Globally Stabilizing Feedback Control of Process Systems in Generalized Lotka-Volterra Form

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Abstract

In the present paper a globally stabilizing feedback controller design method is proposed for process systems when the feedback structure is also assumed to be in a QP-form. It is shown that such feedback structure can always be achieved for process systems. By exploiting the special structure of the controller design problem, existent iterative linear matrix inequality (ILMI) algorithm of [1] is applied to solve the BMI feasibility problem underlying the design.

In addition, some partial results on placing the globally stable equilibrium point with respect to the positive orthant have also been proposed that is only possible in a fully actuated situation when the input variables are the intensive variables at the inlet. Furthermore, some preliminary results in selecting the structure of the QP-type feedback have also been presented.

Key words: nonlinear process systems, quasi-polynomial systems, bilinear matrix inequalities

1 Introduction

It is widely known in process systems engineering that almost all process systems are nonlinear in nature. Therefore, a number of papers and books (see e.g. [2], [3]) have proposed nonlinear controller design methods of various kind for more or less wide classes of nonlinear process systems.

The most popular method is to use model predictive controllers for nonlinear process systems (see [4] for a recent review) where a detailed dynamic process model is used in an optimization framework. This popularity is partially explained by the fact that dynamic models for simulation or prediction purposes are often available for existing plants. Modern heuristic black-box type control approaches, such as neural nets and fuzzy controllers, have also appeared recently even in industrial practice.

At the same time, the results and approaches of modern nonlinear control theory have not earned a wide acceptance in the field of process control except for a few attempts (see e.g. [5], [6]). The reason for this lies partially in the fact that modern nonlinear control methods are computationally hard, and are only feasible for small scale systems in the general case. The problems with nonlinear control techniques applied in the general case indicate that a solid knowledge of the special characteristics of the nonlinear system in question may significantly help in developing nonlinear controllers for process systems with reasonably realistic complexity.

Quasi-polynomial (QP) systems play an important role in the modelling of dynamical systems because a wide class of smooth nonlinear systems can be easily transformed to QP form [7]. The stability properties of QP systems have also been studied intensively recently [8], [9]. It has been shown [10], [11] that local and global stability analysis of QP systems and their zero dynamics can be efficiently performed by solving LMIs.

At the same time, there are only a few papers found in the literature about the stabilizing control of Lotka-Volterra systems and only in special cases (see, e.g. [12],[13]), but to the best of the authors' knowledge no one has tried to use the above mentioned theoretical and numerical tools of QP systems in the framework of nonlinear control systems. The possibility of designing globally stabilizing QP feedback to QP systems has been explored in our previous conference paper [14] where the stability of zero dynamics of such systems has also been investigated. The design problem was found to be a BMI feasibility problem and the importance for feedback structure design has also been recognized.

Meanwhile, some computationally effective numerical methods have been developed lately, that allow us to practically perform the stability analysis of QP systems [15] by LMIs and the design of globally stabilizing feedback by BMIs. A summary of linear and bilinear matrix inequalities and the available software tools for solving them can be found in [16].

Thus the present paper aims to offer a computationally feasible globally stabilizing feedback design method for process systems where both of the above mentioned computationally effective numerical methods and the specialities of the problem for process systems that are embedded in QP form are exploited.

The outline of the paper is as follows. We start with some essential notions on quasi-polynomial and Lotka-Volterra systems in Section 2. Linear and bilinear matrix inequalities, and stability analysis of quasi-polynomial and Lotka-Volterra systems are also summarized in this section. The main contribution is Section 3 which is about stabilizing feedback controller design, and its numerical aspects. Thereafter, Section 4 presents some examples. Finally, Section 5 summarizes the main results of the paper and aims our future work.

2 Basic Notions

The elementary notions in the field of quasi-polynomial (QP) and Lotka-Volterra (LV) systems are introduced in this section, together with the basic results of their stability analysis and that of linear and bilinear matrix inequalities. In order to emphasize the similarity of QP and LV systems, QP systems are also called *generalized Lotka-Volterra (GLV) systems.*

2.1 Quasi-Polynomial and Lotka-Volterra Models

2.1.1 QP models

Let us denote the element of an arbitrary matrix W with row index i and column index j by W_{ij}. Quasi-polynomial models are systems of ODEs of the following form

$$\dot{y}_i = y_i \left(L_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} y_k^{B_{jk}} \right), \quad i = 1, \ldots, n. \tag{1}$$

where $y \in \text{int}(\mathbf{R}^n_+)$, $A \in \mathbf{R}^{n\times m}$, $B \in \mathbf{R}^{m\times n}$, $L_i \in \mathbf{R}$, $i = 1, \ldots, n$. Furthermore, $L = [L_1 \ \ldots \ L_n]^T$. Let us denote the equilibrium point of interest of (1) as $y^* = [y_1^* \ \ y_2^* \ \ \ldots \ \ y_n^*]^T$. Without the loss of generality we can assume that $\text{Rank}(B) = n$ and $m \geq n$ (see [7]).

2.1.2 Lotka-Volterra models

The above family of models is split into classes of equivalence [17] according to the values of the products $M = B \cdot A$ and $N = B \cdot L$. The *Lotka-Volterra form* gives the representative elements of these classes of equivalence. If $rank(B) = n$, then the set of ODEs in (1) can be embedded into the following m-dimensional set of equations, the so called Lotka-Volterra model:

$$\dot{z}_j = z_j \left(N_j + \sum_{i=1}^{m} M_{ji} z_i \right), \qquad j = 1, \ldots, m \tag{2}$$

where

$$M = B \cdot A, \quad N = B \cdot L,$$

and each z_j represents a so called *quasi-monomial*:

$$z_j = \prod_{k=1}^{n} y_k^{B_{jk}}, \qquad j = 1, \ldots, m. \tag{3}$$

2.1.3 Input-affine QP system models

An input-affine nonlinear system model with state vector y, input vector u and output vector η

$$\begin{aligned} \dot{y} &= f(y) + \sum_{i=1}^{p} g_i(y) u_i \\ \eta &= h(y) \end{aligned} \tag{4}$$

is in QP-form if all of the functions f, g and h are in QP-form. Then the general form of the state equation of an input-affine QP system model with p-inputs is:

$$\begin{aligned} \dot{y}_i = y_i &\left(L_{0_i} + \sum_{j=1}^{m} A_{0_{ij}} \prod_{k=1}^{n} y_k^{B_{jk}} \right) + \\ &+ \sum_{l=1}^{p} y_i \left(L_{l_i} + \sum_{j=1}^{m} A_{l_{ij}} \prod_{k=1}^{n} y_k^{B_{jk}} \right) u_l \end{aligned} \tag{5}$$

where

$$i = 1, \ldots, n, \quad A_0, A_l \in \mathbf{R}^{n \times m}, \quad B \in \mathbf{R}^{m \times n},$$

$$L_0, L_l \in \mathbf{R}^n, \quad l = 1, \ldots, p.$$

The corresponding input-affine Lotka-Volterra model is in the form

$$\dot{z}_j = z_j \left(N_{0_j} + \sum_{k=1}^{m} M_{0_{jk}} z_k \right) + \sum_{l=1}^{p} z_j \left(N_{l_j} + \sum_{k=1}^{m} M_{l_{jk}} z_k \right) u_l \tag{6}$$

where

$$j = 1, \ldots, m, \quad M_0, M_l \in \mathbf{R}^{m \times m}, \quad N_0, N_l \in \mathbf{R}^m, \quad l = 1, \ldots, p,$$

and the parameters can be obtained from the input-affine QP system's ones in the following way

$$
\begin{aligned}
M_0 &= B \cdot A_0 \\
N_0 &= B \cdot L_0 \\
M_l &= B \cdot A_l \\
N_l &= B \cdot L_l \qquad l = 1, \ldots, p.
\end{aligned}
\tag{7}
$$

2.1.4 *Rewriting non-QP ODE models into QP-form*

A wide class of nonlinear autonomous systems with smooth nonlinearities can be embedded into QP-form [18] if they satisfy two requirements.

(1) The set of nonlinear ODEs should be in the form:

$$
\dot{y_s} = \sum_{i_{s1},\ldots,i_{sn},j_s} a_{i_{s1}\ldots i_{sn}j_s} y_1^{i_{s1}} \ldots y_n^{i_{sn}} f(\overline{y})^{j_s}, \tag{8}
$$

$$
y_s(t_0) = y_s^0, \qquad s = 1, \ldots, n
$$

where $f(\overline{y})$ is some scalar valued function, which is not reducible to *quasi-monomial form* containing terms in the form of $\prod_{k=1}^n y_k^{\Gamma_{jk}}, \; j = 1, \ldots, m$ with Γ being a real matrix.

(2) Furthermore, we require that the partial derivatives of the model (8) fulfil:

$$
\frac{\partial f}{\partial y_s} = \sum_{e_{s1},..,e_{sn},e_s} b_{e_{s1}..e_{sn}e_s} y_1^{e_{s1}} \ldots y_n^{e_{sn}} f(\overline{y})^{e_s}
$$

The embedding is performed by introducing a *new auxiliary variable*

$$
\eta = f^q \prod_{s=1}^n y_s^{p_s}, \qquad q \neq 0. \tag{9}
$$

Then, instead of the non-quasi-polynomial nonlinearity f we can write the original set of equations (8) into QP-form:

$$
\dot{y_s} = \left(y_s \sum_{i_{s1},\ldots,i_{sn},j_s} \left(a_{i_{s1}\ldots i_{sn}j_s} \eta^{j_s/q} \prod_{k=1}^n y_k^{i_{sk} - \delta_{sk} - j_s p_k/q} \right) \right), \qquad s = 1, \ldots, n \tag{10}
$$

where $\delta_{sk} = 1$ if $s = k$ and 0 otherwise. In addition, a new quasi-polynomial ODE appears for the new variable η:

$$\dot{\eta} = \eta \left[\sum_{s=1}^{n} \left(p_s y_s^{-1} \dot{y}_s + \sum_{\substack{i_{s\alpha},j_s \\ e_{s\alpha},e_s}} a_{i_{s\alpha},j_s} b_{e_{s\alpha},e_s} q \eta^{(e_s+j_s-1)/q} \times \right.\right.$$
$$\left.\left. \times \prod_{k=1}^{n} y_k^{i_{sk}+e_{sk}+(1-e_s-j_s)p_k/q} \right) \right], \qquad \alpha = 1, \ldots, n. \tag{11}$$

It is important to observe that the embedding is not unique, because we can choose the parameters p_s and q in (9) in many different ways: the simplest is to choose $(p_s = 0,\ s = 1, ..., n;\ q = 1)$.

If we set the initial values of the newly introduced variables according to (9) then the dynamics of the embedded system is equivalent to the original non-QP system described in (8). Since the embedded QP system includes the original differential variables y_i, $i = 1, \ldots, n$, it is clear that the stability of the embedded system (10)-(11) implies the stability of the original system (8).

It is important to note that QP models originate from embedding have some unusual dynamic properties because their trajectories range only a lower dimensional manifold of the QP state space. Thus they can be regarded as "hidden" differential-algebraic (DAE) system models with *rank deficient* A parameter matrices.

2.1.5 QP models of process systems

The nonlinearities of a lumped parameter process system model are of two types from the viewpoint of their QP-form representation. The nonlinearities originating from the sources (e.g. reaction or transfer rates) appear in the f function of the input-affine state-space model (4) and they are not necessarily in QP-form. Therefore, the above described embedding of such models into QP-form is of great practical importance.

The specialities of the input function g_i The specialities of the input function g_i of the input-affine state-space model (4) originate from the fact that the inputs of process systems are most often realized through either inlet mass or component mass flow-rates, or alternatively, intensive variables at the inlet, like temperatures or concentrations. This means that they act through the inlet convection term [19] of the conservation balances that are transformed into state equations. As convection is bilinear in a mass flow-rate and an intensive variable (such as concentration, temperature or pressure), the nonlinear input function $g_{il}(y)$ is most often a simple homogeneous linear function of the corresponding state variable y_i:

(1) $g_{il}(y) = const \cdot y_i$ when the mass flow-rates are the input variables, or
(2) $g_{il}(y) = const^*$ when the intensive variables at the inlet are the inputs.

Case (1) implies that the parameters $A_l = 0$ in (5) and $M_l = 0$ in (6).

The above special form is, of course, not valid, when a QP state equation originates from variable embedding.

2.1.6 A simple fermentation example

A simple fermentation example illustrates the way of embedding non-QP system models into QP-form and the special properties of process system models in QP-form. Consider a simple fermentation process with non-monotonous reaction kinetics that is described by the non-QP input-affine state-space model

$$\begin{aligned} \dot{X} &= \mu(S)X + \tfrac{(X_F - X)F}{V} \\ \dot{S} &= -\tfrac{\mu(S)X}{Y} + \tfrac{(S_F - S)F}{V} \\ \mu(S) &= \mu_{max}\tfrac{S}{K_S + S}, \end{aligned} \tag{12}$$

where the inlet substrate and biomass concentrations denoted by S_F and X_F, are the manipulated inputs. The variables and parameters of the model together with their units and parameter values are given in Table 1.

The system has a unique locally stable equilibrium point in the positive orthant:

$$\begin{bmatrix} \bar{X} \\ \bar{S} \end{bmatrix} = \begin{bmatrix} 0.6500 \\ 0.4950 \end{bmatrix} \tag{13}$$

with steady-state inputs

$$\begin{bmatrix} \bar{X}_F \\ \bar{S}_F \end{bmatrix} = \begin{bmatrix} 0.6141 \\ 4.3543 \end{bmatrix}.$$

By introducing a new differential variable $Z = \frac{1}{K_S + S}$ one arrives at a third differential equation

$$\begin{aligned} \dot{Z} &= -\tfrac{1}{(K_S + S)^2} \cdot \tfrac{dS}{dt} = -Z^2 \cdot \left(-\tfrac{\mu_{max}}{Y} SXZ + \tfrac{(S_F - S)F}{V}\right) = \\ &= Z\left(\tfrac{\mu_{max}}{Y} SXZ^2 + \tfrac{F}{V} SZ - S_F \tfrac{F}{V} Z\right) \end{aligned} \tag{14}$$

that completes the ones for X and S. Thus the original system (12) can be represented by three differential equations in input-affine QP-form characterized by the following matrices:

$$A_0 = \begin{bmatrix} \mu_{max} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu_{max}}{Y} & 0 & 0 & 0 \\ \frac{F}{V} & 0 & 0 & 0 & \frac{\mu_{max}}{Y} & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & \frac{F}{V} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{F}{V} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{F}{V} \end{bmatrix} \tag{15}$$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \; L_0 = \begin{bmatrix} -\frac{F}{V} \\ -\frac{F}{V} \\ 0 \end{bmatrix} \quad L_1 = L_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The six quasi-monomials of the QP system model given by the matrices (15) are

$$SZ, \;\; X^{-1}, \;\; XZ, \;\; S^{-1}, \;\; SXZ^2, \;\; Z.$$

2.2 Linear and bilinear matrix inequalities

A (non-strict) linear matrix inequality (LMI) is an inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \leq 0, \tag{16}$$

where $x \in \mathbf{R}^m$ is the variable and $F_i \in \mathbf{R}^{n \times n}$, $i = 0, \ldots, m$ are given symmetric matrices. The inequality symbol in (16) stands for the negative semidefiniteness of $F(x)$.

One of the most important properties of LMIs is the fact, that they form a convex constraint on the variables, i.e. the set $\mathcal{S} = \{x \mid F(x) \leq 0\}$ is convex and thus many different kinds of convex constraints can be expressed in this way [20], [21]. It is important to note that a particular point from the convex solution set $\mathcal{S}$ can be selected using additional criteria (e.g. different kinds of objective functions) [20]. Standard LMI optimization problems are e.g.:

- linear function minimization:

$$\text{minimize} \;\; c^T x$$
$$\text{subject to} \;\; F(x) > 0$$

- generalized eigenvalue problem:

$$\text{minimize} \;\; \lambda$$
$$\text{subject to} \; \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0$$

- convex problem:

$$\text{minimize} \;\; \log\det A(x)^{-1}$$
$$\text{subject to} \; A(x) > 0, \quad B(x) > 0,$$

where $c \in \mathbf{R}^m$, A, B and F are symmetric matrices that are affine functions of x.

A bilinear matrix inequality (BMI) is a diagonal block composed of q matrix inequalities of the following form

$$G_0^i + \sum_{k=1}^{p} x_k G_k^i + \sum_{k=1}^{p} \sum_{j=1}^{p} x_k x_j K_{kj}^i \leq 0, \quad i = 1, \ldots, q \tag{17}$$

where $x \in \mathbf{R}^p$ is the decision variable to be determined and G_k^i, $k = 0, \ldots, p$, $i = 1, \ldots, q$ and K_{kj}^i, $k, j = 1, \ldots, p$, $i = 1, \ldots, q$ are symmetric, quadratic matrices.

The main properties of BMIs are that they are non-convex in x (which makes their solution numerically much more complicated than that of linear matrix inequalities), and their solution is NP-hard [16], so the size of the tractable problems is limited. However, there exist practically applicable and effective algorithms for BMI solution [22], [23], or [1]. Similarly to the LMIs, *additional criteria can be used to select a preferred solution point of a feasible BMI from its solution set.*

2.3 Global stability analysis of QP and LV models

Henceforth it is assumed that y^* is a positive equilibrium point, i.e. $y^* \in \text{int}(\mathbf{R}_+^n)$ in the QP case and similarly $z^* \in \text{int}(\mathbf{R}_+^m)$ is a positive equilibrium point in the LV case. For LV systems there is a well known candidate Lyapunov function family [9],[8], which is in the form:

$$V(z) = \sum_{i=1}^{m} c_i \left(z_i - z_i^* - z_i^* \ln \frac{z_i}{z_i^*} \right), \tag{18}$$

$$c_i > 0, \quad i = 1 \dots m,$$

where $z^* = (z_1^*, \dots, z_m^*)^T$ is the equilibrium point corresponding to the equilibrium y^* of the original QP system. The time derivative of the of the Lyapunov function (18) is:

$$\dot{V}(z) = \frac{1}{2}(z - z^*)(CM + M^T C)(z - z^*) \tag{19}$$

where $C = \text{diag}(c_1, \dots, c_m)$ and M is the invariant characterizing the LV form. Therefore the non-increasing nature of the Lyapunov function is equivalent to a feasibility problem over the following set of LMI constraints:

$$\begin{aligned} CM + M^T C &\leq 0 \\ C &> 0 \end{aligned} \tag{20}$$

where the unknown matrix is C, which is diagonal and contains the coefficients of (18). It is important to note that the strict positivity constraint on c_i can be somewhat relaxed in the following way [8]: if the equations of the model (1) are ordered in such a way that the first n rows of B are linearly independent, then $c_i > 0$ for $i = 1, \dots, n$ and $c_j = 0$ for $j = n + 1, \dots, m$ still guarantee global stability.

It is examined and proved in [8] and [9] that the global stability of (2) with Lyapunov function (18) implies the boundedness of solutions and global stability of the original QP system (1). It is stressed that *global stability is restricted to the positive orthant int*($\mathbf{R}_+^n$) *only for QP and LV models*, because it is their original domain (see the definition in (1)).

It is also important that the global stability of the equilibrium points of (1) with Lyapunov function (18) does not depend on the value of the vector L as long as the equilibrium points are in the positive orthant [8]. This fact will allow us to place the equilibrium point of the closed loop system during the stabilizing controller design (see section 3.3).

The possibilities to find a Lyapunov function that proves the global asymptotic stability of a QP system can be increased by using time-reparametrization [11].

3 Globally Stabilizing Feedback Design

If the state feedback is in QP-form then the closed loop system will also be in QP-form and its stability can be conveniently investigated by using LMIs if the feedback parameters are known and fixed. However, the solution of the QP feedback design problem with its structure fixed requires to solve a

BMI problem, that is the subject of this section. In addition, some structural feedback design results are also proposed in this section.

Unfortunately, the solution of the feedback design problem does not automatically provide tools for the design of the steady-state point of the system. Therefore, the basic conditions of steady-state point placing is also discussed here.

3.1 The controller design problem

Globally stabilizing QP state feedback design problem for QP systems can be formulated as follows. Consider arbitrary quasi-polynomial inputs in the form:

$$u_l = \sum_{i=1}^{r} k_{il}\hat{q}_i, \quad l = 1 \ldots, p \tag{21}$$

where $\hat{q}_i = \hat{q}_i(y_1, \ldots, y_n),\ i = 1, ..., r$ are arbitrary quasi-monomial functions of the state variables of (5) and k_{il} is the constant gain of the quasi-monomial function $\hat{q}_i$ in the l-th input u_l. The closed loop system will also be a QP system with matrices

$$\hat{A} = A_0 + \sum_{l=1}^{p}\sum_{i=1}^{r} k_{il}A_{il}, \quad \hat{B}, \tag{22}$$

$$\hat{L} = L_0 + \sum_{l=1}^{p}\sum_{i=1}^{r} k_{il}L_{il}. \tag{23}$$

Note that the number of quasi-monomials in the closed-loop system (i.e. the dimension of the matrices) together with the matrix $\hat{B}$ may significantly change depending on the choice of the feedback structure, i.e. on the quasi-monomial functions $\hat{q}_i$.

Furthermore, the closed loop LV coefficient matrix $\hat{M}$ can also be expressed in the form:

$$\hat{M} = \hat{B} \cdot \hat{A} = M_0 + \sum_{l=1}^{p}\sum_{i=1}^{r} k_{il}M_{il}.$$

Then the global stability analysis of the closed loop system with unknown feedback gains k_{il} leads to the following *bilinear matrix inequality*

$$\hat{M}^T C + C\hat{M} = M_0^T C + CM_0 + \sum_{l=1}^{p}\sum_{i=1}^{r} k_{il}\left(M_{il}^T C + CM_{il}\right) < 0. \tag{24}$$

The variables of the BMI are the $p \times r$ k_{il} feedback gain parameters and the $c_j,\ j = 1, .., m$ parameters of the Lyapunov function. If the BMI above is feasible then there exists a globally stabilizing feedback with the selected structure.

3.2 Numerical solution of the controller design BMI

This section deals with the numerical aspects of the globally stabilizing controller design problem.

3.2.1 Numerical solution based on bilinear matrix inequalities

There are just few software tools available for solving general bilinear matrix inequalities that is a computationally hard problem. In some rare fortunate cases with a suitable change of variables quadratic matrix inequalities can be rewritten as linear matrix inequalities (see e.g. [20]). Unfortunately, the structure of the matrix variable of (24) does not fall into this fortunate problem class, so the previously mentioned idea cannot be used.

In Matlab environment the TomLab/PENBMI solver [24] can be used effectively to solve bilinear matrix inequalities. Rewriting the above matrix inequality (24) in the form (17) one gets the following expression which can be directly solved by [24] as a BMI feasibility problem:

$$\begin{aligned} \sum_{j=1}^{m} c_j \bar{M}_{0,j} + \sum_{j=1}^{m}\sum_{l=1}^{p}\sum_{i=1}^{r} c_j k_{il} \bar{M}_{il,j} &< 0 \\ -c_1 &< 0 \\ &\vdots \\ -c_m &< 0. \end{aligned} \tag{25}$$

The two disjoint sets of BMI variables are the c_j parameters of the Lyapunov function and the k_{il} feedback parameters. The parameters of the problem $\bar{M}_{0,j}$ ($\bar{M}_{il,j}$, respectively) are the symmetric matrices obtained from M_0 (M_{il}, respectively) by adding the $m \times m$ matrix that contains only the j-th column of M_0 (M_{il}, respectively) to its transpose:

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1j} & \cdots & m_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ m_{m1} & \cdots & m_{mj} & \cdots & m_{mm} \end{bmatrix}$$

$$\downarrow$$

$$\bar{M}_j = \begin{bmatrix} 0 & \cdots & m_{1j} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ m_{1j} & \cdots & 2m_{jj} & \cdots & m_{mj} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & m_{mj} & \cdots & 0 \end{bmatrix}.$$

Note that for low dimensions (i.e. for $m < 3$) there are practically feasible methods for circumvent the BMI feasibility problem [25] but these cannot be extended to the practically important higher dimensional case.

3.2.2 Numerical solution based on iterative LMIs

Because of the NP-hard nature of the general BMI solution problem, it is worthwhile to search for an approximate but numerically efficient alternative way of solution. As shown below, the special structure of the QP stabilizing feedback design BMI feasibility problem allows us to apply a computationally feasible method for its solution that solves an LMI in each of its iterative approximation step. The iterative LMI (ILMI) algorithm used for static output feedback stabilization (see e.g. in [1]) will be used for this purpose.

In order to be able to use the ILMI algorithm, it is necessary to write up the QP stabilizing feedback design problem as a static output feedback stabilization problem for LTI systems. In what follows the globally stabilizing feedback design BMI (24) is used in the form

$$(M_0 + \Theta K)^T C + C(M_0 + \Theta K) < 0. \tag{26}$$

where

$$\Theta = \left[\overbrace{M_1, \dots, M_p}^{\text{1st}}, \dots, \overbrace{M_1, \dots, M_p}^{r\text{th}} \right], \quad K = \begin{bmatrix} k_{11} \cdot I_{m \times m} \\ \vdots \\ k_{1p} \cdot I_{m \times m} \\ \vdots \\ k_{r1} \cdot I_{m \times m} \\ \vdots \\ k_{rp} \cdot I_{m \times m} \end{bmatrix}.$$

The above problem is equivalent to a LTI output feedback stabilization problem

$$(A + BFC)^T P + P(A + BFC) < 0$$

with M_0 corresponding to the state matrix A, Θ playing the role of the input matrix B, and K serving as FC and P is the unknown matrix variable of the problem. It is apparent that the matrix parameters and variables have a special structure for quasi-polynomial systems.

The ILMI algorithm does not aim at finding the complete feasible set of the BMI (26) but computes an optimal solution point with minimal trace of C if the BMI is feasible. The ILMI algorithm solves a linear objective function minimizing LMI and a generalized eigenvalue problem in each step. The scheme of the algorithm is the following:

Step 1: Let $Q > 0$, the parameter of the algorithm. Solve the Riccati equation

$$M_0^T C + C M_0 - C\Theta\Theta^T C + Q = 0, \tag{27}$$

for C (not necessarily diagonal).

$$i = 1, \quad X_1 = C.$$

Step 2: Solve the following optimization problem for C_i, K and α_i:
Minimize α_i subject to the LMI constraint

$$\begin{bmatrix} M_0^T C_i + C_i M_0 - X_i\Theta\Theta^T C_i - C_i\Theta\Theta^T X_i + X_i\Theta\Theta^T X_i - \alpha_i C_i & (\Theta^T C_i + K)^T \\ \Theta^T C_i + K & -I \end{bmatrix} < 0,$$

$$C_i = \mathrm{diag}(c_{i1}, \dots, c_{im}) > 0 \tag{28}$$

α_i^* denotes the minimized α_i.

Step 3: If $\alpha_i^* \leq 0$, K is a stabilizing feedback gain. STOP.
Step 4: Solve the following optimization problem for C_i and K:
Minimize trace(C_i) subject to the LMI constraints (28) using $\alpha_i = \alpha_i^*$. Denote C_i^* as the C_i that minimizes trace(C_i).
Step 5: If $\|X_i - C_i^*\| < \delta$, GOTO Step 6. Else set $i = i + 1$ and $X_i = C_i^*$ and GOTO Step 2.
Step 6: The system may not be stabilizable by a quasi-polynomial feedback. STOP.

It is important to note that for QP systems with rank deficient $M_0 = B \cdot A$ some additional techniques are needed because the algorithm fails for singular M_0 matrices. One possible way is using singular perturbation on M_0:

$$\tilde{M}_0 = M_0 - \varepsilon \cdot I_{m\times m}, \qquad \varepsilon > 0.$$

If this way $(\tilde{M}_0, \Theta)$ become stabilizable then the algorithm can be applied.

According to [1] the algorithm is convergent although sometimes we may not achieve a solution because α not always converges to its minimum. The proper selection of initial Q affects the convergence of the algorithm, a suitable selection of Q that guarantees the immediate convergence can be found in [1].

It is important to emphasize here, that the computationally feasible ILMI algorithm can be used to test the feasibility of the associated BMI, and then the final design can be performed by a constrained optimization method using a suitable controller performance criterion in the feasible case.

3.3 Placing the equilibrium point of the QP system

After solving the globally stabilizing feedback design BMI the resulting Lotka-Volterra system has a globally asymptotically stable equilibrium point in the positive orthant. This steady-state equilibrium point y^* can be determined from the steady-state version of the closed loop quasi-polynomial system (1)

$$0 = y_i \left(\hat{L}_i + \sum_{j=1}^{m} \hat{A}_{ij} \prod_{k=1}^{n} y_k^{\hat{B}_{jk}} \right), \quad i = 1, \ldots, n. \tag{29}$$

By excluding the non strictly positive equilibrium states one only has to deal with the equation:

$$0 = \hat{L}_i + \sum_{j=1}^{m} \hat{A}_{ij} \prod_{k=1}^{n} y_k^{\hat{B}_{jk}}, \quad i = 1, \ldots, n \tag{30}$$

where the parameters $\hat{L}_i$ and $\hat{A}_{ij}$ depend linearly on the feedback parameters according to the equations (22) and (23).

However, with the BMI (24) it is not possible to prescribe the equilibria of the closed loop system but only to globally stabilize it. So it is necessary to introduce extra parameters to the feedback in order to be able to place the positive steady state point anywhere in the positive orthant as needed. The feedback structure has to be constructed in a way that the parameters that are used in the *steady state point placing problem* appear in the vector $\hat{L}$ of the closed loop quasi-polynomial system. This way the parameters of the equilibrium placing are separated from the stabilizing feedback design BMI's parameters. The feedback has the form

$$u = K(k, y) + D(\delta, y) \tag{31}$$

where $K(k, y)$ is the feedback structure with the parameters for the BMI, and $D(\delta, y)$ has the form so that the components of the parameter vector δ appear in the vector L of the closed loop QP system. It is important to note that the QP input (21) is linear in both of the parameters k and δ.

One can further simplify the QP input structure (21) for process systems if the input variables are selected to be the intensive variables at the inlet, i.e. $g_i(y) = const^*$ (see sub-section 2.1.5). Then we can use a linear term $D_i(\delta_i, y_i) = \delta_i y_i$ in the feedback (31) to take care of the placing of the steady-state point, and the other term for stabilizing the closed loop system.

3.3.1 Fully actuated case

In this case the QP system has at least one designated input for each of the n state equations. The steady state point of these systems can be put anywhere in the positive orthant.

$$0 = L_i(\delta) + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} y_k^{*B_{jk}}, \quad i = 1, \ldots, n \tag{32}$$

where $L_i(\delta)$ is a linear function of the δ parameters of the problem and $y^* = (y_1^*, \ldots, y_n^*)^T$ is the desired equilibrium. That is, δ can be determined from a linear system of equations.

3.3.2 Partially actuated case

If the system has $k < n$ different inputs, then there are no general results for QP models. However, in the Lotka-Volterra case there is some possibility of shifting some components of the equilibrium point. If the LV coefficient matrix M can be transformed into an upper block triangular matrix by row and column changes then it means that the first k coordinates of the equilibrium point can be prescribed at will independently of the remaining $n - k$.

Note that if the system does not belong to the above two classes then it is not possible to redesign its equilibrium point with the above technique.

3.3.3 Rank deficient (embedded) systems

In case of systems that are not originally in quasi-polynomial form (see subsection 2.1.4 for embedding into QP-form) all the above hold with some specialities. It is known that for such QP systems that their trajectories range only a lower dimensional manifold of the QP state space and their parameter matrix A is rank deficient. With this understanding one has to design the equilibrium point of the system (if it is possible to design at all, see section 3.3.2) into this lower dimensional manifold.

3.4 Feedback structure design

Of course, the feedback structure selection affects heavily the solution of the BMI. The results of zero dynamics analysis of QP-systems [14] indicate that a fortunate choice of a QP-type feedback can simplify the dynamics of a closed-loop system in such a way that the number of quasi-monomials may drastically decrease. This way the dimension of the LV system, and the size of the BMI to be solved can also be drastically reduced.

In certain special cases it is possible to change the entire system dynamics to a desired one while this possibility depends on the number of available inputs. An example of this is shown in section 4.

3.4.1 Fully actuated case

Suppose, that we have an input affine QP system in the form:

$$\begin{aligned} \dot{y}_i = f_i(y) + g_i(y)u_i = y_i \left(L_{0_i} + \sum_{j=1}^{m} A_{0_{ij}} \prod_{k=1}^{n} y_k^{B_{jk}} \right) + \\ + y_i \left(L_{i_i} + \sum_{j=1}^{m} A_{i_{ij}} \prod_{k=1}^{n} y_k^{B_{jk}} \right) u_i, \quad i = 1 \ldots, n, \end{aligned} \tag{33}$$

i.e. every equation has a *designated input*. Suppose in addition that *the desired closed-loop system dynamics is given in the form*:

$$\dot{y}_i = h_i(y), \quad i = 1, \ldots, n \tag{34}$$

where h_i are quasi-polynomial functions.

It is obvious that (33) can be transformed into (34) with the following feedback structure:

$$u_i = -\frac{f_i(y)}{g_i(y)} + \frac{h_i(y)}{g_i(y)}, \quad g_i(y) \neq 0. \tag{35}$$

It can be seen that in general case the expression fed back to the input is not a QP, but a rational function.

Fortunately, the input function g_i in the denominator of the above formulae (35) is a simple linear function $g_i(y) = const \cdot y_i$ or $g_i(y) = const^*$ for process systems (see sub-section 2.1.5), therefore the *feedback remains a QP function for process systems implying the closed-loop system dynamics to remain in the QP system class.*

3.4.2 Partially actuated case

The other case is when there are not as many *different* inputs as equations i.e. the QP system can be arranged into the form

$$\begin{aligned} \dot{y}_p = f_i(y) + g_p(y)u_p = y_p \left(L_{0_p} + \sum_{j=1}^{m} A_{0_{pj}} \prod_{k=1}^{n} y_k^{B_{jk}} \right) + \\ + y_p \left(L_{p_p} + \sum_{j=1}^{m} A_{p_{pj}} \prod_{k=1}^{n} y_k^{B_{jk}} \right) u_p, \quad p = 1 \ldots, k \end{aligned} \tag{36}$$

$$\dot{y}_q = f_q(y) = y_q \left(L_{0_q} + \sum_{j=1}^{m} A_{0_{qj}} \prod_{k=1}^{n} y_k^{B_{jk}} \right), \quad q = k+1 \ldots, n. \tag{37}$$

This way only the first k equations can be modified freely:

$$u_p = -\frac{f_p(y)}{g_p(y)} + \frac{h_p(y)}{g_p(y)}, \quad g_p(y) \neq 0, \quad p = 1, \ldots, k.$$

The closed loop system with the above structure is

$$\begin{aligned} \dot{y}_p &= h_p(y), \quad p = 1, \ldots, k \\ \dot{y}_q &= f_q(y), \quad q = k+1, \ldots, n. \end{aligned} \tag{38}$$

3.4.3 Degenerated case

When there is an input that is assigned to more than one equations the above change of dynamics cannot be used in general. Choosing one equation to change with the input one can destroy the QP form of the other equations having the same input. Of course in special cases it is possible to have useful results, for example in the case of zero dynamics [14].

3.5 Output performance estimation

Using the advantageous form of the derivative of the control Lyapunov function, it is possible to give an upper bound on the norm of certain output functions of the monomials. The requirements for this are the following (see [20] or [26]):

$$\phi_1(\|z - z^*\|) \leq V(z) \leq \phi_2(\|z - z^*\|) \tag{39}$$

where ϕ_1 and ϕ_2 are class $\mathcal{K}$ functions (a continuous function α belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$), and

$$\dot{V}(t) \leq -w^T(t)w(t), \ \forall t > 0 \tag{40}$$

where w is an appropriately selected (performance) output of the system, i.e., $w = h(z)$, with $h : \mathbf{R}^m \mapsto \mathbf{R}^k$, $k \leq m$.

If (39) and (40) are satisfied, then the following inequality holds for the 2-norm of the output:

$$\|w\|_2^2 = \int_{t_0}^{\tau} w^T(t)w(t)dt \leq V(z(t_0)), \quad \forall \tau > t_0 \tag{41}$$

It is easy to see from the special form of (18) that the lower and upper estimates in (39) can be given (e.g. componentwise) for V on any open neighborhood $\mathcal{U} \subset \mathbf{R}_+^m$ of z^*.

Let us choose the performance output w as a linear function of the centered monomials, i.e.

$$w = E(z - z^*) \tag{42}$$

where $E \in \mathbf{R}^{k \times m}$. Using (19) and (42), the condition (40) for the closed-loop system can be written as

$$\frac{1}{2}(z - z^*)(C\tilde{M} + \tilde{M}^T C)(z - z^*) \leq -(z - z^*)^T E^T E(z - z^*) \tag{43}$$

that is equivalent to the feasibility of the following LMI:

$$\frac{1}{2}(C\tilde{M} + \tilde{M}^T C) + E^T E \leq 0 \tag{44}$$

Using (44), it is possible to check whether the solution of a feedback design problem satisfies a given performance criteria (defined by matrix E). If not, then by solving an LMI feasibility problem, it can be easily examined (with fixed feedback parameters), whether there exists such a positive definite diagonal C matrix that satisfies both the stability and the performance criteria.

4 Examples

In the following, some simple process system examples are proposed for the BMI based stabilizing controller design problem discussed so far. The first two are simple continuously stirred tank reactor (CSTR) examples with second order chemical reactions where the system model is naturally in a QP-form. The last one is the simple fermentation example described in sub-section 2.1.6 that has an embedded rank deficient QP model.

4.1 Partially actuated process system example in QP-form

The system of this example is a simpler variant of the fermentation process of subsection (2.1.6) with S_F being the manipulable input:

$$\begin{aligned} \dot{X} &= \mu_{max} S X - \tfrac{F}{V} X \\ \dot{S} &= -\tfrac{\mu_{max}}{Y} S X + \tfrac{F}{V}(S_F - S). \end{aligned} \tag{45}$$

The parameter values can be seen in Table 2. The quasi-polynomial form of the model is:

$$\begin{aligned} \dot{X} &= X\,(S-2) \\ \dot{S} &= S\,(-X + 2S^{-1}S_F - 2)\,. \end{aligned} \tag{46}$$

The system has an asymptotically stable wash-out type equilibrium point

$$\begin{bmatrix} X^* \\ S^* \\ S_F^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The feedback structure was chosen to be

$$S_F = k_1 S^2 + \delta_1 S.$$

The closed loop system with the above structure is

$$\begin{aligned} \dot{X} &= X\,(S-2) \\ \dot{S} &= S\,(-X + 2k_2 S + 2(\delta_1 - 1))\,. \end{aligned} \tag{47}$$

It is apparent, that the above QP model (47) is also the Lotka-Volterra model of the system. The LV matrices of the system are the following ones:

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 2k_1 \end{bmatrix}, \quad N = \begin{bmatrix} -2 \\ 2(\delta_1 - 1) \end{bmatrix}.$$

It is noticeable that matrix M is not upper triangular, i.e. the equilibrium cannot be manipulated partially based on the results of section 3.3.2. However, with a fortunate choice of δ_1 (e.g. $\delta_1 = 2.5$) one can modify the value of the (non wash-out type) equilibrium of system (47). It is important to note, that in this case the equilibrium will be positive, but one cannot decide its value. The other free parameter (k_1) can be used for stabilizing this equilibria. So k_1 and the two parameters of the Lyapunov function are given to the ILMI algorithm. It gives the following results:

$$k_1 = -0.0013, \quad C = \begin{bmatrix} 1.2822 & 0 \\ 0 & 1.2822 \end{bmatrix}.$$

Fig. 1. shows the feasibility region of the globally stabilizing BMI problem and the solution given by the ILMI algorithm. The obtained feedback with parameters k_1 and δ_1 globally stabilizes the system in the positive orthant. Indeed, the closed loop system has a unique equilibrium state in the positive orthant $\mathrm{int}(\mathbf{R}^2_+)$ with eigenvalues having strictly negative real part:

$$\begin{bmatrix} \bar{X} \\ \bar{S} \end{bmatrix} = \begin{bmatrix} 2.9948 \\ 2.0000 \end{bmatrix}.$$

4.2 Fully actuated process system example in QP-form

The second process system example is of the same fermentation process examined in the previous example but this time biomass is also fed to the reactor with manipulable inlet concentration X_F. The parameters of the system are

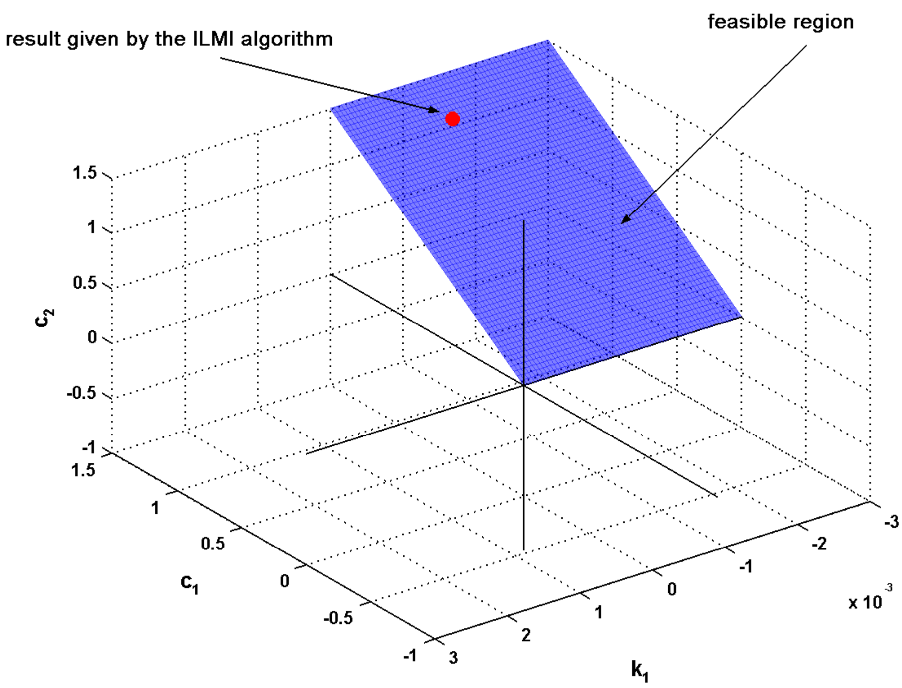

Fig. 1. BMI feasibility region for Example 4.1

the same as in the previous case.

$$
\begin{aligned}
\dot{X} &= \mu_{max} S X + \tfrac{F}{V}(X_F - X) \\
\dot{S} &= -\tfrac{\mu_{max}}{Y} S X + \tfrac{F}{V}(S_F - S)
\end{aligned}
\tag{48}
$$

The quasi-polynomial form of the model is:

$$
\begin{aligned}
\dot{X} &= X\left(S + 2X^{-1}X_F - 2\right) \\
\dot{S} &= S\left(-X + 2S^{-1}S_F - 2\right).
\end{aligned}
\tag{49}
$$

Note that (49) is also the Lotka-Volterra model of the system. The manipulable inputs are X_F and S_F. The system has no equilibrium in the strictly positive orthant but has one asymptotically stable wash-out equilibrium on the boundary

$$
\begin{bmatrix} X^* \\ S^* \\ X_F^* \\ S_F^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.
$$

The feedback structure is chosen to be

$$
\begin{aligned}
X_F &= k_1 X^2 + \delta_1 X \\
S_F &= k_2 S^2 + \delta_2 S.
\end{aligned}
$$

Parameters k_1 and k_2 are to stabilize the system, δ_1 and δ_2 will be used to shift the equilibrium. The closed loop system is

$$
\begin{aligned}
\dot{X} &= X\left(2(\delta_1 - 1) + S + 2k_1 X\right) \\
\dot{S} &= S\left(2(\delta_2 - 1) - X + 2k_2 S\right).
\end{aligned}
$$

The iterative BMI algorithm yielded the following parameters for the feedback and the Lyapunov function:

$$
k_1 = -1.0004, \quad k_2 = -1.0004, \quad C = \begin{bmatrix} 1.0004 & 0 \\ 0 & 1.0004 \end{bmatrix}.
$$

We would like to prescribe a strictly positive equilibrium instead of the original one. Suppose that the desired equilibrium is at

$$
\begin{bmatrix} \tilde{X} \\ \tilde{S} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.
$$

Expressing the values of δ_1 and δ_2 from the state equations in which the desired equilibrium point is substituted in yields

$$
\delta_1 = 1.2502, \quad \delta_2 = 1.7502.
$$

Indeed, the closed loop system with the determined parameters $k_1, k_2, \delta_1, \delta_2$ has an asymptotically stable equilibria in $[\tilde{X}, \tilde{S}]^T$.

It is apparent that in this example with a higher degree of freedom it was possible to shift the steady state point of the system.

4.3 Feedback design for the simple fermentation example

The following example is of the fermentation process (12). The QP-embedded model of the fermenter is the following 3 dimensional system:

$$\dot{X} = X \cdot \left(-\tfrac{F}{V} + \mu_{max} SZ + \tfrac{F}{V} X^{-1} X_F\right)$$

$$\dot{S} = S \cdot \left(-\tfrac{F}{V} - \tfrac{\mu_{max}}{Y} XZ + \tfrac{F}{V} S^{-1} S_F\right)$$

$$\dot{Z} = Z \cdot \left(\tfrac{F}{V} SZ + \tfrac{\mu_{max}}{Y} SXZ^2 - \tfrac{F}{V} ZS_F\right).$$

Using a wise choice of the feedback structure, the quasi-monomials of the closed loop system may decrease. In our case the feedback structure is chosen to be

$$\begin{aligned} X_F &= k_1 SXZ + \delta_1 X \\ S_F &= k_2 SXZ + \delta_2 S. \end{aligned}$$

The closed loop QP system is then

$$\dot{X} = X \cdot \left(-\tfrac{F}{V} + \left(\mu_{max} + k_1 \tfrac{F}{V}\right) SZ\right)$$

$$\dot{S} = S \cdot \left(-\tfrac{F}{V} + \left(-\tfrac{\mu_{max}}{Y} + k_2 \tfrac{F}{V}\right) XZ\right)$$

$$\dot{Z} = Z \cdot \left(\tfrac{F}{V} SZ + \left(\tfrac{\mu_{max}}{Y} - k_2 \tfrac{F}{V}\right) SXZ^2\right).$$

Note, that for the globally stabilizing feedback design phase parameters δ_1, and δ_2 are set to zero, since they will be used for shifting the equilibrium of the closed loop system to the original fermenter's one. It is apparent that the closed loop system has only 3 quasi-monomials: SZ, XZ, SXZ^2.
The solution of the BMI problem gives the feedback gain parameters

$$\begin{aligned} k_1 &= -1.5355 \\ k_2 &= 43.6516, \end{aligned}$$

which makes the system globally asymptotically stable (in the positive orthant) with the Lyapunov function (18) having parameters:

$$c_1 = 0.0010, \quad c_2 = 0.0761, \quad c_3 = 0.0760.$$

The equilibrium (13) of the open loop fermenter can be reset by expressing δ_1, and δ_2 from the steady-state equations. This gives $\delta_1 = 1.7152$, $\delta_2 =$

-20.9293, so the equilibrium point (13) of the fermentation process (12) is globally stabilized.

5 Conclusions and future work

An optimization based globally stabilizing controller design technique for process systems in QP form was presented in this paper. The problem with QP-type state feedback structure is equivalent to a bilinear matrix inequality feasibility problem with one variable set for the controller parameters and another one for the Lyapunov function parameters. The use of an existent iterative LMI algorithm is possible because of the special structure of the problem.

In addition, some partial results on placing the globally stable equilibrium point have also been proposed that is only possible in a fully actuated situation when the input variables are the intensive variables at the inlet.

The results concerning output performance estimation gives a solid basement for the selection of an appropriate objective function that - supplemented with the BMI feasibility problem - gives rise to a controller design procedure that also takes performance specifications into account.

Although some preliminary results in selecting the structure of the QP-type feedback have also been presented, the development of a systematic method for feedback structure selection based on the present results is the target of future research.

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Tables

Table 1
Variables and parameters of the fermenter model (12)

X	biomass concentration		$[\frac{g}{l}]$
S	substrate concentration		$[\frac{g}{l}]$
S_F	substrate feed concentration		$[\frac{g}{l}]$
X_F	biomass feed concentration		$[\frac{g}{l}]$
F	inlet feed flow-rate	1.0000	$[\frac{l}{h}]$
V	volume	97.8037	$[l]$
Y	yield coefficient	0.0097	-
$\mu_{max},$	kinetic parameter	0.0010	$[\frac{1}{h}]$
K_s	kinetic parameter	0.5	$[\frac{l}{g}]$

Table 2
Variables and parameters of the fermenter model (45)

X	biomass concentration		$[\frac{g}{l}]$
S	substrate concentration		$[\frac{g}{l}]$
F	inlet feed flow-rate	2	$[\frac{l}{h}]$
V	volume	1	$[l]$
S_F	substrate feed concentration		$[\frac{g}{l}]$
Y	yield coefficient	1	-
$\mu_{max},$	kinetic parameter	1	$[\frac{1}{h}]$