Stabilizing kinetic feedback design using semidefinite programming

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Abstract A novel state feedback design method is proposed in this paper for the stabilization of polynomial systems with linear input structure. Using a static nonlinear feedback, the open loop system is transformed to a complex balanced kinetic closed loop system with stoichiometric subspace having maximum dimension, that is known to be stable. The feedback law is computed using semidefinite programming where the objective function is used to adjust the performance of the closed-loop system by tuning the largest eigenvalue of the state matrix of the linearized closed-loop system. The approach is illustrated on a purely computational example followed by a simple process system example.

Keywords: non-negative systems, kinetic systems, optimization, chemical reaction networks, feedback equivalence, feedback design

1. INTRODUCTION

Several kinds of important dynamical phenomena in nature or technology can be modelled in the framework of nonnegative systems having the property that the nonnegative orthant is invariant for the dynamics. Notable examples are biochemical reaction networks, models of disease and population dynamics, a wide range of models in the process industries, and certain economical or transportation processes (Takeuchi, 1996; Érdi and Tóth, 1989; Hangos and Cameron, 2001).

Nonnegative systems have several interesting and useful properties that can be utilized in dynamical analysis and control design (Haddad et al., 2010; Farina and Rinaldi, 2000). Kinetic systems form an important class within the family of nonnegative models with increasing research interest in the last decade. The main reasons for this are the following. Firstly, they are suitable for the modelling of complex nonlinear dynamical behaviour, but have a mathematically simple and therefore computationally appealing structure. Secondly, there are numerous (and continuously increasing number of) strong results in the literature on the relation between the graph structure and important dynamical properties of kinetic systems (Feinberg, 1987; Sontag, 2001; Angeli, 2009).

A central notion in our current work is the *complex balanced* property of kinetic systems. Roughly speaking, complex balancing means that the sum of the signed reaction rates corresponding to any complex is zero at equilibrium, and it was originally introduced to characterize the thermodynamic compatibility of reaction networks (Horn and Jackson, 1972). It was shown that the equilibria of complex balanced networks are at least locally asymptotically stable within the so-called stoichiometric compatibility classes, and it was conjectured for more than 40 years (formulated in the Global Attractor Conjecture) that stability is actually global with respect to the nonnegative orthant (Craciun et al., 2009). An important and well-known special case is formed by the class of deficiency zero weakly reversible reaction networks that are complex balanced for any set of positive reaction rate coefficients ensuring a robust stability property for such systems (Feinberg, 1987). A recent fundamental result is the possible general proof of the Global Attractor Conjecture (Craciun, 2015). Kinetic systems are known to have a structural non-uniqueness property meaning that different reaction graphs may give rise to the same kinetic ODEs, where these graphs - the kinetic realizations - can be determined efficiently using optimization (Szederkényi and Hangos, 2011). It is also known that important features like weak reversibility or deficiency, and complex balance are realization properties.

Motivated by the above results, computation methods were proposed in (Lipták et al., 2016) to transform nonlinear polynomial systems into kinetic form via full state feedback. One sub-problem successfully solved in the framework of linear programming was to achieve a complex balance closed loop system with a prescribed equilibrium point. The purpose of this paper is to improve this solution by prescribing new performance specifications that will be shown to lead to semidefinite programming (SDP).

2. BASIC NOTIONS

Kinetic systems are special types of positive polynomial systems, their structure and properties form the basis of the feedback design presented here.

2.1 Polynomial systems

Let us consider a polynomial dynamical system in the form

$$\dot{x} = f(x) = M\,\psi(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $M \in \mathbb{R}^{n\times m}$ is the coefficient matrix. The monomial mapping $\psi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is given by

$$\psi_j(x) = \prod_{i=1}^{n} x_i^{Y_{ij}}, \quad j = 1, \ldots, m, \tag{2}$$

where $Y \in \mathbb{Z}_{\geq 0}^{n\times m}$ is called the *monomial or complex composition matrix.*

2.2 Kinetic systems, their dynamics and structure

We say that the system (1) is kinetically realizable with the complex composition matrix Y if and only if

$$f(x) = Y\,A_k\,\psi(x), \quad \forall x \in \mathbb{R}_{\geq 0}^n, \tag{3}$$

where $\psi(x)$ is generated by Y and A_k is a *Kirchhoff* matrix, i.e. its off-diagonals are non-negative

$$[A_k]_{ij} \geq 0, \quad \forall i \neq j,$$

and its column-sums are zero, i.e.

$$\mathbf{1}^T A_k = \mathbf{0}.$$

The pair of matrices (Y, A_k) is called a *kinetic realization* of the system (1). If there exists a complex composition matrix Y such that the system (1) is kinetically realizable, then this system is called a *kinetic system.* Note, that the realization (Y, A_k) may not be unique even for a fixed Y (see Szederkényi et al. (2011), Szederkényi (2010) and Lipták et al. (2015)).

The non-negative orthant $\mathbb{R}_{\geq 0}^n$ is an invariant subspace of kinetic systems, i.e. if $x_0 \in \mathbb{R}_{\geq 0}^n$, then $x(t) \in \mathbb{R}_{\geq 0}^n$ for all $t \geq 0$ (Haddad et al. (2010)). Therefore, we are only interested in the nonnegative orthant as state space.

The reaction graph The realization (Y, A_k) can be described by a directed, weighted graph. The graph has m vertices where the jth vertex V_j corresponds to the jth column of the matrix Y. The edges are described by the Kirchhoff matrix A_k. There exists an edge $V_j \to V_i$ in the graph with the weight $[A_k]_{ij}$ if and only if $[A_k]_{ij} > 0$.

Weak reversibility and complex balance The dynamic properties of a kinetic system depend on some of the structural properties of the reaction graph, most notably on its connectivity and on its strong components.

A kinetic system is *weakly reversible* whenever there exists a directed path from V_i to V_j in its reaction graph, then there also exists a directed path from V_j to V_i. In graph theoretic terms this means that all components of the reaction graph are strongly connected components. It is equivalent to the algebraic condition

$$A_k\,p = \mathbf{0}, \tag{4}$$

where p is an arbitrary positive vector (see Theorem 3.1 of Gatermann and Huber (2002) and Proposition 4.1 of Feinberg (1979)).

Clearly, the positive vector $x^* \in \mathbb{R}_{>0}^n$ is an equilibrium point of the kinetic system (Y, A_k) if and only if

$$Y\,A_k\,\psi(x^*) = \mathbf{0}. \tag{5}$$

When $\psi(x^*) \in \ker(A_k)$, then the equilibrium point x^* is called a *complex balanced* equilibrium point. It is well-known that if a system (Y, A_k) has a complex balanced equilibrium point, then all of its positive equilibrium points are complex balanced (Horn and Jackson (1972)). Therefore, we can call a kinetic system complex balanced, if (5) is confirmed for any equilibrium point x^*.

Remark: If a kinetic system is complex balanced then it is weakly reversible, too (Horn (1972)).

The *balanced Laplacian matrix* $\mathcal{L}(x^*)$ at a complex balanced equilibrium point x^* of a kinetic system (Y, A_k) is defined in (van der Schaft et al. (2015)) as

$$\mathcal{L}(x^*) = -A_k\,D(\psi(x^*)) \tag{6}$$

where $D(\cdot)$ stands for $diag(\cdot)$. The left and right kernels of the matrix $\mathcal{L}(x^*)$ are equal. Therefore, it is not only column, but row conservative, too.

Stoichiometric subspace The state space of a kinetic system (Y, A_k) can be partitioned into invariant affine subspaces. The stoichiometric subspace $\mathcal{S}$ is defined as

$$\mathcal{S} = \operatorname{span}\left(\{Y_i - Y_j \mid [A_k]_{ij} > 0, \ \forall i \neq j\}\right), \tag{7}$$

where Y_i denotes the ith column of matrix Y. When the system (Y, A_k) is weakly reversible (Feinberg (1979)), then

$$\mathcal{S} = \operatorname{im}(Y\,A_k). \tag{8}$$

The positive stoichiometric compatibility classes are defined as

$$\mathcal{S}_{x_0} = (x_0 + \mathcal{S}) \cap \mathbb{R}_{>0}^n, \tag{9}$$

where $x_0 \in \mathbb{R}_{>0}^n$ is an arbitrary element of the state space. The manifold $\mathcal{S}_{x_0}$ is an invariant of the kinetic system (Y, A_k).

2.3 Stability of a complex balanced kinetic system

The Global Attractor Conjecture (GAC) says the following (Craciun et al. (2009)): a complex balanced kinetic system (Y, A_k) has a unique positive equilibrium point in each positive stoichiometric compatibility class $\mathcal{S}_{x_0}$. Moreover, the equilibrium points are globally asymptotically stable for all positive initial condition $x_0 \in \mathbb{R}_{>0}^n$ in its positive stoichiometric compatibility class $\mathcal{S}_{x_0}$ with the following Lyapunov function

$$V(x) = \sum_{i=1}^{n} x_i(\ln(x_i) - \ln(x_i^*) - 1) + x_i^*. \tag{10}$$

Important special cases were proven in Anderson (2011) and Gopalkrishnan et al. (2013). Moreover, a possible general proof of the conjecture has recently appeared in Craciun (2015).

Remark: The number of the positive equilibrium points depends only on the dimension of $\mathcal{S}$. Therefore, the additional condition

$$\mathcal{S} = \mathbb{R}^n \tag{11}$$

is equivalent to the stability of the unique equilibrium point in the positive orthant.

2.4 Linearization of a complex balanced kinetic system

The linearized version of the complex balanced kinetic system (Y, A_k) around its positive equilibrium point x^* is in the form (Johnston (2011))

$$\begin{aligned} \Delta\dot{x} &= Y A_k D(x^*) Y^T D(1/x^*) \Delta x \\ &= -Y \mathcal{L}(x^*) Y^T D(1/x^*) \Delta x, \end{aligned} \tag{12}$$

where $\Delta x = x - x^*$.

3. FORMULATION AND SOLUTION OF THE FEEDBACK DESIGN PROBLEM

In this section, a stabilizing feedback design method for polynomial systems is presented. The stability of the closed-loop system is guaranteed as a novel semidefinite constraint. To improve the local convergence rate of the closed-loop system, the largest eigenvalue of the linearized system is minimized. The obtained problem is formulated as a semidefinite programming (SDP) problem. Assuming that the proof in (Craciun, 2015) is correct, the asymptotic stability of the equilibrium x^* of the closed loop system will actually be global.

3.1 Unique positive equilibrium point

In this subsection, the stoichiometric subspace is characterized by a positive-semidefinite matrix. The direct consequence of this result is the uniqueness of the complex balanced equilibrium point. If the GAC holds, then the unique equilibrium point will be globally asymptotically stable.

Lemma 1. Let us consider a complex balanced kinetic system (Y, A_k) with its balanced Laplacian matrix $\mathcal{L}(x^*)$. Then the matrix $Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T$ is positive-semidefinite and satisfies

$$\mathrm{im}(Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T) = \mathcal{S}. \tag{13}$$

Proof. The transpose and sum of balanced Laplacian matrices remain Laplacian matrices. Hence, the matrix $\mathcal{L}(x^*) + \mathcal{L}(x^*)^T$ is a balanced and symmetric Laplacian matrix which is positive-semidefinite (Mohar and Poljak (1993)). Therefore, the matrix $Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T$ is positive-semidefinite, too. Then eq. (13) is a direct consequence of the Theorem 4.3.3 in the article of Johnston (2011). □

The following Theorem is a direct consequence of Lemma 1. It gives a positive-definite condition of the full dimensional stoichiometric subspace in the complex balanced case.

Theorem 2. Let us consider a complex balanced kinetic system (Y, A_k) with its balanced Laplacian matrix $\mathcal{L}(x^*)$. Then, $\mathcal{S} = \mathbb{R}^n$ if and only if

$$Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T > 0. \tag{14}$$

Proof. $\Rightarrow$ If $\mathrm{im}(Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T) = \mathbb{R}^n$, then $Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T$ is invertible. Therefore, $Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T$ is positive-definite.
$\Leftarrow$ If $Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)Y^T$ is positive-definite, then it is invertible. Therefore, $\mathrm{im}(Y(\mathcal{L}(x^*) + \mathcal{L}(x^*)T)Y^T) = \mathbb{R}^n$. □

3.2 Asymptotically stabilizing feedback design

The open-loop model We assume for the feedback design that the equations of the open loop polynomial system with linear input structure are given as

$$\dot{x} = M\,\psi_p(x) + Bu, \tag{15}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^r$ is the input, $\psi_p \in \mathbb{R}^n \to \mathbb{R}^{m_p}$ contains the monomials of the open-loop system generated by $Y_p \in \mathbb{Z}_{\geq 0}^{n \times m_p}$, $M \in \mathbb{R}^{n \times m_p}$ and $B \in \mathbb{R}^{n \times r}$.

The state feedback law We assume a polynomial feedback of the form

$$u = K_p \psi_p(x) + K_c \psi_c(x) = K\,\psi(x), \tag{16}$$

where $\psi_c \in \mathbb{R}^n \to \mathbb{R}^{m_c}$ contains the additional monomials generated by $Y_c \in \mathbb{Z}_{\geq 0}^{n \times m_c}$. The matrices $K_p \in \mathbb{R}^{r \times m_p}$ and $K_c \in \mathbb{R}^{r \times m_c}$ are the feedback gains. Then, the closed-loop system can be written as

$$\dot{x} = [M + BK_p \mid BK_c]\,\psi(x) = \overline{M}(K)\psi(x), \tag{17}$$

where $\overline{M}(K)$ is the coefficients matrix of the closed loop system which depends in an affine way on the feedback gain $K = [K_p \mid K_c]$.

3.3 Feedback computation

The goal of the feedback is to transform the given open-loop system (15) into a complex balanced closed-loop system with a given/desired equilibrium point x^* using a suitably extended monomial set. The problem will be formulated as a semidefinite programming problem.

The monomials of the feedback Before the optimization, we have to determine the new monomimals of the feedback $\psi_c(x)$. In the case of $\mathcal{S} = \mathbb{R}^n$, a *necessary condition* is

$$\mathbb{R}^n = \mathrm{span}(\{\overline{Y}_i - \overline{Y}_j \mid \forall i \neq j\}), \tag{18}$$

where $\overline{Y}_i$ is the ith column of the matrix $\overline{Y} = [Y_p \mid Y_c]$ while Y_p and Y_c describe the monomials corresponding to $\psi_p(x)$ and $\psi_c(x)$, respectively. When eq. (18) is not fulfilled, then there does not exist a closed loop system which satisfies eq. (11).

Therefore, if the monomials of the open-loop system represented by the columns of matrix Y_p are not rich enough, i.e.

$$\mathrm{span}(\{[\overline{Y}_p]_i - [\overline{Y}_p]_j \mid \forall i \neq j\}) \subset \mathbb{R}^n$$

the additional monomials, i.e. additional columns in $\overline{Y}$ are needed to achieve eq. (18).

It is important to note that the choice of the new monomials is generally not unique, and it has an impact on the achievable control performance. Therefore, the selection of the new monomials is an important tuning knob of the proposed feedback design method.

The basic constraints The first constraint is used to guarantee that the solution will be a kinetic realization of the closed-loop system. It is in the form

$$\overline{M}(K) = \overline{Y}\, A_k, \tag{19}$$

where $A_k \in \mathbb{R}^{(m_p+m_c)\times(m_p+m_c)}$ and $K \in \mathbb{R}^{r\times(m_p+m_c)}$ are decision variables of the problem and $\overline{Y}$ is given. The Kirchhoff property is required for matrix A_k, so the following constraints are included as well:

$$\mathbf{1}^T A_k = \mathbf{0} \tag{20}$$

$$[A_k]_{ij} \geq 0 \quad, \forall i \neq j. \tag{21}$$

The resulting system $(\overline{Y}, A_k)$ should be complex balanced, which is ensured by the following constraint:

$$A_k \psi(x^*) = \mathbf{0}, \tag{22}$$

that is a linear constraint in A_k, because the equilibrium point x^* is given before the optimization.

Uniqueness of the desired equilibrium point We can guarantee the uniqueness of the desired equilibrium point x^* with setting the stoichiometric subspace $\mathcal{S} = \mathbb{R}^n$. This can be formulated as a semidefinite constraint (see Theorem 2.)

$$\overline{Y}(\mathcal{L}(x^*) + \mathcal{L}(x^*)^T)\overline{Y}^T > 0, \tag{23}$$

where $\mathcal{L}(x^*)$ is the balanced Laplacian matrix of A_k in the point x^*.

Performance Since the constraint set is formulated in a way that all the solutions are guaranteed to be complex balanced with $\mathcal{S} = \mathbb{R}^n$. Therefore, we can be sure that all eigenvalues of the closed-loop system have negative real parts. Hence, a suitable performance of the feedback design can be achieved by minimizing the largest eigenvalue of the linearized closed-loop system

$$\min \left\{ \mathrm{Re}(\lambda_{\max}(-\overline{Y}\mathcal{L}(x^*)\overline{Y}^T D(1/x^*))) \right\}, \tag{24}$$

where $\lambda_{\max}$ denotes the eigenvalue of its argument with the largest real part.

In the general case, the problem (24) is a non-smooth optimization problem. Therefore, we consider the relaxed version (Boyd and Vandenberghe (2004)) of the above objective (24):

$$\min \left\{ \lambda_{\max}(-\overline{Y}\mathcal{L}(x^*)\overline{Y}^T D(1/x^*) - D(1/x^*)\overline{Y}\mathcal{L}(x^*)^T\overline{Y}^T) \right\}. \tag{25}$$

This objective contains a symmetric matrix in the argument, therefore its eigenvalues are negative real numbers.

Using the above relaxed objective a semidefinite programming problem can be formulated for the feedback design as follows:

$$\min -t \tag{26}$$

subject to

$$\overline{Y}\mathcal{L}(x^*)\overline{Y}^T D(1/x^*) + D(1/x^*)\overline{Y}\mathcal{L}(x^*)^T\overline{Y}^T - tI \geq 0, \tag{27}$$

where t is a variable of the optimization. Choosing larger t values lead to solutions with smaller $\lambda_{\max}$ values, hence closed-loop systems with faster local convergence.

By putting together the constraints described in (19)-(23), (27) and considering (26) as the objective function, the SDP optimization problem can be constructed. The parameters of the optimization are the monom composition matrix $\overline{Y}$ and the desired equilibrium point x^*. The decision variables of the optimization are the Kirchhoff matrix A_k, the set of feedback gains K and the auxiliary variable t.

Note that the formulated optimization problem may not be feasible. In this case we may choose additional monomials in the polynomial feedback to make the problem feasible.

4. EXAMPLES

In the following, we present the applicability of the proposed design technique on two different examples. The algorithms were implemented in MATLAB (2012) using the YALMIP modelling language Löfberg (2004). MOSEK (2015) was used to solve the SDP problems.

4.1 Computational example

In this section, the proposed design method is demonstrated by a computational example. The open-loop model is unstable and does not have any complex balanced realization.

Let the open-loop system be given as

$$\dot{x} = \underbrace{\begin{bmatrix} 5 & -5 & -3 \\ 4 & 3 & -5 \\ -1 & 0 & 2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_1x_2x_3 \\ x_1^2x_2 \\ x_1x_2^2 \end{bmatrix}}_{\psi_p(x)} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{B} u. \tag{28}$$

Let the desired equilibrium point be chosen as $x^* = [1\ 2\ 4]^T$. It is easy to see that condition (18) is not fulfilled. Therefore, we choose a new monomial $\psi_c(x) = x_3$ to be able to achive $\mathcal{S} = \mathbb{R}^n$.

Now let us apply the feedback design method which is proposed in this paper. The optimization problem described by (19)-(23), (27) and the objective function (26) is formulated by plugging in x^* and

$$\overline{Y} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

The result of the SDP problem is the following feedback gain:

$$K = [\, -10.1912\ \ 1.0000\ \ 3.0000\ \ 12.3824\,]. \tag{30}$$

The closed-loop system is complex balanced with the equilibrium point x^* and the locally linearized model has the eigenvalues $\lambda = \{-2.8562, -7.0820, -78.3564\}$. Fig. 1 shows the time domain simulation of the closed loop system with five different initial values.

The effect of the chosen feedback monomials As it was mentioned in section 3.3.1, we can choose different new feedback monomials $\psi_c(x)$ such that the condition (18) is fulfilled. In order to illustrate the effect of this choice on the controller design, we present two different choices and their resulting feedback design.

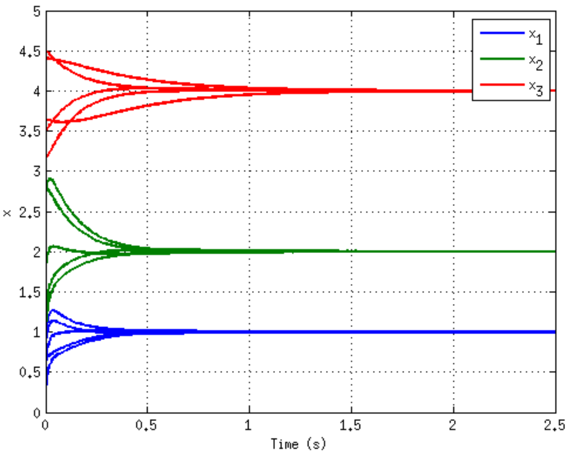

Figure 1. Time domain simulation of the closed loop system from Subsection 4.1 with 5 different initial values.

- The choice of the monomial $\psi_c(x) = x_1x_3$ results in an infeasible problem.
- With the choice $\psi_c(x) = x_1x_2$, the problem will be feasible, but the achieved performance in terms of the largest closed loop eigenvalue $\lambda_{\max} = -2$ is larger than that of the above design (with $\psi_c(x) = x_3$), that is -2.8562. This results a slower local convergence of the design with $\psi_c(x) = x_1x_2$.

4.2 Process control example

Hereby we consider the open chemical reaction network which is presented in Lipták et al. (2016) Section 5.2. Let us recall the set of chemical reactions involved:

$$\begin{aligned} &2X_1 + X_3 \mapsto X_1 + X_2, \quad X_1 + X_2 \mapsto X_1 + X_3, \\ &X_1 + X_3 \mapsto X_1 + X_2, \quad X_1 + 2X_2 \mapsto X_1 + X_2, \\ &X_1 + 2X_2 \mapsto X_1 + X_3 \end{aligned}$$

where all the reaction rate constants are equal to 1 and isothermal conditions are held. We assume constant volume, hence the outflow appears in open-loop system model as three different linear reaction in the form $X_i \mapsto \emptyset$, where $\emptyset$ is the zero complex.

The inlet concentrations of the species X_1 and X_2 are considered as input variables. Their nonnegativity is ensured by requiring non-negative feedback gains.

The open-loop system is defined as

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & -3 & 1 & 0 & -1 & 0 \\ -1 & 2 & 2 & -2 & 0 & 0 & -1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_1^2x_3 \\ x_1x_2 \\ x_1x_2^2 \\ x_1x_3^2 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\psi_p(x)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{B} u. \tag{31}$$

Note that for perfectly stirred process systems with outflow there appears always a unit matrix I as a block of the open loop complex composition matrix Y, therefore condition (18) is always fulfilled. Thus there is no need for adding new monomials in the feedback, unless we find that the optimization problem is not feasible or the obtained control performance is not satisfactory.

Let the desired equilibrium point be $x^* = [1\ 1\ 1]$. In Lipták et al. (2016) the following feedback is proposed:

$$u_1 = x_1 + x_3 \quad , \quad u_2 = x_1^2x_3 + x_2 + x_3 \tag{32}$$

which transforms the system into a complex balanced one. Note, that in this case $\psi_c(x)$ is an empty vector. The corresponding non-negative feedback gain K is

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{33}$$

The obtained closed loop system is linearized around the equilibrium point x^* to get its eigenvalues $\lambda = \{-1, -2, -12\}$.

Now let us apply the feedback design method which is proposed in this paper. While considering the same open-loop system, our aim is to find a non-negative feedback K' which ensures faster local convergence while keeping the feedback structure unchanged. To accomplish that, the optimization problem described by (19)-(23), (27) and the objective function (26) is formulated by plugging in x^* and

$$\overline{Y} = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}. \tag{34}$$

By solving the resulting SDP problem the following feedback gain is obtained:

$$K' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.3402 & 1.0969 & 0.5629 \\ 0 & 0 & 0 & 0 & 2.3472 & 0.2157 & 0.4371 \end{bmatrix}. \tag{35}$$

Again, the eigenvalues of the linearized closed loop system are computed resulting in $\lambda' = \{-1.9144, -2.8688, -11.6609\}$.

As it can be seen $\lambda'_{max} < \lambda_{max}$ showing that feedback gain K' ensures faster local convergence.

The time-domain behaviour of the two closed-loop systems are simulated using the two different feedback gains (K, K') starting with the initial condition $x_0 = [0.7679, 1.1746, 1.2170]$. The simulation results are shown in Fig. 2, where we can see that indeed a faster local convergence could be achieved by the new feedback design method.

Note, that the closed loop system has a stoichiometric subspace with dimension 3 so the equilibrium point x^* is unique and asymptotically stable.

5. CONCLUSIONS

A novel state feedback design method is proposed in this paper for the asymptotically stabilization of polynomial systems with linear input structure. A static nonlinear feedback structure is selected with possibility to include new monomials in the feedback gain. This way the open loop system can be transformed to a complex balanced kinetic closed loop system with a stoichiometric subspace of maximal dimension, that is known to be (at least) asymptotically stable.

The feedback law is computed using semidefinite programming. The objective function is chosen to be the largest

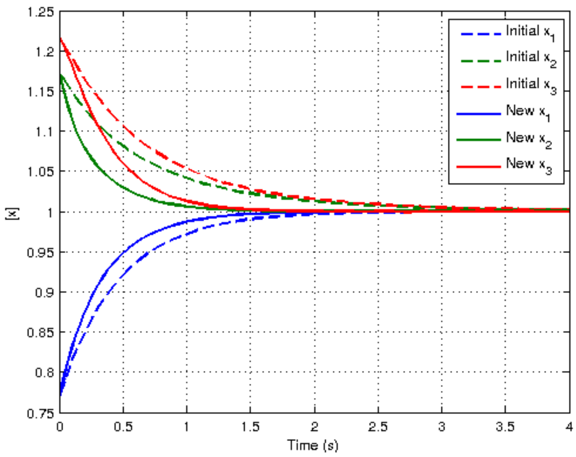

Figure 2. Time domain simulation of the closed loop systems from Subsection 4.2. Trajectories called "Initial" are obtained from the system described in Lipták et al. (2016) while "New" trajectories are generated by the controller designed by the method proposed in the this paper, using feedback gains K and K', respectively. Note the difference in the speed of convergence.

eigenvalue of the state matrix of the linearized closed-loop system, that enables to adjust the performance of the closed-loop system by finding the fastest local convergence to the specified equilibrium point.

The approach is illustrated on two simple examples. The first one is a computational example where the effect of selecting the additional new monomials in the feedback is illustrated. Finally, a simple process system example is also given.

Further work will be focused on extending our method to handle parametric uncertainty, similarly to our previous results Lipták et al. (2016).

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