Quasi-polynomial and Lotka-Volterra representation in nonlinear systems and control theory

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Nonlinear input-affine systems in quasi-polynomial (QP) and Lotka-Volterra (LV) forms are investigated in this paper. It is shown that both the global stability analysis with an entropy-like Lyapunov function candidate and the local quadratic stability region determination can be performed by solving linear matrix inequalities (LMIs). The invariance transformations preserving the form of the description are also described and their use for stability analysis is discussed. It is also shown that zero dynamics analysis can be performed by solving LMIs but the design of globally stabilizing feedback controllers leads to a bilinear matrix inequality (BMI) problem.

1 Introduction

The class of quasi-polynomial (QP) systems plays an increasingly important role in the modelling of dynamical systems since the majority of smooth nonlinear systems occurring in practice can be easily transformed to QP form [12]. The QP and LV description forms are shown to be invariant under certain nonlinear state transformations, therefore this nonlinear system class can be split into equivalence classes with the same dynamical properties.

At the same time, the stability properties of QP systems have been intensively studied recently [5], [10], and a simple but physically motivated Lyapunov function candidate has also been proposed. Furthermore, some computationally effective numerical methods have been developed lately, that allow us to practically perform the stability analysis of QP systems [8].

However, almost all of the literature on QP and LV systems and their stability properties are only about their representation and analysis, while a systematic explosion of their control theoretical properties is missing. Therefore, this paper aims to give a short overview about QP systems and their possible use in nonlinear systems and control theory.

2 Basic notions

2.1 QP and LV systems

Let $[W]_{i,j}$ denote the element at the i-th row and j-th column of a matrix W. Quasi-polynomial (QP) systems are systems of ODEs of the following form

$$\dot{y}_i = y_i \left(l_i + \sum_{j=1}^{m} [M]_{i,j} \prod_{k=1}^{n} y_k^{[B]_{j,k}} \right), \quad i = 1, \ldots, n. \tag{1}$$

where $y \in \text{int}(\mathbb{R}_+^n)$, $M \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, $l_i \in \mathbb{R}$, $i = 1, \ldots, n$. Furthermore, $l = [l_1 \ \ldots \ l_n]^T$. Without the loss of generality we can assume that $\text{Rank}(B) = n$ and $m \geq n$ (see [12]).

It is important to note that QP systems are *autonomous systems* in systems and control theoretical point of view.

Let us denote the *monomials* of (1) as

$$z_j = \prod_{k=1}^{n} y_k^{[B]_{j,k}}, \quad j = 1, \ldots, m. \tag{2}$$

Let $z = [z_1 \ z_2 \ \ldots \ z_m]^T$. It can be easily calculated that the time derivatives of the monomials form a Lotka-Volterra (LV) system i.e.

$$\dot{z}_i = z_i(\lambda_i + \sum_{j=1}^{m}[A]_{i,j} \cdot z_j), \quad i = 1,\ldots,m \tag{3}$$

where $A = B \cdot M \in \mathbb{R}^{m \times m}$, $\lambda = B \cdot l \in \mathbb{R}^{m \times 1}$, $\lambda_i = [\lambda]_i$, and $z_i > 0$, i=1,..., m.

Let us denote the equilibrium point of interest of (3) with

$$z^* = [z_1^* \ z_2^* \ \ldots \ z_m^*]^T \in \mathrm{int}(\mathbb{R}_+^m)$$

We note that the matrix A of an LV system originating from a QP system is often rank deficient since the number of monomials is larger than the number of QP variables in many cases. It is visible that LV systems form a proper subset of QP systems with B being the unit matrix of size $m \times m$.

It is often useful to represent (3) in its *homogeneous* form. This form can be obtained by introducing a new variable z_{m+1}, such that $\dot{z}_{m+1} = 0$ and $z_{m+1}(0) = 1$. Using the new variable, (3) can be written as

$$\dot{z}_i = z_i \left(\sum_{j=1}^{m+1}[E]_{i,j}z_j \right), \quad i = 1,\ldots,m+1 \tag{4}$$

with

$$E = \begin{bmatrix} A & \lambda \\ 0 & 0 \end{bmatrix} \tag{5}$$

2.2 Rewriting non-QP systems into QP form

A set of nonlinear ODEs can be embedded to QP form if the non-QP elements are multiplicative functions f appearing in the QP-terms and a QP-type ODE can be found such that f is a solution of it [11].

The embedding is performed by introducing a *new auxiliary variable x* for each non-QP function f which is in the simplest case $x = f$. One can differentiate this algebraic equation in order to arrive at a new ODE in QP-form that completes the embedded QP-ODE model.

It is important to note that the embedding is not unique, because we can choose the new variables in a different, more complicated way as compared to $x = f$.

2.3 Linear and bilinear matrix inequalities

A (nonstrict) linear matrix inequality (LMI) is an inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0, \tag{6}$$

where $x \in \mathbb{R}^m$ is the variable and $F_i \in \mathbb{R}^{n \times n}$, $i = 0,\ldots,m$ are given symmetric matrices. The inequality symbol in (6) stands for the positive semidefiniteness of $F(x)$.

One of the most important properties of LMIs is the fact, that they form a convex constraint on the variables i.e. the set $\{x \mid F(x) \geq 0\}$ is convex. LMIs have been playing an increasingly important role in the field of optimization and control theory since a wide variety of different problems (linear and convex quadratic inequalities, matrix norm inequalities, convex constraints etc.) can be written as LMIs and there are computationally stable and effective (polynomial time) algorithms for their solution [2], [16].

A *bilinear matrix inequality (BMI)* is a diagonal block composed of q matrix inequalities of the following form

$$G_0^i + \sum_{k=1}^{p} x_k G_k^i + \sum_{k=1}^{p}\sum_{j=1}^{p} x_k x_j K_{kj}^i \leq 0, \tag{7}$$

$$i = 1,\ldots,q$$

where $x \in \mathbb{R}^p$ is the decision variable to be determined and G_k^i, $k = 0,\ldots,p$, $i = 1,\ldots,q$ and K_{kj}^i, $k,j = 1,\ldots,p$, $i = 1,\ldots,q$ are symmetric, quadratic matrices.

The main properties of BMIs are that they are non-convex in x (which makes their solution numerically much more complicated than that of linear matrix inequalities), and their solution is NP-hard [13]. However, there exist practically applicable and effective algorithms for BMI solution [21], [14].

3 Stability analysis of QP and LV systems

3.1 Global stability with the entropy-like Lyapunov function

Most often, the following, so-called "entropy-like" Lyapunov function candidate is used for examining the global stability of QP systems

$$V(z) = \sum_{i=1}^{m} c_i \left(z_i - z_i^* - z_i^* \ln \frac{z_i}{z_i^*} \right) \tag{8}$$

where $c_i > 0, \ i = 1, \ldots, m$.

It can be easily shown that V is nonincreasing (i.e. the equilibrium z^* and y^* is globally stable with Lyapunov function V) if and only if the following linear matrix inequality

$$A^T C + CA \leq 0 \tag{9}$$

can be solved for a positive definite diagonal matrix $C \in \mathbb{R}^{m \times m}$. In this case, the c_i coefficients in (8) are the diagonal elements of C (see, e.g. [7] or [20]) and the matrix A is called *diagonally stabilizable* [1] or *admissible* [5].

We note that the solvability of the LMI (9) and thus the existence and nonincreasing nature of the Lyapunov function (8) is not a necessary condition for the global stability of (1). However, this Lyapunov function is used most frequently for QP systems because it's existence can be tested numerically or even symbolically [15].

We remark that the numerical solution of (9) can be of different difficulty depending on the rank of A. If A is of full rank, then there are several possibilities to check the feasibility and solve (9). One of the most popular tools is the LMI Control Toolbox for the Matlab computing software environment [6]. If A is rank deficient (which is the general case if the number of monomials is greater than the number of QP variables) then the only applicable numerical method known by the authors is described in [8].

3.2 Local quadratic stability

As it was mentioned in the previous section, the diagonal stabilizability of a quadratic matrix is a very special property, so let us consider the more frequent case when there is no diagonal solution for (9). Then one naturally tries to find a quadratic Lyapunov function candidate that is valid locally in some neighborhood of an equilibrium point.

In order to obtain a more comfortable notation for this, let us perform a coordinates shift on the LV-equations, i.e. $x = z - z^*$. Then the LV-equations in the transformed coordinates have the form

$$\dot{x} = (X + Z^*) \cdot A \cdot x, \tag{10}$$

where

$$X = \text{diag}(x_1, \ldots, x_m), \ \ Z^* = \text{diag}(z_1^*, \ldots, z_m^*) \tag{11}$$

and the equilibrium value of x moves to the origin. Let the quadratic Lyapunov function candidate be given in the following form:

$$V(x) = x^T P x \tag{12}$$

where P is a positive definite symmetric matrix of size $m \times m$. The time derivative of V is given by

$$\dot{V} = x^T P \dot{x} + \dot{x}^T P x = \tag{13}$$

$$x^T P(X + Z^*)Ax + x^T A^T (X + Z^*)Px = \tag{14}$$

$$x^T \left\{ P(X + Z^*)A + A^T(X + Z^*)P \right\} x = \tag{15}$$

$$x^T \left\{ PXA + PZ^*A + A^TXP + A^TZ^*P \right\} x \tag{16}$$

The non-increasing nature of the quadratic Lyapunov function in a neighborhood $\mathcal{N}$ of the origin is equivalent to the validity of the following LMI

$$PXA + PZ^*A + A^TXP + A^TZ^*P \leq 0 \tag{17}$$

where $X = \text{diag}(x_1, \ldots, x_m)$ and $[x_1, \ldots, x_m]^T \in \mathcal{N}$.

Therefore the quadratic stability region can be estimated by first solving (17) for P with $X = 0$, i.e.

$$A^T Z^* P + P Z^* A \leq 0 \tag{18}$$

and then finding a convex neighborhood of 0 where (17) is valid.

3.3 Global stability and local Hamiltonian description

Based on [17], an interesting special case of quadratic stability is discussed here. If the P matrix in (12) defining the quadratic Lyapunov function is diagonal, then the LV system can be written locally as a Hamiltonian system with dissipation as it is defined e.g. in [4]. In the autonomous case this system class is defined by the equations

$$\dot{x} = (J(x) - R(x))\mathcal{H}_x^T(x), \tag{19}$$

where $x \in \mathbb{R}^n$, $\mathcal{H} : \mathbb{R}^n \mapsto \mathbb{R}$ is the Hamiltonian function, $J(x)$ is an $n \times n$ skew symmetric matrix (i.e. $J^T(x) = -J(x)$), the energy conserving part of the system, and $R(x) = R^T(x)$ is the so called dissipation matrix. $\mathcal{H}_x$ denotes the gradient of $\mathcal{H}$ (row vector).

The time derivative of the Hamiltonian function is

$$\dot{\mathcal{H}} = \mathcal{H}_x(x)(J(x) - R(x))\mathcal{H}_x^T(x) = \tag{20}$$

$$\underbrace{\mathcal{H}_x(x)J(x)\mathcal{H}_x^T(x)}_{0} - \mathcal{H}_x(x)R(x)\mathcal{H}_x^T(x). \tag{21}$$

It is visible from (21) that if $y^T R(x)y \geq 0$, $\forall y \in \mathbb{R}^n$ (i.e. R is positive semidefinite), then the Hamiltonian function is nonincreasing in time. Of course, this property might be satisfied not globally, but only in some neighborhood of the equilibrium point.

Assume that P in (12) is a diagonal matrix with positive elements in the diagonal. Let us multiply the left hand side of (17) by P^{-1} from the left and $P^{-T} = P^{-1}$ from the right. This operation does not change the definiteness of the left hand side and gives

$$XAP^{-1} + Z^*AP^{-1} + P^{-1}A^T X + P^{-1}A^T Z^* =$$
$$(X + Z^*)AP^{-1} + P^{-1}A^T(X + Z^*)^T < 0 \tag{22}$$

Consider a quadratic Hamiltonian function

$$\mathcal{H}(x) = \frac{1}{2}(p_1 x_1^2 + p_2 x_2^2 + \cdots + p_m x_m^2) \tag{23}$$

where the coefficients $p_i > 0$, $i = 1, \dots m$ are the diagonal elements of P. Furthermore, let us use the notation

$$W(x) = (X + Z^*) \cdot A \cdot P^{-1} \tag{24}$$

Using (24) the original LV model (3) can be written as

$$\dot{x} = W(x) \cdot \mathcal{H}_x^T(x) \tag{25}$$

Therefore the matrices J and R in (19) are the following

$$J(x) = \frac{1}{2}(W(x) - W^T(x)) \tag{26}$$

$$-R(x) = \frac{1}{2}(W(x) + W^T(x)) \tag{27}$$

and the positive definiteness condition on R is equivalent to the feasibility of the LMI (22).

It is well-known that a Hamiltonian description of this kind leads to a simple and elegant solution of many nonlinear analysis and control problems. Note that the diagonality constraint on P can be easily handled in the numerical solution of LMI problems (see, e.g. [16]).

3.3.1 A simple example

Let us consider the LV system characterized by the following matrices

$$A = \begin{bmatrix} -0.5 & 0.1 \\ 1.8 & -1.5 \end{bmatrix}, \ \lambda = \begin{bmatrix} 1.9 \\ -5.7 \end{bmatrix} \tag{28}$$

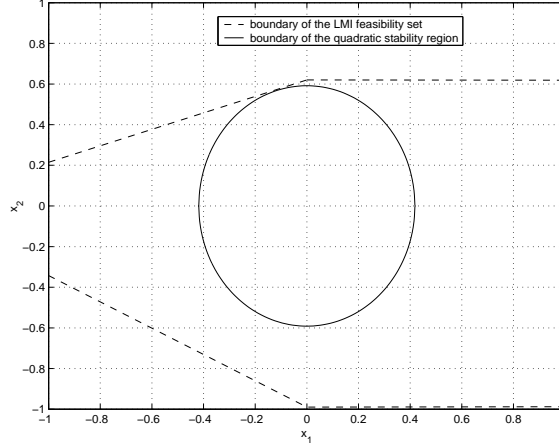The equilibrium of the above system is at

$$z^* = [4 \ 1]^T.$$

Figure 1: LMI feasibility set and quadratic stability region for the system (28)

Let us choose the following weighting matrix

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

It's easy to see that A is diagonally stabilizable with C, since the eigenvalues of $A^T C + CA$ are strictly negative $(-0.4384, -4.5616)$. From this, it's clear that the system (28) is globally stable with the logarithmic Lyapunov-function

$$V(z) = 2(z_1 - 4 - 4\ln\frac{z_1}{4}) + (z_2 - 1 - \ln(z_2)) \tag{29}$$

On the other hand, it's easy to check that the LMI

$$Z^* A P^{-1} + P^{-1} A^T Z^* < 0 \tag{30}$$

(where $Z^* = \mathrm{diag}(z^*)$) has also a diagonal solution

$$P^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

which is identical to C. This means that the system (28) has a local dissipative Hamiltonian description of the form (25) with the quadratic Hamiltonian function (which is also a quadratic Lyapunov function)

$$\mathcal{H}(x) = \frac{1}{2}x_1^2 + x_2^2 \tag{31}$$

where $x = z - z^*$, and $\mathcal{H}_x^T(x) = P \cdot x$. Possible corner points of the convex set where $-R(x) < 0$ are

$$c_1 = (-1.53, \, 0), \quad c_2 = (510, \, 0)$$
$$c_3 = (0, \, -0.99), \quad c_4 = (0, \, 0.62) \tag{32}$$

Note that an estimate for the actual quadratic stability region can be the largest level set of (31) that is inside the polygon defined by the corner points in (32) (see Figure 1).

4 Transformations of QP and LV systems

4.1 Variable scaling

It is clear that under the scaling of differential variables both the QP and the LV form is preserved. Let us assume that (1) has an equilibrium point y^* in the positive orthant. Then with the following variable scaling

$$y_i' = \frac{y_i}{y_i^*}, \quad i = 1, \ldots, n \tag{33}$$

the new equilibrium is at $[1 \ 1 \ \ldots \ 1]^T \in \mathbb{R}^n$.

4.2 The quasi-monomial transformation

The quasi-monomial transformation (QMT) of (1) is defined as

$$y_i' = \prod_{k=1}^{n} y_k^{[C]_{i,k}}, \quad i = 1, \ldots, n \tag{34}$$

where C is an invertible $n \times n$ matrix. The transformed system preserves the QP form and its matrices are given by

$$B' = BC, \ M' = C^{-1}M, \ l' = C^{-1}l \tag{35}$$

The QP systems connected by QM transformations form equivalence classes (Brenig's equivalence classes or BECs), where the products $B \cdot M$ and $B \cdot l$ are invariants of the class [11]. It can be seen from (3) the QP systems belonging to the same BEC can be transformed into the same Lotka-Volterra system. It is shown e.g. in [3] that the inverse of (34) is characterized by the inverse of C, i.e.

$$y_i = \prod_{k=1}^{n} y_k'^{[C^{-1}]_{ik}}, \quad i = 1, \ldots, n \tag{36}$$

4.3 The time-reparametrization transformation

4.3.1 The generic case

Let $\Omega = [\Omega_1 \ \ldots \ \Omega_n]^T \in \mathbb{R}^n$. It is shown e.g. in [5] that the following reparametrization of time

$$dt = \prod_{k=1}^{n} y_k^{\Omega_k} dt' \tag{37}$$

transforms the original QP system (1) into the following (also QP) form

$$\frac{dy_i}{dt'} = y_i \sum_{j=1}^{m+1} [\tilde{M}]_{i,j} \prod_{k=1}^{n} y_k^{[\tilde{B}]_{j,k}}, \ i = 1, \ldots, n \tag{38}$$

where $\tilde{M} \in \mathbb{R}^{n \times (m+1)}$, $\tilde{B} \in \mathbb{R}^{(m+1) \times n}$ and

$$[\tilde{M}]_{i,j} = [M]_{i,j}, \quad i = 1, \ldots, n; \ j = 1, \ldots, m \tag{39}$$

$$[\tilde{M}]_{i,m+1} = l_i, \quad i = 1, \ldots, n \tag{40}$$

and

$$[\tilde{B}]_{i,j} = [B]_{i,j} + \Omega_j, \quad i = 1, \ldots, m; \ j = 1, \ldots, n \tag{41}$$

$$[\tilde{B}]_{m+1,j} = \Omega_j, \quad j = 1, \ldots, n. \tag{42}$$

It can be seen that the number of monomials is increased by one and vector $\tilde{L}$ is zero in the transformed system.

4.3.2 A special (non-generic) case

A special case of the time-reparametrization or new time transformation occurs when the following relation holds:

$$\Omega^T = -b_j, \quad 1 \leq j \leq m, \tag{43}$$

where b_j is an arbitrary row of the matrix B of the original system (1). From Eqs. (41)-(42) we can see that in this case the j-th row of $\tilde{B}$ is a zero vector. This means that the number of monomials in the transformed system (38) remains the same as in the original QP system (1) and a nonzero $\tilde{L}$ vector that is equal to the j-th column of M appears in the transformed system (for an example, see [5]).

4.3.3 The properties of the time-reparametrization transformation

It is shown e.g. in [18] that the transformation (37) has the following important properties.

- The set of monomials $p_1, \ldots, p_{m+1}$ for the reparametrized system can be written up in terms of the original monomials:

$$p_j = \prod_{k=1}^{n} y_k^{\Omega_k} \cdot \prod_{k=1}^{n} y_k^{[B]_{j,k}} = \prod_{k=1}^{n} y_k^{[B]_{j,k}+\Omega_k}, \quad j = 1, \ldots, m$$

and

$$p_{m+1} = \prod_{k=1}^{n} y_k^{\Omega_k}$$

or using a shorter notation:

$$p_j = r \cdot z_j, \quad j = 1, \ldots, m$$

$$p_{m+1} = r$$

where

$$r = \prod_{k=1}^{n} y_k^{\Omega_k} \quad .$$

- The transformation leaves the equilibrium points of the original QP system unchanged.
- Local and global stability is invariant under the transformation.

4.3.4 Using time-reparametrization for proving global stability

The time-reparametrization transformation adds extra degrees of freedom to find a Lyapunov function of the form (8) for proving global stability of the system. As it is shown in [18], the global stability analysis of a QP system with the Lyapunov function (8) and using time-reparametrization requires the feasibility check of the following set of matrix inequalities

$$-C \quad < \quad 0 \tag{44}$$
$$\tilde{A}^T \cdot C + C \cdot \tilde{A} \quad \leq \quad 0 \tag{45}$$

where $\tilde{A} = \tilde{B} \cdot \tilde{M}$ is the LV coefficient matrix of the reparametrized system. It can be seen from (39)-(42) that (44)-(45) is a BMI, where the unknowns are the coefficients of the Lyapunov function contained in C and the parameter vector Ω of the time-reparametrization transformation.

4.4 The logarithmic transformation

The logarithmic coordinates transformation is most often used for studying the set of equilibrium points of QP systems. The transformation is defined as

$$\bar{y}_i = \ln(y_i), \quad i = 1, \ldots, n \tag{46}$$

It is easy to calculate that the differential equations in the transformed coordinates system can be written as

$$\frac{d\bar{y}}{dt} = l + A \cdot z = l + A \cdot \exp(B \cdot \bar{y}) \tag{47}$$

In the case of homogeneous LV systems (4) the transformed equations (47) with $\bar{z} = \ln(z)$ have the following special form

$$\frac{d\bar{z}}{dt} = E \cdot \exp(\bar{z}) \tag{48}$$

Now the structure of the equilibrium points can be studied by examining the kernel of E.

4.5 The family of LV systems having the same phase portrait

Consider an m-dimensional LV-system in the homogeneous form (4) where the variables are scaled so that the equilibrium point z^* is in $\mathbf{1} = [1\ 1\ \ldots\ 1]^T \in \mathbb{R}^m$. Let $k \in \mathbb{R}^m$ such that $\sum_{i=1}^m k_i = 1$. Let us define the following linear change of coordinates (see [9]):

$$\bar{z} = z - (k^T z)\mathbf{1} \tag{49}$$

The time-derivative of $\bar{z}$ is given by

$$\frac{d\bar{z}}{dt} = (E - \mathbf{1}(k^T E))\exp(\bar{z})\exp(k^T z) \tag{50}$$

Let us define the following time-reparametrization transformation

$$dt' = dt \cdot \exp(k^T z) \tag{51}$$

Since

$$\frac{dt'}{dt} = \exp(k^T z), \tag{52}$$

we can see that t' is a strictly monotonously increasing and therefore invertible function of t. Using (51), the model (50) can be rewritten as

$$\frac{d\bar{z}}{dt'} = (E - \mathbf{1}(k^T E))\exp(\bar{z}) \tag{53}$$

The system (53) has the linear first integral $I(\bar{z}) = k^T \bar{z}$. Indeed,

$$k^T \dot{\bar{z}} = (k^T E - k^T \underbrace{\mathbf{1}k^T}_{1} E)\exp(\bar{z}) = 0 \tag{54}$$

The family of systems indexed by k contains m LV systems obtained by taking k as the vectors of the canonical basis in $\mathbb{R}^m$. These differential systems have the same phase portrait on any open bounded domain of the positive orthant [9].

Example

Consider again the 2-dimensional LV system described in (28). It's matrix E in homogeneous form is given by

$$E_1 = \begin{bmatrix} -0.5 & 0.1 & 1.9 \\ 1.8 & -1.5 & -5.7 \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

For $k = [1\ 0\ 0]^T$ the transformed system (after swapping the first and third state variable to place the zero row to the third row of E) is characterized by

$$E_2 = \begin{bmatrix} -1.9 & -0.1 & 0.5 \\ 1.8 & -1.5 & -5.7 \\ 0 & 0 & 0 \end{bmatrix}, \tag{56}$$

while for $k = [0\ 1\ 0]^T$, it is described by

$$E_3 = \begin{bmatrix} -2.3 & 7.6 & 1.6 \\ -1.8 & 5.7 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

It is easy to check that for $k = [0\ 0\ 1]^T$, the transformation leaves E_1 unchanged.

5 Quasi-polynomial control systems

In order to use QP and LV systems for control studies, one first has to extend them by introducing suitable input terms to form so-called input-affine QP control systems.

5.1 Input-affine QP control systems

An input-affine nonlinear system model

$$\begin{aligned} \dot{y} &= f(y) + \sum_{i=1}^{p} g_i(y) u_i \\ \eta &= h(y) \end{aligned} \tag{58}$$

is in QP-form if all of the functions f, g and h are quasi-polynomial functions of y. Then the general form of the state equation of an input-affine QP system model with p-inputs is:

$$\dot{y}_i = y_i \left(l_{0_i} + \sum_{j=1}^{m} [M_0]_{i,j} \prod_{k=1}^{n} y_k^{[B]_{j,k}} \right) + \tag{59}$$

$$+ \sum_{r=1}^{p} y_i \left(l_{r_i} + \sum_{j=1}^{m} [M_r]_{i,j} \prod_{k=1}^{n} y_k^{[B]_{j,k}} \right) u_r$$

where

$$i = 1, \ldots, n, \quad M_0, M_r \in \mathbb{R}^{n \times m}, \quad B \in \mathbb{R}^{m \times n},$$

$$l_0, l_r \in \mathbb{R}^n, \quad r = 1, \ldots, p.$$

The corresponding input-affine Lotka-Volterra model is in the form

$$\dot{z}_j = z_j \left(\lambda_{0_j} + \sum_{k=1}^{m} [A_0]_{j,k} \cdot z_k \right) + \sum_{r=1}^{p} z_j \left(\lambda_{r_j} + \sum_{k=1}^{m} [A_r]_{j,k} \cdot z_k \right) u_r \tag{60}$$

where

$$j = 1, \ldots, m, \quad A_0, A_r \in \mathbb{R}^{m \times m}, \quad \lambda_0, \lambda_r \in \mathbb{R}^m, \quad r = 1, \ldots, p.$$

and the parameters can be obtained from the input-affine QP system's ones in the following way

$$\begin{aligned} A_0 &= B \cdot M_0 \\ \lambda_0 &= B \cdot l_0 \\ A_r &= B \cdot M_r \\ \lambda_r &= B \cdot l_r \end{aligned} \quad r = 1, \ldots, p \tag{61}$$

5.2 Zero dynamics analysis

Let us consider a SISO input-affine QP-model in the form of Eq. (59) with $p = 1$ and with the simplest output $\eta = y_i - w^*$ for some i and $w^* > 0$, i.e. we want to keep the system's output at a positive constant value. Moreover, let us assume that the relative degree of the system equals one and $g_{i1}(y) = g_i(y) = \prod_{j=1}^{n} y_j^{\gamma_{ji}}$, i.e. the input function is of quasi-monomial type. Then the output zeroing input is given in the form

$$u(t) = -\frac{L_f h(y)}{L_g h(y)} = -\frac{f_i(y)}{\prod_{j=1}^{n} y_j^{\gamma_{ji}}} \tag{62}$$

It is seen that the output zeroing input above is in QP-form if $f_i(y)$ is in QP-form.

In order to obtain the zero dynamics, one has to substitute the input (62) to the state equation (59) to obtain an *autonomous* system model. It is easy to compute that the *resulting zero dynamics system model will remain in QP-form with an output zeroing input in QP-form*. Therefore the stability analysis of the zero dynamics can be investigated using the methods described earlier in Section 3.

The above result can be easily generalized to the case of output functions in quasi-monomial form.

Table 1: Variables and parameters of the bioreactor model

X	biomass concentration		$[\frac{g}{l}]$
S	substrate concentration		$[\frac{g}{l}]$
F	inlet feed flow rate		$[\frac{l}{h}]$
V	volume	4	$[l]$
S_F	substrate feed concentration	10	$[\frac{g}{l}]$
Y	yield coefficient	0.5	-
μ_{max},	kinetic parameter	1	$[\frac{1}{h}]$
K_1	kinetic parameter	0.03	$[\frac{g}{l}]$
K_2	kinetic parameter	0.5	$[\frac{l}{g}]$

5.2.1 A simple fermentation example

Consider a simple fermentation process with non-monotonous reaction kinetics that is described by the non-QP input-affine state-space model

$$
\begin{aligned}
\dot{X} &= \mu(S)X - \frac{XF}{V} \\
\dot{S} &= -\frac{\mu(S)X}{Y} + \frac{(S_F - S)F}{V} \\
\mu(S) &= \mu_{max}\frac{S}{K_2 S^2 + S + K_1},
\end{aligned}
\tag{63}
$$

where the inlet feed flow rate denoted by F is the manipulated input. The variables and parameters of the model together with their units and parameter values are given in Table 1.

The investigated equilibrium point of the system is where the outlet biomass flow rate (i.e. biomass production per unit time) is maximal:

$$
\begin{aligned}
S_0 &= \frac{1}{2}\frac{-2K_1 + 2\sqrt{K_1^2 + S_F^2 K_1 K_2 + S_F K_1}}{S_F K_2 + 1} \\
X_0 &= (S_F - S_0)Y
\end{aligned}
\tag{64}
$$
$$
\tag{65}
$$

By introducing a new differential variable $Z = \frac{1}{K_2 S^2 + S + K_1}$ in addition to X and S, the original system (63) can be represented in QP-form characterized by the following matrices:

$$
M_0 = \begin{bmatrix} \mu_{max} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu_{max}}{Y} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\mu_{max}K_2}{Y} & 0 & \frac{\mu_{max}}{Y} & 0 \end{bmatrix}
$$

$$
M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{S_F}{V} & 0 & 0 & 0 & 0 \\ \frac{1-2K_2 S_F}{V} & 0 & 0 & 0 & \frac{2K_2}{V} & 0 & -\frac{S_F}{V} \end{bmatrix}
$$

$$
B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad l_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad l_1 = \begin{bmatrix} -\frac{1}{V} \\ -\frac{1}{V} \\ 0 \end{bmatrix}
$$

The quasi-monomials in the QP system model are:

$$
SZ, \ XZ, \ S^{-1}, \ S^2 XZ^2, \ S^2 Z, \ SXZ^2, \ Z
$$

Let us choose input of the system to be the input flowrate, F, and the output to be the centered substrate-concentration:

$$
\eta = S - S_0
$$

The output zeroing input can be easily computed:

$$F = \frac{\mu_{max} S_0 V}{Y(S_F - S_0)} XZ \tag{66}$$

If the above equations are substituted into the QP-form, one gets the following zero dynamics

$$\dot{X} = X \left(\frac{\mu_{max} S_0}{K_2 S_0^2 + S_0 + K_1} - \frac{S_0 \mu_{max}}{Y(S_F - S_0)(K_2 S_0^2 + S_0 + K_1)} X \right) \tag{67}$$

with QP matrices M_z, B_z and l_z being the following ones:

$$M_z = \left[-\frac{S_0 \mu_{max}}{Y(S_F - S_0)(K_2 S_0^2 + S_0 + K_1)} \right] = \left[-0.1640 \right],$$

$$B_z = \left[1 \right], \quad l_z = \left[\frac{\mu_{max} S_0}{K_2 S_0^2 + S_0 + K_1} \right] = \left[0.8022 \right], \tag{68}$$

Hence, the only monomial of the zero dynamics is X. Note that *the number of quasi-monomials has been drastically reduced.*

In order to study the local stability of the zero dynamics, we first computed the eigenvalue (i.e. the value) of the Jacobian of the zero dynamics at the equilibrium point X_0 that is -0.8022. It is easy to see from (68) that if the condition $S_F > S_0$ holds, then any positive C satisfies the LMI (9). Therefore the global stability of the zero dynamics is proved through the QP description. This result is in good agreement with [19] where the stability of the zero dynamics was proved through nonlinear coordinates-transformations.

5.3   Stabilizing controller design

The output zeroing input (62) can be viewed as a nonlinear static state feedback acting on the QP-form state equation (59). If the state feedback is in QP-form then the closed-loop system will also be in QP-form and its stability can be conveniently investigated by using LMI (9) if the feedback parameters are known and fixed.

Therefore, one can formulate a *globally stabilizing state feedback* design problem for QP systems as follows. Consider arbitrary quasi-polynomial inputs in the form:

$$u_j = \sum_{i=1}^{r} k_{i_j} \hat{q}_i, \quad j = 1, \ldots, p \tag{69}$$

where $\hat{q}_i = \hat{q}_i(y_1, \ldots, y_n)$, $i = 1, ..., r$ are arbitrary quasi-monomial functions of the state variables of (59) and k_i is a p dimensional constant gain vector. The closed loop system will also be a QP system with matrices

$$\hat{M} = \hat{M}_0 + \sum_{j=1}^{p} \sum_{i=1}^{r} k_{i_j} M_{ij}, \quad \hat{B},$$

$$\hat{l} = \hat{l}_0 + \sum_{j=1}^{p} \sum_{i=1}^{r} k_{i_j} l_{ij}.$$

where k_{i_j} is the jth entry of the ith gain vector k_i. Note that the number of quasi-monomials in the closed-loop system (i.e. the dimension of the matrices) together with the matrix $\hat{B}$ may significantly change depending on the choice of the feedback structure, i.e. on the quasi-monomial functions $\hat{q}_i$.

Furthermore, the LV coefficient matrix A is also an affine function of the feedback gain parameters:

$$A = \hat{B} \cdot \hat{M} = \hat{A}_0 + \sum_{j=1}^{p} \sum_{i=1}^{r} k_{i_j} A_{ij}$$

Then the global stability analysis of the closed loop system with unknown feedback gains k_{i_j} leads to the following *bilinear matrix inequality*

$$A^T C + CA = A_0^T C + CA_0 + \sum_{j=1}^{p} \sum_{i=1}^{r} k_{i_j} \left(A_{ij}^T C + CA_{ij} \right) < 0. \tag{70}$$

The variables of the BMI are the $p \times r$ k_{i_j} input-parameters and the c_i, $i = 1, .., \hat{m}$ parameters of the Lyapunov function. If the BMI above is feasible then there exists a globally stabilizing feedback with the selected structure.

5.3.1 Feedback structure design

Clearly, the general feedback structure (69) should be specialized in order to reduce the number of quasi-monomials in the closed-loop system. This can be performed by analyzing carefully the relationship between the quasi-monomials of the open-loop system. Further reduction can be possibly achieved by choosing appropriate feedback gain values from the feasible set.

The following example illustrates the above approach to design globally stabilizing static QP feedback controllers.

5.3.2 Simple numerical example

Our example is the following simple 2 dimensional QP model:

$$
\begin{array}{rcl}
\dot{x} & = & x \left(1.6943 x^{1/2} - 2.1469 y^{4/5} + u_1\right) \\
\dot{y} & = & y \left(-0.1436 x^{1/2} - 0.4283 y^{4/5} + u_2\right)
\end{array}
\tag{71}
$$

with QP matrices

$$
M = \left[\begin{array}{cc} 1.6943 & -2.1469 \\ -0.1436 & -0.4283 \end{array}\right] \quad B = \left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{4}{5} \end{array}\right] \quad l = \left[\begin{array}{c} u_1 \\ u_2 \end{array}\right]
$$

The system (71) has one meaningful (i.e. positive and real in each coordinates) equilibrium point:

$$
\left(\begin{array}{c} \bar{x} \\ \bar{y} \\ \bar{u}_1 \\ \bar{u}_2 \end{array}\right) = \left(\begin{array}{c} 3.1214 \\ 2.8723 \\ 0.0000 \\ 0.0000 \end{array}\right)
$$

The above equilibrium is unstable because of the Jacobian's eigenvalues: $\lambda_1 = 1.6667$, and $\lambda_2 = -0.9999$. The stabilizing feedback was searched for in the form:

$$
\begin{array}{rcl}
u_1 & = & k_1 \cdot x^{1/2} \\
u_2 & = & k_2 \cdot y^{4/5}
\end{array}
\tag{72}
$$

An algorithm used for solving special BMI problems gives the following result for the stabilizing state feedback problem:

$$
k_1 = -5.8504, \quad k_2 = -3.4488,
$$

$$
C = \left[\begin{array}{cc} 16.5043 & 0 \\ 0 & 14.8135 \end{array}\right]
$$

It means that applying the feedback (72) with the above parameters k_1 and k_2 the closed loop QP system

$$
\begin{array}{rcl}
\dot{x} & = & x \left(-4.1560 x^{1/2} - 2.1469 y^{4/5}\right) \\
\dot{y} & = & y \left(-0.1436 x^{1/2} - 3.8771 y^{4/5}\right)
\end{array}
$$

is globally asymptotically stable with Lyapunov function (8) with parameters 16.5043 and 14.8135. Indeed, the system has a unique asymptotically stable equilibria in

$$
\left(\begin{array}{c} \tilde{x} \\ \tilde{y} \end{array}\right) = \left(\begin{array}{c} 0.1029 \\ 0.2317 \end{array}\right)
$$

with eigenvalues $\lambda_1 = -0.6298$ and $\lambda_1 = -1.0000$.

6 Conclusions

Different theoretical and practical aspects of QP and LV systems were studied in this paper.

Nonlinear input-affine systems in quasi-polynomial (QP) and Lotka-Volterra (LV) forms are investigated in this paper. It has been shown that both the global stability analysis with an entropy-like Lyapunov function candidate and the local quadratic stability region determination can be performed by solving linear matrix inequalities (LMIs). The existence of a diagonal weighting matrix for local quadratic stability has been found to

be equivalent to the existence of a local Hamiltonian description of a QP or LV system. The invariance transformations, the quasi-monomial, time-reparametrization and logarithmic transformations that preserve the form of the description are also described and their use for stability analysis is discussed.

For control studies, the original QP and LV system models have been extended by QP and LV input terms, respectively. It has been shown that zero dynamics analysis can be performed by solving LMIs in case of SISO QP systems with relative degree equals one.

It has also been shown that the globally stabilizing controller design problem with quasi-polynomial feedback structure for QP systems having relative degree 1 leads to the feasibility of a bilinear matrix inequality where the unknowns to be determined are the parameters of the Lyapunov function of the closed loop system and the constant coefficients of the monomials in the feedback law.

The proposed theoretical issues and methods and tools were demonstrated using simple illustrative examples.

Acknowledgements

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