Stochastic Signals and Systems

Lecture 5.

Spectral theory, III.

6. November 2020.

REMINDER:

Introducing spectral representation

Interpreting $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, I.

Let us return to the interpretation of the FT of a w.s.st. process (y_n):

$$\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}.$$

Consider a special case: $y_n = \xi e^{in\,\omega_0}$ with $\mathbb{E}\,\xi = 0$, and interpret

$$\sum_{n=-\infty}^{\infty} \xi e^{in\,(\omega_0-\omega)}.$$

Recall the formal Fourier series of the Dirac-delta function $\delta(\omega-\omega_0)$:

$$\int_0^{2\pi} \delta(\omega-\omega_0)\, e^{in\,\omega}\, d\omega = e^{in\,\omega_0}$$

formally implies

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\,\omega_0}\, e^{-in\,\omega} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\,(\omega_0-\omega)} = \delta(\omega-\omega_0).$$

Interpreting $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, II.

Summarizing we get:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \xi e^{in\,(\omega_0-\omega)} = \xi\,\delta(\omega-\omega_0).$$

Consider now another special case: $y_n = \sum_{k=1}^{m} \xi(\omega_k)\,e^{in\,\omega_k}$ where

$$\mathbb{E}\,\xi(\omega_k) = 0 \quad \text{and} \quad \mathbb{E}\,\xi(\omega_k)\,\overline{\xi}(\omega_l) = 0 \quad \text{for} \quad k \neq l.$$

Extending the above formal argument we get (formally):

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} y_n\,e^{-in\omega} = \sum_{k=1}^{m} \xi(\omega_k)\,\delta(\omega-\omega_k).$$

Note: the Dirac delta function $\delta(\omega-\omega_k)$ is a generalized function,

but its integral is an ordinary function: the unit step function $H(\omega-\omega_k)$.

The integral of $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, I.

We are motivated to consider the integrated (and truncated) process

$$\zeta_N(\omega') = \int_0^{\omega'} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega.$$

Rewrite the r.h.s. as

$$\sum_{n=-N}^{N} y_n \int_0^{\omega'} e^{-in\omega} \, d\omega = \sum_{n=-N}^{N} y_n c_n$$

Here

$$c_n = \int_0^{\omega'} e^{-in\omega} \, d\omega = \int_0^{2\pi} \chi_{[0,\omega')}(\omega) \, e^{-in\omega} d\omega,$$

where $\chi_{[0,\omega')}(.)$ is the characteristic function of the interval $[0, \omega')$.

The integral of $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, II.

Repeat: we have

$$c_n = \int_0^{\omega'} e^{-in\omega}\, d\omega = \int_0^{2\pi} \chi_{[0,\omega')}(\omega)\, e^{-in\omega} d\omega.$$

Thus $\frac{1}{\sqrt{2\pi}} c_n$ is the Fourier coefficients of $\chi_{[0,\omega')}(.)$. Hence setting

$$C_N(e^{i\omega}) = \sum_{n=-N}^{N} c_n e^{in\omega} = 2\pi \sum_{n=-N}^{N} \frac{c_n}{(2\pi)^{1/2}}\ \frac{e^{in\omega}}{(2\pi)^{1/2}},$$

and letting $N \to \infty$ we get, with convergence in $L_2[0, 2\pi)$,

$$\lim_{N\to\infty} C_N(e^{i\omega}) = 2\pi\, \chi_{[0,\omega')}(\omega)$$

The variance of $\int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, I.

$$\zeta_N(\omega') = \int_0^{\omega'} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega = \sum_{n=-N}^{N} y_n c_n$$

$$C_N(e^{i\omega}) = \sum_{n=-N}^{N} c_n e^{in\omega} \to 2\pi \, \chi_{[\,0,\omega')}(\omega).$$

Let us now compute $\mathbb{E}\,|\zeta_N(\omega')|^2$. We have by Herglotz's theorem

$$\mathbb{E}\,|\zeta_N(\omega')|^2 = \mathbb{E}\left|\sum_{n=-N}^{N} y_n c_n\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} |C_N(e^{i\omega})|^2 \, f(\omega)\, d\omega.$$

But $f(\omega) \in L_2\,[\,0,2\pi)$, (why?), and the scalar product in $L_2[\,0,2\pi)$ continuous in its variables, hence we conclude:

$$\lim_{N\to\infty} \mathbb{E}\,|\zeta_N(\omega')|^2 = 2\pi \int_0^{2\pi} \chi_{[\,0,\omega')}(\omega)\, f(\omega) d\omega = 2\pi F(\omega'). \qquad (1)$$

The variance of $\int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, II.

By similar arguments we get for any pair of integers $0 < M, N$:

$$\mathbb{E}\,|\zeta_M(\omega') - \zeta_N(\omega')|^2 = \frac{1}{2\pi}\int_0^{2\pi} |C_M(e^{i\omega}) - C_N(e^{i\omega})|^2 \, f(\omega)\, d\omega.$$

Recall that $C_N(e^{i\omega})$ converges in $L_2[0, 2\pi)$ (to $\chi_{[0,\omega')}(\omega)$), hence it is a Cauchy sequence in $L_2[0, 2\pi)$. Assuming that $f(.)$ is bounded infer that $C_N(e^{i\omega})$ is a Cauchy sequence also in $L_2[0, 2\pi, dF)$!

But then $\zeta_N(\omega')$ itself is a Cauchy sequence in $L_2(\Omega, \mathcal{F}, \mathcal{P})$, and thus

$$\lim_{N\to\infty} \zeta_N(\omega') =: \zeta_N(\omega') \quad \text{in} \quad L_2(\Omega, \mathcal{F}, \mathcal{P}). \tag{2}$$

Orthogonality of $\int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, I.

By similar arguments, we get for $0 \leq a < b < 2\pi$:

$$\zeta_N(b) - \zeta_N(a) = \int_a^b \sum_{n=-N}^{N} y_n \, e^{-in\omega} \; d\omega = \sum_{n=-N}^{N} y_n \, g_n .$$

$$G_N(e^{i\omega}) = \sum_{n=-N}^{N} g_n \, e^{in\omega} \; \to \; 2\pi \, \chi_{[\, a,b)}(\omega).$$

Taking another interval $[c, d)$, we can write

$$\zeta_N(d) - \zeta_N(c) = \int_c^d \sum_{n=-N}^{N} y_n \, e^{-in\omega} \; d\omega = \sum_{n=-N}^{N} y_n h_n$$

$$H_N(e^{i\omega}) = \sum_{n=-N}^{N} h_n e^{in\omega} \; \to \; 2\pi \, \chi_{[\, c,d)}(\omega).$$

Orthogonality of $\int \sum_{n=-\infty}^{\infty} y_n e^{-in\omega}$, II.

Take now 2 non-overlapping intervals $[a, b)$ and $[c, d)$.

Applying Herglotz's theorem to FIR filtered processes, we get

$$\mathbb{E}\ (\zeta_N(b) - \zeta_N(a))\ \left(\overline{\zeta}_N(d) - \overline{\zeta}_N(c)\right) = \frac{1}{2\pi} \int_0^{2\pi} G_N(e^{i\omega})\, H_N(e^{-i\omega})\ dF(\omega).$$

Taking limit for N, and noting that $\chi_{[\,a,b)}(\omega) \cdot \chi_{[\,c,d)}(\omega) = 0,$ we get

$$\mathbb{E}\,\zeta(a)(\zeta(a) - \zeta(b))(\overline{\zeta}(d) - \overline{\zeta}(c)) = 0.$$

I.e. the process $\zeta(.)$ has orthogonal increments.

A spectral representation measure

Theorem

Let $y = (y_n)$ be a w.s.st. process with auto-covariance function such that

$$\sum_{\tau=0}^{\infty} r^2(\tau) < \infty \quad \text{and} \quad f(\omega) := \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i\tau\omega} \leq K$$

for all $\omega \in [0, 2\pi)$. Then, with convergence meant in $L_2^c(\Omega, \mathcal{F}, P)$:

$$\lim_{N\to\infty} \int_0^{\omega'} \sum_{n=-N}^{N} y_n e^{-in\omega} \, d\omega =: \zeta(\omega'),$$

where $\zeta(\omega')$ is a process with orthogonal increments. Moreover, denoting the spectral distribution function of $y = (y_n)$ by $F(.)$ we have

$$\mathbb{E} \, |\zeta(\omega')|^2 = 2\pi \, F(\omega').$$

Random measures with orthogonal increments

In search of spectral representation

The challenge: try to extend the def. of singular processes of the form

$$y_n = \sum_{k=1}^{m} \xi_k e^{i\omega_k n} \qquad (3)$$

by letting all freq. in $[0, 2\pi)$ to appear. I.e. find a cont. extension of (3).

A formal extension is sought in the form:

$$y_n = \int_0^{2\pi} e^{i\omega n} d\zeta(\omega), \qquad (4)$$

where $d\zeta(\omega)$ is a random weight or random measure, a substitute for ξ_k.

Recalling the conditions imposed on ξ_k we define random measures with orthogonal increments, $d\zeta(\omega)$. The definition is obvious:

Random measures with orth. increments, I.

Definition

A complex valued stochastic process $\zeta(.)$ defined in $[0, 2\pi)$ is called a process with orthogonal increments, if

$$\mathbb{E}\zeta(\omega) = 0 \quad \text{and} \quad \mathbb{E}\left|\zeta(\omega)\right|^2 =: F(\omega) < \infty \quad \forall \omega \in [0, 2\pi),$$

and for non-overlapping intervals $[a, b)$ and $[c, d)$ in $[0, 2\pi)$ we have

$$\zeta(d) - \zeta(c) \perp \zeta(b) - \zeta(a).$$

We will also assume that $\zeta(0) = 0$ and $\zeta(.)$ is left-cont. in $L^c(\Omega, \mathcal{F}, \mathcal{P})$.

It follows that $F(0) = 0$ and $F(.)$ is left-continuous. $F(.)$ is called the structure function of $\zeta(.)$. The increments of $\zeta(.)$ define a measure $d\zeta(\omega)$, see below, called a random measure with orthogonal increments.

Example

Consider the special case: $y_n = \sum_{k=1}^{m} \xi(\omega_k)\, e^{in\,\omega_k}$ where

$$\mathbb{E}\,\xi(\omega_k) = 0 \quad \text{and} \quad \mathbb{E}\,\xi(\omega_k)\,\bar{\xi}(\omega_l) = 0 \quad \text{for} \quad k \neq l.$$

Then the process $\zeta(\omega)$ below has orthogonal increments:

$$\zeta(\omega) = \sum_{\omega_k \leq \omega} \xi_k.$$

Random measures with orth. increments, II.

Exercise. **HW**

Prove, that for any $0 \leq a < b < 2\pi$ we have

$$F(b) - F(a) = \mathbb{E} \left|\zeta(b) - \zeta(a)\right|^2 .$$

It follows that $F(.)$ is monotone non-decreasing.

(*Hint:* Write $[0, b)$ as the union of $[0, a)$ and $[a, b)$ and apply Pythagoras theorem.)

Integration

Integration, I.

The problem: how do we define the integral $\int_0^{2\pi} h(\omega)\, d\zeta(\omega)$?

This looks difficult, but it is actually very simple.

First, let $h(\omega)$ be a (possibly complex-valued) step-function of the form

$$h(\omega) = \sum_{k=1}^{p} \lambda_k \, \chi_{[a_k, b_k)}(\omega).$$

Here the intervals $[a_k, b_k)$ are non-overlapping. Then define:

$$I(h) := \int_0^{2\pi} h(\omega)\, d\zeta(\omega) = \sum_{k=1}^{p} \lambda_k(\zeta(b_k) - \zeta(a_k)).$$

Thus $I(h)$ is a <u>random variable</u>. Obviously it is an element of $L_2^c(\Omega, \mathcal{F}, P)$.

Integration, II.

Exercise. Let g and h be left-cont. (?) step functions on $[0, 2\pi)$. Then

$$\mathbb{E}\, I(g)\, \overline{I(h)} = \int_0^{2\pi} g(\omega)\overline{h(\omega)}\, dF(\omega). \tag{5}$$

(*Hint*: Take a common subdivision for g and h.)

Let H_s^c be the set of $\mathbb{C}$-valued left-continuous step-functions on $[0, 2\pi)$.

Then (5) can be restated as saying that stochastic integration

as a linear operator $I \ : \ H_s^c \to L_2^c(\Omega, \mathcal{F}, P)$ is an isometry.

Integration, III.

Let us define the Hilbert-space

$$L_2^c([0, 2\pi], dF) = \{h : \int_0^{2\pi} |h(\omega)|^2 \, dF(\omega) < \infty\}.$$

Obviously H_s^c is a linear space, and $H_s^c \subset L_2^c([0, 2\pi], dF)$.

Now recall from functional analysis that H_s^c is <u>dense</u> in $L_2^c([0, 2\pi], dF)$.

Hence the isometry I can be extended from H_s^c to $L_2^c([0, 2\pi], dF)$!

Integration, IV.

Repeat: the isometry I can be extended from H_s^c to $L_2^c([0,2\pi],dF)$:

Reminder: <u>for any</u> $h \in L_2^c([0,2\pi),dF)$ take an approximating sequence $(h_n) \subset H_s^c$ s.t. $\lim h_n = h$, and set $I(h) = \lim_n I(h_n).$

The extended isometry I from $L_2^c([\,0,\,2\pi),dF)$ to $L_2^c(\Omega,\mathcal{F},P)$ is called the <u>stochastic integral</u>, and we write

$$I(h) =: \int_0^{2\pi} h(\omega)\,d\zeta(\omega).$$

Isomporphy restated

Let g and h be (possibly $\mathbb{C}$-valued) functions in $L_2^c([0, 2\pi), dF)$. Then

$$\mathbb{E}\, I(g)\, \overline{I(h)} = \int_0^{2\pi} g(\omega)\overline{h(\omega)}\, dF(\omega). \tag{6}$$

Just an exercise

Exercise. Let $\zeta(\omega)$ be a random measure with orthogonal increments. Define

$$y_n = \int_0^{2\pi} e^{in\omega} d\zeta(\omega).$$

Prove that (y_n) is a wide sense stationary process.

The spectral representation theorem

The spectral representation theorem

The most powerful tool in the theory of w.s.st. processes:

Theorem

Let (y_n) be a w.s.st. process. Then $\exists!$ random measure with orthogonal increments $d\zeta(\omega)$, such that

$$y_n = \int_0^{2\pi} e^{in\omega} d\zeta(\omega).$$

The process $\zeta(.)$ is called *the spectral representation process* of (y_n).

Surprise: the theorem covers all processes (recall Wold decomposition!).

The concept of $d\zeta(\omega)$ is a simplifying abstraction. A bit like $i := \sqrt{-1}$.

Outline of proof, I.

Let us try to match what is known to what is unknown.

Assume that (y_n) can be represented as stated. Then we have

$$\mathbb{E}\left(y_{n+\tau}\overline{y_n}\right) = \int_0^{2\pi} e^{i(n+\tau)\omega} e^{-in\omega} dF(\omega) = \int_0^{2\pi} e^{i\tau\omega} dF(\omega),$$

where $F(.)$ is the structure function of $\zeta(.)$.

On the other hand, by Herglotz's theorem, we can also write

$$\mathbb{E}\left(y_{n+\tau}\overline{y_n}\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tau\omega} dF^y(\omega),$$

where $F^y(.)$ is the spectral distribution of y. It follows that

$$dF(\omega) = dF^y(\omega) \cdot \frac{1}{2\pi}.$$

Outline of proof, II.

Recall: a random measure with orth. incr $\zeta(.)$ defines a linear mapping, an isometry $I(.)$ from $L_2^c(dF)$ to $L_2^c(\Omega, \mathcal{F}, P)$:

$$I(h) = \int_0^{2\pi} h(\omega) \, d\zeta(\omega).$$

Conversely:

the isometry $I(.)$ itself completely determines the random measure $\zeta(.)$:

$$\zeta(\omega) = \int_0^{\omega} d\zeta(\omega') = \int_0^{2\pi} \chi_{[0,\omega)}(\omega') \, d\zeta(\omega') = I(\chi_{[0,\omega)}).$$

Outline of proof, III.

In general: for a linear isometry $I(.)$ from $L_2^c(dF)$ to $L_2^c(\Omega, \mathcal{F}, P)$ define

$$\zeta(\omega) := I(\chi_{[0,\omega)}).$$

Exercise. Prove that the random measure $\zeta(.)$ defined above has indeed orthogonal increments, morover its structure function is $F(.)$.

Thus finding $\zeta(.)$ is equivalent to finding the linear isometry $I(.)$.

What is known of $I(.)$? We must have $I(e^{in\omega}) = y_n$!

Let's take a finite linear combination $g = \sum_n c_n e^{in\omega}$ with $c_n \in \mathbb{C}$, and set

$$I(g) := \sum_n c_n y_n.$$

Outline of proof, IV.

Repeat: for an arbitrary finite linear combination $g = \sum_n c_n e^{in\omega}$ we set

$$I(g) := \sum_n c_n y_n.$$

Claim: the extension of $I(.)$ is well-defined, i.e. $I(g)$ is independent of the representation of g. Equivalently:

$$g = 0 \quad \text{in} \quad L_2^c(dF) \quad \Rightarrow \quad I(g) = 0 \quad \text{in} \quad L_2^c(\Omega, \mathcal{F}, P).$$

Exercise. Prove the above implication.

Thus $I(.)$ is well-defined on a dense subset of $L_2^c(dF)$.

Extend it by continuity to a linear isometry defined on the whole $L_2^c(dF)$.

Linear filters

Linear filters, I.

Question: what is the effect of *linear filters* on the spectral repr. process?

Let (u_n) be a w.s.st. process with spectral representation process $d\zeta^u(\omega)$.

Define the process (y_n) via a FIR filter as

$$y_n = \sum_{k=0}^{m} h_k u_{n-k}.$$

Then (y_n) is a wide sense stationary process. We can write:

$$y_n = \sum_{k=0}^{m} h_k \int_0^{2\pi} e^{i(n-k)\omega}\, d\zeta^u(\omega) = \int_0^{2\pi} \left(e^{in\omega} \sum_{k=0}^{m} h_k e^{-ik\omega} \right) d\zeta^u(\omega).$$

Linear filters, II.

Repeat and continue:

$$y_n = \int_0^{2\pi} \left(e^{in\omega} \sum_{k=0}^{m} h_k e^{-ik\omega} \right) \, d\zeta^u(\omega) = \int_0^{2\pi} \left(e^{in\omega} H(e^{-i\omega}) \right) \, d\zeta^u(\omega),$$

where $H(e^{-i\omega}) := \sum_{k=0}^{m} h_k e^{-ik\omega}$. Question: can we re-bracket and write

$$\int_0^{2\pi} \left(e^{in\omega} H(e^{-i\omega}) \right) \, d\zeta^u(\omega) = \int_0^{2\pi} e^{in\omega} \left(H(e^{-i\omega}) \, d\zeta^u(\omega) \right) ?$$

Does it follow that

$$d\zeta^y(\omega) = H(e^{-i\omega}) \, d\zeta^u(\omega) ?$$

And what is $H(e^{-i\omega}) \, d\zeta^u(\omega)$?

Change of measure, I.

Let $\zeta(.)$ be a random measure with orthogonal increments on $[0, 2\pi)$ with structure function F. Let $g \in L_2^c(dF)$, and define

$$\eta(\omega) = \int_0^\omega g(\omega')\, d\zeta(\omega') \qquad 0 \leq \omega < 2\pi.$$

Exercise. Show that $\eta(\omega)$ is a random measure with orthogonal increments with structure function

$$G(\omega) = \int_0^\omega |g(\omega')|^2 dF(\omega') \quad \text{or} \quad dG(\omega) = |g(\omega)|^2 dF(\omega).$$

The random measure with orthogonal increments corresponding to $\eta(\omega)$ will be written as

$$d\eta(\omega) = g(\omega)\, d\zeta(\omega).$$

Change of measure, II.

$d\eta(\omega) = g(\omega)\, d\zeta(\omega)$ has structure function $dG(\omega) = |g(\omega)|^2\, dF(\omega).$

Let now $h(\omega)$ be a function in $L_2^c(dG)$ (rather than $L_2^c(dF)$!). Then

$$\int_0^{2\pi} h(\omega)\, d\eta(\omega)$$

is well-defined. We have the following, intuitively obvious-looking result:

Proposition

We have

$$\int_0^{2\pi} h(\omega)\ d\eta(\omega) = \int_0^{2\pi} \Big(h(\omega) \cdot g(\omega) \Big)\ d\zeta(\omega).$$

Exercise: Verify the above when h is a characteristic function $\chi_{[0,a)}(\omega)$.

THE END of LECTURE 5.