Comments in English to the hand–written pages No.I. – No.XX. on Nonlinear Dynamical Systems

ad I.: HOW TO SKETCH PHASE PORTRAITS?
examples and more geometry less algebra
The trace–determinant diagram
and THE METHOD OF LINEARIZATION ARE TRULY FUNDAMENTAL
figures are presented separately
ENCIRCLED 1
since $\lambda_{1,2} = -1 \pm i \notin \mathbb{R}$, we pass[1] to a second order equation

ad II.: PROJECTION to the x–y plane along the t axis
the final Figure is the PHASE PORTRAIT
one can do the same for systems of the form $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$

ad III.: a simple method
spiralling towards the origin
clockwise or counterclockwise?
it is enough to investigate the vector field at a single point
clockwise means rotation to the left
are given
at each point of the phase portrait, we know
the tangent vector of the solution curve that passes through the given point

this method leads to all non–degenerate cases
of the trace–determinant diagram
if

ad IV.: ENCIRCLED 2 degenerate cases: $\forall k$ such that $\operatorname{Re} \lambda_k = 0$
the method of linearization does not work,
higher order terms cannot be neglected
not properly chosen numerical methods
may lead to false conclusions, too

Newton second law for the spring
encircled L in the special case $m = 1$, $k = 1$

energy stored in the spring + kinetic energy = constant[2]
but
encircled N

[1] from a system of two first order (differential) equations
[2] along an arbitrary trajectory: different constants for different trajectories, depending on the initial conditions

and ... as

but
encircled M
... as

ad V.: The

a scalar product having a nice geometrical meaning, i.e.,
the scalar product between the normal vector of a level surface of the energy and the tangent vector of a trajectory of the differential equation ...
at an arbitrary point $\binom{x}{y} \in \mathbb{R}^2$

encircled L encircled N encircled M
centrum global attraction global repulsion

IN GENERAL: (E) as an abbreviation of *equation* & V being a LIAPUNOV FUNCTION

obtuse angle[3]
"inward intersection"

ad VI.:

explicit Euler method, with stepsize $h > 0$

the numerical energy at time instant kh

... fixed ... if ...
... fixed ... if ...

Does the explicit Euler method[4] conserve the energy?
no, at least not in a "good enough" way

ENCIRCLED 3 not only the energy can be a Liapunov function[5]

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[3] between the normal vector of a level surface of the Liapunov function V and the vector field f, i.e., the tangent vector of a trajectory of the differential equation (E)

[4] applied for the system $\dot{x} = y$, $\dot{y} = -x$ — in other words, applied to the centrum case L (introduced on pages IV-V) which is a degenerate case

[5] downward intersections with a family of horizontal line segments and leftward intersections with a family of vertical line segments — OBSERVATION: the vector field is horizontal on the line $y = x - 1$ and vertical on the line $y = 1 - \frac{x}{2}$

ad VII.: Jacobian

saddle point $O = \binom{0}{0}$ saddle point P attracting focus N approached by spirals[6] on the bases of arrow directions

the huge rectangle[7] captures all the trajectories

(horizontal and vertical segments as) pieces of Liapunov surfaces Is[8] it hard?

For the time being, the existence of periodic orbits around N cannot be excluded. We need non–local methods to this end.[9]

ad VIII.:
V turns out to be a strong Liapunov function[10] on $\mathrm{int}\mathbb{R}^2_+$
$\Rightarrow$ N is a globally attracting equilibrium point excluding the possibility of any periodic orbit
In fact,
function V attains its minimum at $N = \binom{4/3}{1/3}$, all the remaining level sets are simple closed curves, and all intersections by the trajectories are inward

Where did we get function V from? How did we come up to this idea?
The reason is this:

a separable differential equation[11]:
(excepting N, and the four trajectories on the boundary half–lines) all the trajectories of (E) are periodic orbits around N

ad IX.: ENCIRCLED 4 Sometimes an elementary argument is enough

on the boundary[12] $\partial\mathbb{R}^2_+$

a little bit above equilibrium P

a little bit to the right of equilibrium Q

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[6] more precisely, by spirals rotating in the + (in other words, in the counterclockwise) direction

[7] the positively invariant subset of the non–negative orthant $[0,\infty)\times[0,\infty)$ on the previous page — see also page No.XVI

[8] the global phase portrait of the prey–predator Lotka–Volterra system (E) $\dot{x} = x(1 - \frac{x}{2} - y)$, $\dot{y} = y(-1 - y + x)$ (for the biologically relevant non–negative orthant)

[9] Function V defined in the first line of the forthcoming page VIII will help.

[10] $\dot{V}_{(E)}(x,y)$, the derivative of V along the trajectories of equation (E) is strictly negative for $\binom{4/3}{1/3} \neq \binom{x}{y} \in (0,\infty)\times(0,\infty)$

[11] the simplified Lotka–Volterra prey–predator system $\dot{x} = x(\frac{1}{3} - y)$, $\dot{y} = y(-\frac{4}{3} + x)$ reduces to a separable differential equation which can be solved explicitely and implies that

[12] $\partial\mathbb{R}^2_+ = \{(x,0) \in \mathbb{R}^2_+ \mid x \geq 0\} \cup \{(0,y) \in \mathbb{R}^2_+ \mid y \geq 0\}$

in the vicinity of N as well as on the whole $\mathrm{int}\mathbb{R}^2_+$

N is a globally attracting focus

equilibria O, P, Q are saddle points

ad X.: ENCIRCLED 5

repelling node   attracting node   attracting node

saddle point

the stable (ingoing) and unstable (outgoing) curves[13] at N as
the essence of the global phase portrait

ad XI.: The above result was obtained gradually[14],
via combining and extending the local phase portraits around equilibria,
and using the repelling property of the "point at infinity"

it is intuitively evident,
that the two outgoing trajectories at N approach P and Q
and that the two ingoing trajectories at N arrives from O and from the "point at infinity"

Essentially, the fourfold intuitive observation is basically enough.

The detailed argument is as follows:

I.) trajectories entering the shaded triangle remain there forever

II.) and are attracted by the equilibrium point P

III.) There is a full trajectory repelled by N and remaining in the shaded triangle forever

IV.) The nonexistence of periodic orbits follows from the Poincaré–Bendixson Theorem[15].

ad XII.: In order to make the phase portrait "nicer" (and more appropriate), observe that

[13] in more general and more precise mathematical terms: the stable manifold and the unstable manifold of the saddle point N of a two–species Lotka–Volterra system with competitive exclusion

[14] WHEN DRAWING THE PHASE PORTRAIT, GEOMETRY AND ALGEBRA GO STEPWISE HAND IN HAND

[15] discussed on page XVII: in fact, the interior of any periodic orbit in 2D contains an "extra" equilibrium point and this is impossible by I.) – III.)

trajectories near O in $\text{int}\mathbb{R}^2_+$ are tangent to the horizontal eigenvector $\underline{s}_2$

trajectories near $P = \binom{2}{0}$ in $\text{int}\mathbb{R}^2_+$ are tangent to the horizontal eigenvector $\underline{s}_1$

remark:

AND NOW THE CRITICAL FIGURE
BELONGING TO THE $\mu = 2$ BIFURCATION VALUE

ENCIRCLED 6
above P a half–saddle
below P a half–node

[we are facing a degenerate case within a transcritical bifurcation]

ad XIII.: ENCIRCLED 7

where $\mu > 0$ is a parameter

equilibrium points

local phase portraits about O, P, Q, N
and some further characteristics of the vector field

the following subsets of $\text{int}\mathbb{R}^2_+$ are attracted by P and Q, respectively

the entire set for $0 < \mu < 1$, a decreasing subset for $1 < \mu < 2$, the empty set for $2 < \mu$

the empty set for $0 < \mu < 1$, an increasing subset for $1 < \mu < 2$, the entire set for $2 < \mu$

The larger $\mu \geq 0$, the better the competitiveness of species y

ad XIV.:
the rise of parameter μ can be interpreted as a larger birthrate

and as a larger carrying capacity[16], too

$\mu = 1$: $N(\mu)$ enters $\mathbb{R}^2_+$
$\mu = 2$: $N(\mu)$ exits $\mathbb{R}^2_+$

[16]in other words, more natural resources for the second species y

$\Leftrightarrow$ the method of linearization about equilibrium N does not work alone

N is an attracting focus	if $\mu \in (-\infty, 1-\sqrt{2})$
N is an attracting node	if $\mu \in (1-\sqrt{2}, 1)$
N is a saddle point	if $\mu \in (1, 2)$
N is an attracting node	if $\mu \in (2, 1+\sqrt{2})$
N is an attracting focus	if $\mu \in (1+\sqrt{2}, \infty)$

the motion of $N = N(\mu)$ on the T–D diagram[17]
there is no bifurcation at parameter $\mu = 1 + \sqrt{2}$

the motion of $N = N(\mu)$ on the plane $\mathbb{R}^2$ of the x, y variables[18]
with a transcritical bifurcation at $\mu = 2$
the (essence of this transcritical) bifurcation[19]

ad XV.: Vocabulary for planar dynamical systems
stable node/focus $\Leftrightarrow$ attracting node/focus
unstable node/focus $\Leftrightarrow$ repelling node/focus

for general equilibria on the plane[20]: stability $\not\Rightarrow$ attractivity and attractivity $\not\Leftarrow$ stability

Assume[21] existence, uniqueness, and continuous dependence (on initial conditions) for the autonomous differential equation $\dot{x} = f(x)$

Definitions:

$x_0 \in \mathbb{R}^n$ is an equilibrium point $\Leftrightarrow$ ...
stable $\Leftrightarrow$ ...[22]
attractive $\Leftrightarrow$ ...
asymptotically stable $\Leftrightarrow$ both stable and attractive
region of attraction[23] $\Leftrightarrow$ ...
unstable $\Leftrightarrow$ not stable

[17] explained on the half–line $T = -1$, $D \geq -\frac{1}{4}$ as a downward–upward motion on the trace–determinant diagram (the case $\mu < 0$ is not displayed)

[18] explained on the half–line $\frac{x}{2} + \frac{y}{1} = 1$, $x \geq -2$ as a downward and rightward motion (the case $\mu < 0$ is not displayed)

[19] is shown by the trajectories of equation $\dot{y} = y(\mu - 2 + y)$ on three vertical lines of the (y, μ) plane (the separation of cases corresponds to parameter values $\mu < 2$, $\mu = 2$, $\mu > 2$, respectively)

[20] the first example is given both in standard orthogonal (Cartesian) and polar coordinates, the second example is given only in polar coordinates

[21] We are speaking about the autonomous differential equation $\dot{x} = f(x)$ and its solution operator $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ — Notation $x_{0,x_0}(\cdot) : \mathbb{R} \to \mathbb{R}^n$ is used for the individual solution of the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$ — "Tfh" is the Hungarian abbreviation for "assume that"

[22] — in a mathematical text, "hogy" is the Hungarian equivalent for "such that"

[23] of an asymptotically stable equilibrium point $x_0 \in \mathbb{R}^n$

ad XVI.: as a continuation of page No.XV:
further basic definitions and Bolzano–Weierstrass type theorems

(X, d) is a metric space
$\Phi : \mathbb{R} \times X \to X$ is a dynamical system if (the following three axioms are all satisfied)
(i) Φ is [in its joint variables (t, x)] continuous

$M \subset X$ is an invariant set if ... or, equivalently, ...
$\gamma(x) = ...$ is the trajectory passing through x

positive/negative half–trajectory: ...
$\omega(x) = ...$such that...and... $\omega(x)$ is the ω–limit set of the point $x \in X$
$\alpha(x) = ...$such that...and... $\alpha(x)$ is the α–limit set of the point $x \in X$

From now on, let $X = \mathbb{R}^d$ and let $\Phi : \mathbb{R} \times X \to X$ be a dynamical system
and let $M \subset \mathbb{R}^d$ be a closed invariant set with respect to Φ

ad XVII.:
Theorem: Let $\gamma^+(x)$ be a bounded subset of X
$\Rightarrow$ $\omega(x)$ is a non–empty, bounded and closed, invariant, and connected subset (of X)
and ... if ... .

Theorem (Poincaré–Bendixson): we have more consequences in the $d = 2 \Leftrightarrow$ planar case.
assume that the number of equilibria is finite and
assume that the positive half–trajectory $\gamma^+(x)$ is bounded in $\mathbb{R}^2$.
$\Rightarrow$ there are three alternatives for the ω–limit set $\omega(x)$: $\omega(x) = P$ an equilibrium point
Γ a periodic orbit
H a heteroclinic cycle
and, in addition, there exists at least one equilibrium point in the interior of Γ and of H, respectively.

One more theorem for the planar case: a 2D Lotka–Volterra system has
no isolated periodic orbit. Moreover, isolatedness can be replaced by one–sided isolatedness, too:
Actually, if there is a periodic orbit,
then $(0, \infty) \times (0, \infty)$ is filled by periodic orbits encircling a single equilibrium of centrum type.

ad XVIII.: Example:
equilibrium ponts ... Jacobian ...
a stable node P ... trajectories near P are tangent to eigenvector $\underline{s}_1$
Q is a saddle point

Now a Bolzano–Weierstrass argument (in France: a Darboux–Weierstrass argument) shows that
level curves in increasing order ...

equilibria P and Q are connected by a trajectory inside the crescent–shape region

existence of a $Q \to P$ connecting orbit is provided by the boundary behavior of the vector field

ad XIX.: An alternative argument:
... via level curves of a Liapunov function

rather upward, than leftward ... the global phase portrait[?!]
in the third quarter of the plane
[there are no upward escape in the second quarter]

strong symmetry properties
simplify the task of drawing global phase portraits considerably
this is also easy

ad XX.: In a small vicinity of non–degenerate[24] equilibria, both linearization and discretization are near–to–identity coordinate transformations (mapping continuous and discrete trajectory segments with time–orientation to trajectory segments with time–orientation, preserving time[25] and moving points as little as desired)

continuous, with continuous inverse

(N) nonlinear (L) linear (D) discretized

example

the z axis (of equation $y = 0$) is invariant for the nonlinear, linear and discretized[26] dynamics alike

$z = 0$ [unstable subspace (of the linear dynamics), $z = u(y)$, $z = u_h(y)$ unstable manifolds]
(of the nonlinear and the discretized dynamics, respectively[27])

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[24] $0 \in \mathbb{R}^n$ is a degenerate equilibrium of the differential equation $\dot{x} = Ax + a(x)$ (where $a \in C^1$, $a(0) = 0$, $a'(0) = 0 \in$ matrices of order n) if $\mathrm{Re}\,\lambda_k \neq 0$ for each $k = 1, 2, \ldots, n$

[25] this is the geometry behind formulas $\mathcal{H}(\Phi(t,x)) = e^{At}\mathcal{H}(x)$ and $\mathcal{H}_h(\Phi(h,x)) = \varphi(h, \mathcal{H}_h(x))$—please observe that $\mathcal{H}(0) = 0$, $\mathcal{H}_h(0) = 0$ for each $0 < h \leq h_0 \ll 1$, and $\mathcal{H}_h(\Phi(kh,x)) = \varphi^k(h, \mathcal{H}_h(x))$ by induction on k

[26] the explicit Euler method $\varphi_{EE}(h,x) = x + h(Ax + a(x))$ can be replaced by any reasonable p–th order one–step discretization operator $\varphi(h,x)$ with stepsize $0 < h \leq h_0 \ll 1$

[27] The abstract result on discretizations stated on this page guarantee that—provided $\mathrm{Re}\,\lambda_k \neq 0$ for each $k = 1, 2, \ldots, n$ (and rounding errors aside)—THE DYNAMICS SHOWN ON THE COMPUTER SCREEN IS A (both qualitatively and quantitatively) RELIABLE APPROXIMATION OF THE EXACT DYNAMICS NEAR NON–DEGENERATE EQUILIBRIA. Birth of new and death of old qualitative properties in a parametrized family of autonomous differential equations $\dot{x} = f(\mu, x)$, $\mu \in \mathbb{R}$ are called bifurcations. The simplest and most frequently occuring bifurcations of equilibria and of periodic orbits are well understood. Please consider the normal form $\dot{x} = \mu - x^2$, $\mu \lesseqgtr 0$ of the saddle–node and the normal form $\dot{x} = \mu x + y - x(x^2 + y^2)$, $\dot{y} = -x + \mu y - y(x^2 + y^2)$ $\Leftrightarrow$ $\dot{r} = \mu r - r^3$, $\dot{\varphi} = -1$, $\mu \lesseqgtr 0$ of the Hopf bifurcation, respectively. See also footnote No.19.