega'} \sum_{n=-N}^{N} y_n e^{-in\omega} \ d\omega =: \zeta(\omega'),$$

where $\zeta(\omega')$ is a process with orthogonal increments. Moreover, denoting the spectral distribution function of $y = (y_n)$ by $F(.)$ we have

$$\mathbb{E} |\zeta(\omega')|^2 = 2\pi F(\omega').$$
Random measures
with orthogonal increments
In search of spectral representation

The challenge: try to extend the def. of singular processes of the form

\[ y_n = \sum_{k=1}^{m} \xi_k e^{i\omega_k n} \]  

by letting all freq. in \([0, 2\pi)\) to appear. I.e. find a cont. extension of (3).

A formal extension is sought in the form:

\[ y_n = \int_{0}^{2\pi} e^{i\omega n} d\zeta(\omega), \]  

where \(d\zeta(\omega)\) is a random weight or random measure, a substitute for \(\xi_k\).

Recalling the conditions imposed on \(\xi_k\) we define random measures with orthogonal increments, \(d\zeta(\omega)\). The definition is obvious:
Random measures with orth. increments, I.

Definition

A complex valued stochastic process \( \zeta(.) \) defined in \([0, 2\pi)\) is called a process with orthogonal increments, if

\[
\mathbb{E}\zeta(\omega) = 0 \quad \text{and} \quad \mathbb{E}|\zeta(\omega)|^2 =: F(\omega) < \infty \quad \forall \omega \in [0, 2\pi),
\]

and for non-overlapping intervals \([a, b)\) and \([c, d)\) in \([0, 2\pi)\) we have

\[
\zeta(d) - \zeta(c) \perp \zeta(b) - \zeta(a).
\]

We will also assume that \( \zeta(0) = 0 \) and \( \zeta(.) \) is left-cont. in \( L^c(\Omega, F, \mathcal{P}) \).

It follows that \( F(0) = 0 \) and \( F(.) \) is left-continuous. \( F(.) \) is called the structure function of \( \zeta(.) \). The increments of \( \zeta(.) \) define a measure \( d\zeta(\omega) \), see below, called a random measure with orthogonal increments.
Example

Consider the special case: 

\[ y_n = \sum_{k=1}^{m} \xi(\omega_k) e^{in\omega_k} \]

where

\[ \mathbb{E} \xi(\omega_k) = 0 \quad \text{and} \quad \mathbb{E} \xi(\omega_k) \bar{\xi}(\omega_l) = 0 \quad \text{for} \quad k \neq l. \]

Then the process \( \zeta(\omega) \) below has orthogonal increments:

\[ \zeta(\omega) = \sum_{\omega_k \leq \omega} \xi_k. \]
Exercise. **HW**

Prove, that for any $0 \leq a < b < 2\pi$ we have

\[
F(b) - F(a) = \mathbb{E} |\zeta(b) - \zeta(a)|^2.
\]

It follows that $F(.)$ is monotone non-decreasing.

*(Hint: Write $[0, b)$ as the union of $[0, a)$ and $[a, b)$ and apply Pythagoras theorem.)*
Integration
Integration, I.

The problem: how do we define the integral \( \int_0^{2\pi} h(\omega) d\zeta(\omega) \)?

This looks difficult, but it is actually very simple.

First, let \( h(\omega) \) be a (possibly complex-valued) step-function of the form

\[
h(\omega) = \sum_{k=1}^{p} \lambda_k \chi_{[a_k, b_k]}(\omega).
\]

Here the intervals \([a_k, b_k]\) are non-overlapping. Then define:

\[
l(h) := \int_0^{2\pi} h(\omega) d\zeta(\omega) = \sum_{k=1}^{p} \lambda_k (\zeta(b_k) - \zeta(a_k)).
\]

Thus \( l(h) \) is a random variable. Obviously it is an element of \( L_2^\zeta(\Omega, \mathcal{F}, P) \).
Exercise. Let $g$ and $h$ be left-cont. (?) step functions on $[0, 2\pi)$. Then

$$\mathbb{E} I(g) I(h) = \int_{0}^{2\pi} g(\omega) h(\omega) \, dF(\omega).$$  \hspace{1cm} (5)$$

(Hint: Take a common subdivision for $g$ and $h$.)

Let $H^c_s$ be the set of $\mathbb{C}$-valued left-continuous step-functions on $[0, 2\pi)$. Then (5) can be restated as saying that stochastic integration as a linear operator $I : H^c_s \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an isometry.
Integration, III.

Let us define the Hilbert-space

\[ L_2^c([0, 2\pi], dF) = \{ h : \int_0^{2\pi} |h(\omega)|^2 dF(\omega) < \infty \}. \]

Obviously \( H_s^c \) is a linear space, and \( H_s^c \subset L_2^c([0, 2\pi], dF) \).

Now recall from functional analysis that \( H_s^c \) is dense in \( L_2^c([0, 2\pi], dF) \).

Hence the isometry \( I \) can be extended from \( H_s^c \) to \( L_2^c([0, 2\pi], dF) \)!
Integration, IV.

Repeat: the isometry $I$ can be extended from $H_*^c$ to $L_*^c([0, 2\pi], dF)$:

Reminder: for any $h \in L_*^c([0, 2\pi], dF)$ take an approximating sequence $(h_n) \subset H_*^c$ s.t. $\lim h_n = h$, and set $I(h) = \lim I(h_n)$.

The extended isometry $I$ from $L_*^c([0, 2\pi], dF)$ to $L_*^c(\Omega, \mathcal{F}, P)$ is called the stochastic integral, and we write

$$I(h) =: \int_0^{2\pi} h(\omega) \, d\zeta(\omega).$$
Isomporphy restated

Let $g$ and $h$ be (possibly $\mathbb{C}$-valued) functions in $L^c_2([0, 2\pi), dF)$. Then

$$\mathbb{E} I(g) \overline{I(h)} = \int_0^{2\pi} g(\omega) \overline{h(\omega)} \, dF(\omega). \quad (6)$$
Just an exercise

Exercise. Let $\zeta(\omega)$ be a random measure with orthogonal increments. Define

$$y_n = \int_0^{2\pi} e^{in\omega} \, d\zeta(\omega).$$

Prove that $(y_n)$ is a wide sense stationary process.
The spectral representation theorem
The spectral representation theorem

The most powerful tool in the theory of w.s.st. processes:

**Theorem**

Let \((y_n)\) be a w.s.st. process. Then \(\exists!\) random measure with orthogonal increments \(d\zeta(\omega)\), such that

\[
y_n = \int_0^{2\pi} e^{in\omega} d\zeta(\omega).
\]

The process \(\zeta(.)\) is called the spectral representation process of \((y_n)\).

Surprise: the theorem covers all processes (recall Wold decomposition!).

The concept of \(d\zeta(\omega)\) is a simplifying abstraction. A bit like \(i := \sqrt{-1}\).
Outline of proof, I.

Let us try to match what is known to what is unknown.

Assume that \((y_n)\) can be represented as stated. Then we have

\[
\mathbb{E}(y_{n+\tau}y_n) = \int_0^{2\pi} e^{i(n+\tau)\omega} e^{-in\omega} dF(\omega) = \int_0^{2\pi} e^{i\tau\omega} dF(\omega),
\]

where \(F(.)\) is the structure function of \(\zeta(.)\).

On the other hand, by Herglotz’s theorem, we can also write

\[
\mathbb{E}(y_{n+\tau}y_n) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tau\omega} dF^y(\omega),
\]

where \(F^y(.)\) is the spectral distribution of \(y\). It follows that

\[
dF(\omega) = dF^y(\omega) \cdot \frac{1}{2\pi}.
\]
Outline of proof, II.

Recall: a random measure with orth. incr $\zeta(.)$ defines a linear mapping, an isometry $I(.)$ from $L^2_\zeta(dF)$ to $L^2_\zeta(\Omega, \mathcal{F}, P)$:

$$I(h) = \int_0^{2\pi} h(\omega) \, d\zeta(\omega).$$

Conversely:

the isometry $I(.)$ itself completely determines the random measure $\zeta(.)$:

$$\zeta(\omega) = \int_0^\omega d\zeta(\omega') = \int_0^{2\pi} \chi_{[0,\omega)}(\omega') \, d\zeta(\omega') = I(\chi_{[0,\omega)}).$$
Outline of proof, III.

In general: for a linear isometry $I(.)$ from $L_2^\xi(dF)$ to $L_2^\xi(\Omega, \mathcal{F}, P)$ define

$$
\zeta(\omega) := I(\chi_{[0,\omega]}).
$$

**Exercise.** Prove that the random measure $\zeta(.)$ defined above has indeed orthogonal increments, moreover its structure function is $F(.)$.

Thus finding $\zeta(.)$ is equivalent to finding the linear isometry $I(.)$.

What is known of $I(.)$? We must have $I(e^{in\omega}) = y_n$.

Let’s take a finite linear combination $g = \sum_n c_ne^{in\omega}$ with $c_n \in \mathbb{C}$, and set

$$
I(g) := \sum_n c_n y_n.
$$
Outline of proof, IV.

Repeat: for an arbitrary finite linear combination \( g = \sum_n c_n e^{in\omega} \) we set

\[ I(g) := \sum_n c_n y_n. \]

Claim: the extension of \( I(.) \) is well-defined, i.e. \( I(g) \) is independent of the representation of \( g \). Equivalently:

\[ g = 0 \text{ in } L_2^c(dF) \Rightarrow I(g) = 0 \text{ in } L_2^c(\Omega, \mathcal{F}, P). \]

**Exercise.** Prove the above implication.

Thus \( I(.) \) is well-defined on a dense subset of \( L_2^c(dF) \).

Extend it by continuity to a linear isometry defined on the whole \( L_2^c(dF) \).
Linear filters
Linear filters, I.

Question: what is the effect of linear filters on the spectral repr. process?

Let \((u_n)\) be a w.s.st. process with spectral representation process \(d \zeta^u(\omega)\).

Define the process \((y_n)\) via a FIR filter as

\[
y_n = \sum_{k=0}^{m} h_k u_{n-k}.
\]

Then \((y_n)\) is a wide sense stationary process. We can write:

\[
y_n = \sum_{k=0}^{m} h_k \int_0^{2\pi} e^{i(n-k)\omega} d \zeta^u(\omega) = \int_0^{2\pi} \left( e^{in\omega} \sum_{k=0}^{m} h_k e^{-ik\omega} \right) d \zeta^u(\omega).
\]
Linear filters, II.

Repeat and continue:

\[ y_n = \int_0^{2\pi} \left( e^{i\omega} \sum_{k=0}^m h_k e^{-ik\omega} \right) d\zeta^u(\omega) = \int_0^{2\pi} \left( e^{i\omega} H(e^{-i\omega}) \right) d\zeta^u(\omega), \]

where \( H(e^{-i\omega}) := \sum_{k=0}^m h_k e^{-ik\omega} \). Question: can we re-bracket and write

\[ \int_0^{2\pi} \left( e^{i\omega} H(e^{-i\omega}) \right) d\zeta^u(\omega) = \int_0^{2\pi} e^{i\omega} \left( H(e^{-i\omega}) d\zeta^u(\omega) \right) ? \]

Does it follow that

\[ d\zeta^y(\omega) = H(e^{-i\omega}) d\zeta^u(\omega) ? \]

And what is \( H(e^{-i\omega}) d\zeta^u(\omega) ? \)
Let $\zeta(\cdot)$ be a random measure with orthogonal increments on $[0, 2\pi)$ with structure function $F$. Let $g \in L_2^2(dF)$, and define

$$\eta(\omega) = \int_0^\omega g(\omega') \, d\zeta(\omega') \quad 0 \leq \omega < 2\pi.$$ 

**Exercise.** Show that $\eta(\omega)$ is a random measure with orthogonal increments with structure function

$$G(\omega) = \int_0^\omega |g(\omega')|^2 \, dF(\omega') \quad \text{or} \quad dG(\omega) = |g(\omega)|^2 \, dF(\omega).$$

The random measure with orthogonal increments corresponding to $\eta(\omega)$ will be written as

$$d\eta(\omega) = g(\omega) \, d\zeta(\omega).$$
Change of measure, II.

\[ d\eta(\omega) = g(\omega) \, d\zeta(\omega) \]  has structure function  \[ dG(\omega) = |g(\omega)|^2 \, dF(\omega). \]

Let now \( h(\omega) \) be a function in \( L_2^c(dG) \) (rather than \( L_2^c(dF) \)). Then

\[ \int_0^{2\pi} h(\omega) \, d\eta(\omega) \]

is well-defined. We have the following, intuitively obvious-looking result:

**Proposition**

We have

\[ \int_0^{2\pi} h(\omega) \, d\eta(\omega) = \int_0^{2\pi} \left( h(\omega) \cdot g(\omega) \right) \, d\zeta(\omega). \]

**Exercise:** Verify the above when \( h \) is a characteristic function \( \chi_{[0,a]}(\omega) \).
THE END of LECTURE 5.