Nonlinear Oscillators: from circuit models to applications

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PPCU - Budapest - November 25th, 2015

Aims of this lecture

- Study of the asymptotic oscillatory behavior in nonlinear dynamic systems
- Spectral methods (Harmonic Balance and Describing Function technique)
- Examples of continuous-time nonlinear dynamic systems (Van der Pol circuit and Chua circuit)

Nonlinear systems/circuits: limit cycles

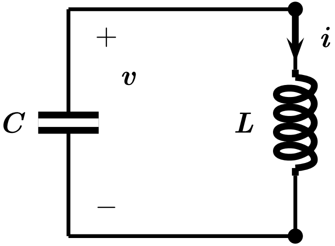
$$\frac{d\ x}{d\ t} = \boldsymbol{f}(x, t) \qquad x \in R^n, \quad t \in R^+$$

A solution $x(t) = \boldsymbol{\Phi}(t, x_0)$ is said to be periodic if there exists T such that:

$$\forall\ t: \quad \boldsymbol{\Phi}(t + T, x_0) = \boldsymbol{\Phi}(t, x_0)$$

The image of $\boldsymbol{\Phi}(t, x_0)$ in the state-space (or phase-space) R^n is called periodic trajectory or limit cycle γ of period T.

Linear systems/circuits: limit cycles

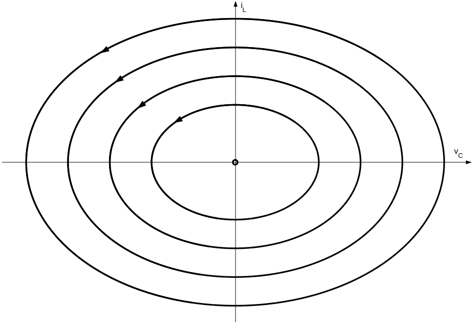


$$\frac{dv}{dt} = -\frac{i}{C}$$
$$\frac{di}{dt} = \frac{v}{L}$$

Eigenvalues: $\lambda_{12} = \pm j\omega \qquad \omega = \frac{1}{\sqrt{LC}}$

The circuit presents infinitely many non-isolated cycles with the same frequency. The cycle amplitude depends on the initial conditions.

Linear systems/circuits: limit cycles



The energy in the circuit is:

$$w(t) = 0.5\, L\, i_L^2 + 0.5\, C\, v_C^2, \quad \Rightarrow \quad \frac{dw}{dt} = C\, v_C \frac{dv_C}{dt} + L\, i_L \frac{di_L}{dt} = 0$$

The following trajectories satisfy the state equation

$$0.5\, L\, i_L^2(t) + 0.5\, C\, v_C^2(t) = 0.5\, L\, i_L^2(0) + 0.5\, C\, v_C^2(0) = costant$$

Nonlinear systems/circuits: limit cycles

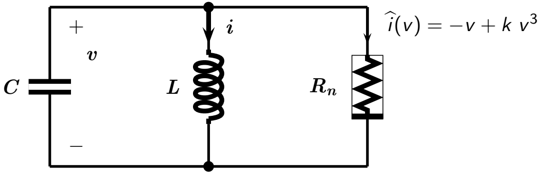


Figure: Van der Pol circuit

$$\hat{\imath}(v) = -v + k\ v^3$$

$$\frac{dv}{dt} = -\frac{1}{C}\left[i + \hat{\imath}(v)\right]$$

$$\frac{di}{dt} = \frac{v}{L}$$

The circuit presents a single limit cycle, that attracts all the trajectories.

Equilibrium point analysis

Equilibrium point $\bar{x} = (v, i) = (0, 0)$

$$\boldsymbol{J} = \begin{pmatrix} \frac{1}{C} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{pmatrix}$$

$$\lambda_{12} = \frac{1}{2C} \pm \frac{1}{2LC}\sqrt{L^2 - 4LC}$$

$$\begin{cases} L < 4C & \text{unstable focus} \\ L > 4C & \text{unstable node} \end{cases}$$

The circuit has no stable equilibrium points. It can be shown that voltage and current are bounded as $t \to \infty$.

Bounded state

Let us consider the state function

$$V(v,i) = 0.5Cv^2 + 0.5Li^2 \;\Rightarrow\; \frac{dV}{dt} = v(v - kv^3)$$

It follows that

$v(t)$ is bounded

1. if $v(t) \to \infty$ then $V(v,i) \to \infty$, but this is impossible as $\frac{dV}{dt} < 0,\ \forall |v| < \frac{1}{\sqrt{k}}$
2. As a consequence, $v(t) \nrightarrow \infty$ as $t \to \infty$, i.e.
3. $\exists M$ such that $|v(t)| < M,\ \forall t$

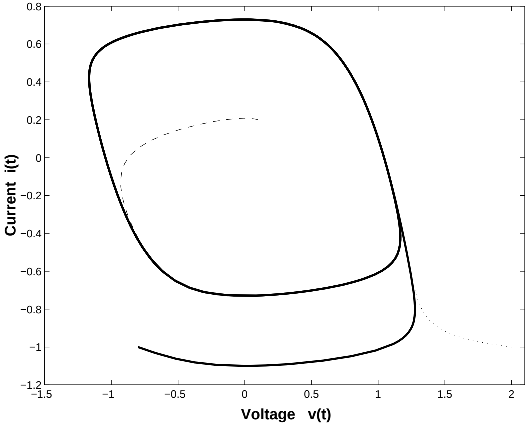
Bounded state

$i(t)$ is bounded

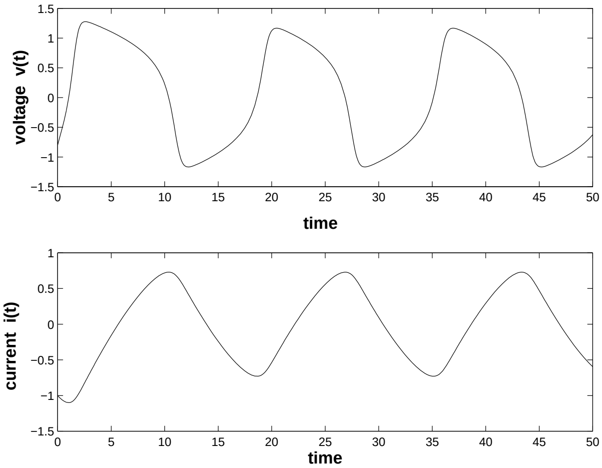
1. if $i(t) \to \pm\infty$ then $\frac{dv}{dt} \sim -\frac{i}{C}$, because $v(t)$ is bounded.
2. if $i(t) \to -\infty$ then $\frac{dv}{dt} > 0$, $\Rightarrow$ $v(t) \to +\infty$ is impossible (as $v(t)$ is bounded)
3. if $i(t) \to +\infty$ then $\frac{dv}{dt} < 0$, $\Rightarrow$ $v(t) \to 0$, $\Rightarrow$ $\frac{di}{dt} = \frac{v}{L} \to 0$ is impossible (due to the assumption $i(t) \to +\infty$)
4. it follows that $\exists M$ such that $|i(t)| < M$, $\forall t$

Limit cycle

$$L = \frac{9}{2} \quad C = 1 \quad k = 1$$



Time waveforms



Computation of limit cycles

- Determination of all the periodic limit cycles (either stable and unstable) and their stability properties (Floquets multipliers FMs)
  - In large scale dynamical systems the sole **time–domain** numerical simulation does not allow to identify all the limit cycles (either stable and unstable)
  - It would require to consider infinitely many initial conditions
  - Unstable limit cycles cannot be detected through simulation
- By means of Spectral methods, the computation of all the limit cycles is reduced to non-differential (sometimes algebraic) problem.
  - Harmonic Balance Technique
  - Describing Function Technique

Computation of limit cycles: Time-domain methods

- If the system possesses a stable cycle γ, we can try to find it by numerical integration (simulation). If the initial point for the integration belongs to the basin of attraction of γ, the computed orbit will converge to γ in forward time. Such a trick will fail to locate a saddle cycle, even if we reverse time.
- there exist different time domain methods especially to directly locate periodic orbits even if they are saddle or unstable cycles. The problem of finding the steady state is converted into a boundary-value problem, to which the standard approaches, such as shooting methods and finite-difference methods, can be applied.
- $\Phi(t_0 + T, x_0) = \Phi(t_0, x_0) = x_0$, where the minimum cycle period T is usually unknown. An extra phase condition has to be added in order to select a solution among all those corresponding to the cycle.

Computation of limit cycles: Spectral methods

1. *Harmonic Balance (HB)* and *Describing function (DF)* techniques permit
    - to determine the set of all stable and unstable limit cycles.
    - to provide an accurate characterization of each limit cycle
        - A. I. Mees, *Dynamics of feedback systems*, John Wiley, New York, 1981.
2. *Floquet's multipliers* permit
    - to investigate limit cycle stability and bifurcations. They can be determined by exploiting either a time-domain or a frequency domain approach.
        - F. Bonani and M. Gilli, "Analysis of stability and bifurcations of limit cycles in Chua's circuit through the harmonic balance approach," *IEEE Transactions on Circuits and Systems: Part I*, vol. 46, pp. 881-890, 1999.

Computation of limit cycles: Spectral methods

Fundamental concepts

1) A periodic solution $\Phi(t, x_0) = x(t) = x(t+T) \in R^n$ can be expanded through the Fourier series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega t) + B_k \sin(k\omega t) \qquad \omega = \frac{2\pi}{T}$$

$$\begin{aligned} A_0 &= \frac{1}{T}\int_0^T x(t)dt \in R^n \\ A_k &= \frac{2}{T}\int_0^T x(t)\cos(k\omega t)dt \in R^n \\ B_k &= \frac{2}{T}\int_0^T x(t)\sin(k\omega t)dt \in R^n \end{aligned}$$

Computation of limit cycles: Spectral methods

Fundamental concepts

2) Approximated representation of $x(t)$ by means of a finite number of N harmonics

$$x(t) \approx A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)$$

Computation of limit cycles: Spectral methods

Fundamental concepts

3) Given a non autonomous nonlinear dynamical system $\frac{dx}{dt} = f(x,t)$, the r.h.s $f(x,t)$ can be expanded through the Fourier series by using N harmonics

$$f(x,t) = F_0 + \sum_{k=1}^{N} F_{A_k} \cos(k\omega t) + F_{B_k} \sin(k\omega t) \qquad \omega = \frac{2\pi}{T}$$

$$\begin{aligned}
F_0 &= \frac{1}{T}\int_0^T f(x,t)dt \in R^n \\
F_{A_k} &= \frac{2}{T}\int_0^T f(x,t)\cos(k\omega t)dt \in R^n \\
F_{B_k} &= \frac{2}{T}\int_0^T f(x,t)\sin(k\omega t)dt \in R^n
\end{aligned}$$

Computation of limit cycles: Spectral methods

Fundamental concepts

4) Substitute $x(t)$ and $f(x,t)$ in $\frac{dx}{dt} = f(x,t)$, i.e.

$$\frac{d}{dt}\left(A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)\right) = F_0 + \sum_{k=1}^{N} F_{A_k} \cos(k\omega t) + F_{B_k} \sin(k\omega t)$$

A set of $2N+1$ **nonlinear algebraic equations** is obtained, by equating the coefficients of the constant term and of the harmonics $\cos(k\omega t)$, $\sin(k\omega t)$

$$\begin{aligned} F_0(A_0, A_1, \ldots, A_N, B_1, \ldots, B_N) &= 0 \\ F_{B_k}(A_0, A_1, \ldots, A_N, B_1, \ldots, B_N) &= -k\omega A_k \\ F_{A_k}(A_0, A_1, \ldots, A_N, B_1, \ldots, B_N) &= k\omega B_k \end{aligned}$$

with $1 \leqslant k \leqslant N$.

Computation of limit cycles: Spectral methods

Harmonic Balance ($1 \leqslant k \leqslant N$)

$$
\begin{aligned}
F_0(A_0, A_1, \ldots, A_N, B_1, \ldots, B_N) &= 0 \\
F_{B_k}(A_0, A_1, \ldots, A_N, B_1, \ldots, B_N) &= -k\omega A_k \\
F_{A_k}(A_0, A_1, \ldots, A_N, B_1, \ldots, B_N) &= k\omega B_k
\end{aligned}
$$

$2N + 1$ equations in $2N + 1$ unknowns

Describing Function ($N = 1$)

$$
\begin{aligned}
F_0(A_0, A_1, B_1) &= 0 \\
F_{B_1}(A_0, A_1, B_1) &= -\omega A_1 \\
F_{A_1}(A_0, A_1, B_1) &= \omega B_1
\end{aligned}
$$

$2N + 1$ equations in $2N + 1$ unknowns

$$x(t) = A_0 + A_1 \cos(\omega t) + B_1 \sin(\omega t)$$

DF Technique: Duffing

$$\frac{d^2y(t)}{dt^2} + \alpha\frac{dy(t)}{dt} + n(y(t)) = \beta\cos(\omega t),$$

where nonlinearity $n(y(t)) = y(t)^3$.

- Let us approximate $y(t)$ with one harmonic only:

$$y(t) \approx A_1\cos(\omega t) + B_1\sin(\omega t)$$

Note: $A_0 = 0$ from simulations.

Unknowns are A_1 and B_1 (ω is given).

- Express nonlinearity $n(y(t))$ as a Fourier series:

$$n(y(t)) \approx N_1^B\sin(\omega t) + N_1^A\cos(\omega t) + N_3^B\sin(3\omega t) + N_3^A\cos(3\omega t),$$

Duffing

Coefficients N_1^A, N_1^B, N_3^A and N_3^B are given by

$$N_1^A = \frac{2}{T}\int_0^T n(y(t))\cos(\omega t)dt = \frac{3}{4}A_1(A_1^2 + B_1^2)$$

$$N_1^B = \frac{2}{T}\int_0^T n(y(t))\sin(\omega t)dt = \frac{3}{4}B_1(A_1^2 + B_1^2)$$

$$N_3^A = \frac{2}{T}\int_0^T n(y(t))\cos(3\omega t)dt = \frac{1}{4}A_1(A_1^2 - 3B_1^2)$$

$$N_3^B = \frac{2}{T}\int_0^T n(y(t))\sin(3\omega t)dt = \frac{1}{4}B_1(3A_1^2 - B_1^2)$$

Duffing

- Using $y(t) \approx A_1 \cos(\omega t) + B_1 \sin(\omega t)$

$$\frac{dy(t)}{dt} = -A_1\omega \sin(\omega t) + B_1\omega \cos(\omega t), \qquad \frac{d^2y(t)}{dt^2} = -A_1\omega^2 \cos(\omega t) - B_1\omega^2 \sin(\omega t)$$

Neglect $n(y(t))$'s 3^{rd} harmonic: $n(y(t)) \approx N_1^A \cos(\omega t) + N_1^B \sin(\omega t)$.

Equating coefficients in $\frac{d^2y(t)}{dt^2} + \alpha \frac{dy(t)}{dt} + n(y(t)) = \beta \cos(\omega t)$ yields

$$\begin{aligned} -A_1\omega^2 + \alpha B_1\omega + N_1^A - \beta &= 0 \\ -B_1\omega^2 - \alpha A_1\omega + N_1^B &= 0 \end{aligned}$$

Duffing

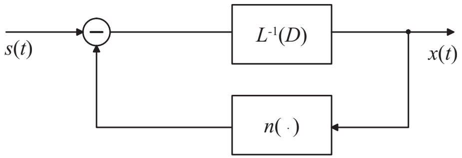
Use $N_1^A = \frac{3}{4}A_1(A_1^2 + B_1^2)$, $N_1^B = \frac{3}{4}B_1(A_1^2 + B_1^2)$:

$$A_1\omega^2 - \frac{3}{4}A_1^3 - \frac{3}{4}A_1 B_1^2 - \alpha B_1\omega + \beta = 0$$
$$B_1\omega^2 - \frac{3}{4}B_1^3 - \frac{3}{4}A_1^2 B_1 + \alpha A_1\omega = 0$$

Letting $\omega = 1$, $\alpha = 0.08$ and $\beta = 0.2$ yields

$$A = 1.07287$$
$$B = 0.608554$$

Computation of limit cycles for Lur'e systems



$$L(D)x(t) + \boldsymbol{n}[x(t)] = s(t), \qquad x(t) \in R$$

If the systems admits of a periodic solution of period T, then $x(t)$ can be expanded through the Fourier series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega t) + B_k \sin(k\omega t) \qquad \omega = \frac{2\pi}{T}$$

Examples

Third order oscillator

$$L(D) = \frac{D^3 + (1+\alpha)D^2 + \beta D + \alpha\beta}{\alpha\left(D^2 + D + \beta\right)} \qquad \boldsymbol{n}(x) = -\frac{8}{7}x + \frac{4}{63}x^3$$

Second order oscillator

$$L(D) = \frac{LCD^2 - LD + 1}{kLD} \qquad \boldsymbol{n}(x) = x^3$$

The harmonic balance (HB) technique

1. The state is represented through a **finite (N)** number of harmonics

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)$$

2. The term $L(D)x(t)$ yields:

$$\begin{aligned} L(D)x(t) \quad = \quad L(0)A_0 \quad &+ \quad \sum_{k=1}^{N} \{ \boldsymbol{Re}[L(jk\omega)]A_k + \boldsymbol{Im}[L(jk\omega)]B_k \} \cos(k\omega t) \\ &+ \quad \sum_{k=1}^{N} \{ \boldsymbol{Re}[L(jk\omega)]B_k - \boldsymbol{Im}[L(jk\omega)A_k \} \sin(k\omega t) \end{aligned}$$

The harmonic balance (HB) technique

3. The term $\boldsymbol{n}[x(t)]$ yields (by truncating the series to N harmonics):

$$\boldsymbol{n}[x(t)] = C_0 + \sum_{k=1}^{N} C_k \cos(k\omega t) + D_k \sin(k\omega t)$$

$$\begin{aligned}
C_0 &= \frac{1}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \sin(k\omega t) + B_k \cos(k\omega t)\right] dt \\
C_k &= \frac{2}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \sin(k\omega t) + B_k \cos(k\omega t)\right] \cos(k\omega t) \\
D_k &= \frac{2}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \sin(k\omega t) + B_k \cos(k\omega t)\right] \sin(k\omega t)
\end{aligned}$$

The harmonic balance (HB) technique

3. The term $s(t)$ yields (by truncating the series to N harmonics):

$$s(t) = P_0 + \sum_{k=1}^{N} P_k \cos(k\omega t) + Q_k \sin(k\omega t)$$

$$P_0 = \frac{1}{T}\int_0^T s(t)\, dt$$

$$P_k = \frac{2}{T}\int_0^T s(t)\cos(k\omega t)\, dt$$

$$Q_k = \frac{2}{T}\int_0^T s(t)\, \sin(k\omega t)\, dt$$

The harmonic balance (HB) technique

4. A set of $2N+1$ **nonlinear equations** is obtained, by equating the coefficients of the constant term and of the harmonics $\cos(k\omega t)$, $\sin(k\omega t)$

$$\begin{array}{lcccll} L(0)A_0 & + & C_0(A_0,\dots,B_N) & = & P_0 & \\ \boldsymbol{Re}[L(jk\omega)]A_k - \boldsymbol{Im}[L(jk\omega)]B_k & + & C_k(A_0,\dots,B_N) & = & P_k & 1 \leqslant k \leqslant N \\ \boldsymbol{Im}[L(jk\omega)]A_k + \boldsymbol{Re}[L(jk\omega)]B_k & + & D_k(A_0,\dots,B_N) & = & Q_k & 1 \leqslant k \leqslant N \end{array}$$

5. **Autonomous systems**: the term A_1 is assumed to be equal to zero (i.e. the phase of the first harmonic of $x(t)$ is arbitrarily fixed); since ω is unknown, the system has an equal number $[(2N+1)]$ of equations and unknowns.

DF Technique: Van der Pol oscillator

The Van der Pol equation (see Fig. 1) is described by the following autonomous system:

$$\dot{x} = \frac{1}{C}\left(\frac{x}{R} - k\,n(x) - y\right)$$
$$\dot{y} = \frac{1}{L}x,$$

where nonlinearity $n(x) = x^3$.

Step 1: Expression for $L(s)$ is

$$L(s) = \frac{kLRs}{LCRs^2 - Ls + R}$$

Step 2: Approximate $x(t)$ as $x(t) \approx B_1 \sin(\omega t)$.

$A_0 = 0$ from simulations, $A_1 = 0$ since phase may be chosen arbitrarily.

Unknowns are B_1 and ω.

Van der Pol oscillator

Step 3: express $n(x(t)) = x^3$ as a Fourier Series:

$$n(x(t)) = (B_1 \sin(\omega t))^3 \approx N_1 \sin(\omega t) + N_3 \sin(3\omega t),$$

where N_1 and N_3 are

$$N_1 = \frac{2}{T}\int_0^T n(x(t))\sin(\omega t)dt = \frac{3}{4}B_1^3$$

$$N_3 = \frac{2}{T}\int_0^T n(x(t))\sin(3\omega t)dt = -\frac{1}{4}B_1^3$$

Step 4: Write differential equation for Lur'e system:

$$x(t) + L(D)n(x(t)) = 0,$$

where $x(t) \approx B_1 \sin(\omega t)$ and $n(x(t)) \approx N_1 \sin(\omega t)$ (3ω neglected, since Fourier series for $n(x(t)$ and $x(t)$ are of same order)

Van der Pol oscillator

Step 4 (contd): express $L(D)n(x(t))$ as

$$L(D)n(x(t)) = L(D)N_1 \sin(\omega t) = N_1 \mathfrak{Re}\{L(j\omega)\} \sin(\omega t) + N_1 \mathfrak{Im}\{L(j\omega)\} \cos(\omega t).$$

Insert this into $B_1 \sin(\omega t) + L(D)n(x(t)) = 0$ and equate coefficients:

$$B_1 + N_1 \mathfrak{Re}\{L(j\omega)\} = 0,$$
$$N_1 \mathfrak{Im}\{L(j\omega)\} = 0$$

Expressions for $\mathfrak{Re}\{L(j\omega)\}$ and $\mathfrak{Im}\{L(j\omega)\}$ are

$$\mathfrak{Re}\{L(j\omega)\} = \frac{-kRL^2\omega^2}{R^2(1-LC\omega^2)^2 + (L\omega)^2}, \qquad \mathfrak{Im}\{L(j\omega)\} = \frac{kLR^2\omega(1-LC\omega^2)}{R^2(1-LC\omega^2)^2 + (L\omega)^2}$$

$\mathfrak{Im}\{L(j\omega)\} = 0$ yields $\omega = \frac{1}{\sqrt{LC}}$

$B_1 + \frac{3}{4} B_1^3 \mathfrak{Re}\{L(j\omega)\} = 0$ gives $B_1 = \frac{2}{\sqrt{3kR}}$

The circuit

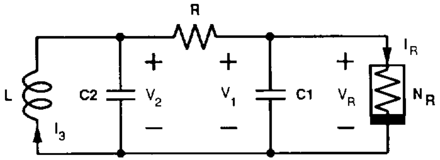


Figure: *Chua's circuit*

$$C_1 \frac{dV_1}{dt} = \frac{V_2 - V_1}{R} - i_R(V_1)$$
$$C_2 \frac{dV_2}{dt} = \frac{V_1 - V_2}{R} + I_3$$
$$L \frac{dI_3}{dt} = -V_2$$

The circuit model

Chua's circuit is described by the following autonomous nonlinear dynamical systems

$$\begin{aligned}
\dot{x}_1 &= \frac{dx_1}{d\tau} = \alpha\left(-x_1 + x_2 - n(x_1)\right) \\
\dot{x}_2 &= \frac{dx_2}{d\tau} = x_1 - x_2 + x_3 \\
\dot{x}_3 &= \frac{dx_3}{d\tau} = -\beta x_2,
\end{aligned}$$

where $\tau = tR^{-1}C_2^{-1}$ and

$$\begin{aligned}
x_1 &= V_1,\ x_2 = V_2,\ x_3 = RI_3, \\
n(x_1) &= Ri_R(V_1) = \gamma x_1 + \delta x_1^3.
\end{aligned}
\qquad
\begin{aligned}
\alpha &= C_2 C_1^{-1},\ \beta = R^2 C_2 L^{-1}, \\
\gamma &= -8/7,\ \delta = 4/63.
\end{aligned}$$

Equilibria

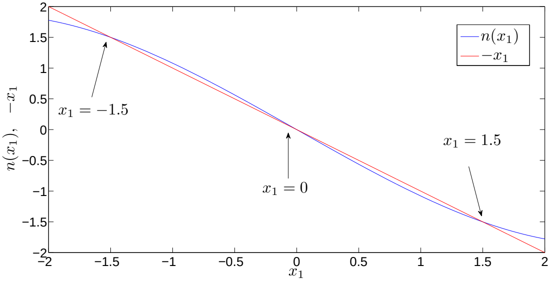


Figure: *Equilibria: x_1 values*

There are three equilibria:

$$\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = (-1.5, 0, 1, 5),$$
$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (0, 0, 0),$$
$$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) = (1.5, 0, -1, 5).$$

Matlab code

```
function [y, t]=Chua

γ = −8/7; δ = 4/63; α = 5; β = 15;
x0 = [0.5, 0.1, −0.5];
tspan=[0 : 0.01 : 200];
options = odeset('RelTol',1e − 12,'AbsTol',1e − 12,'Jacobian',@J);
[t, y] = ode15s(@f,tspan,x0,options);
plot(y(:, 1),y(:, 2),'b')

function dydt = f(t, x)
n = [−α ∗ g(x(1)) 0 0]';
A = [−α α 0 ; 1 − 1 1 ;0 − β 0];
dydt = A ∗ x + n;
end

function dfdx = J(t, x)
dfdx = [−α ∗ (1 + δ + 3 ∗ δ ∗ x(1).^2) α 0 ; 1 − 1 1 ; 0 − β 0];
end

function y = g(x)
y = γ ∗ x + δ ∗ x.^3;
end

end
```

Numerical simulations for $\alpha = 5$, $\beta = 15$

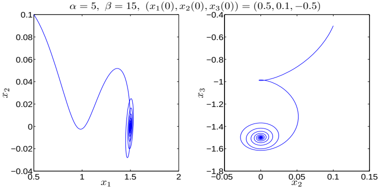


Figure: *Evolution towards* $\hat{\mathbf{x}} = (1.5, 0, -1, 5)$.

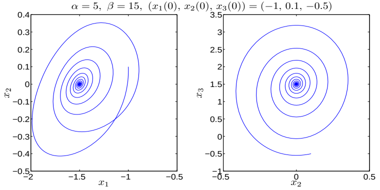


Figure: *Evolution towards* $\bar{\mathbf{x}} = (-1.5, 0, 1, 5)$.

Numerical simulations for $\alpha = 8$, $\beta = 15$

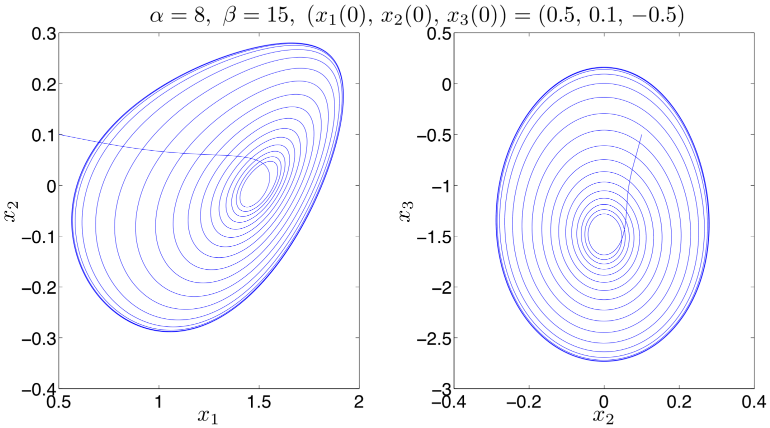


Figure: *Evolution towards a stable limit cycle located at* $x_1 > 1$ *and* $x_3 < -1$.

Numerical simulations for $\alpha = 8$, $\beta = 15$

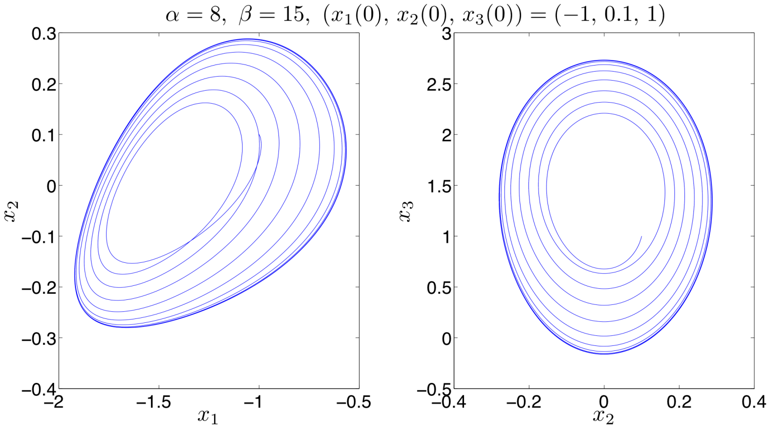


Figure: *Evolution towards a stable limit cycle located at* $x_1 < -1$ *and* $x_3 > 1$.

Numerical simulations for $\alpha = 9$, $\beta = 15$

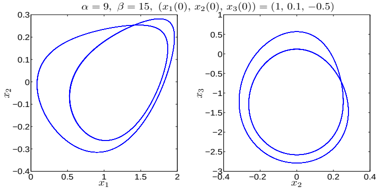


Figure: *Stable period-2 cycle mostly located at $x_1 > 1$ and $x_3 < -1$.*

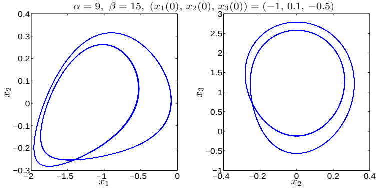


Figure: *Stable period-2 limit cycle mostly located at $x_1 < -1$ and $x_3 > 1$.*

Numerical simulations for $\alpha = 9.1$, $\beta = 15$

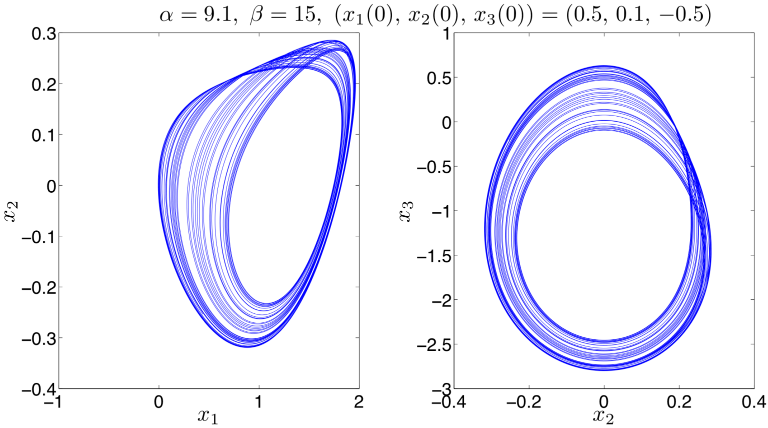


Figure: *Stable single-scroll chaotic attractor mostly located at* $x_1 > 1$ *and* $x_3 < -1$.

Numerical simulations for $\alpha = 9.1$, $\beta = 15$

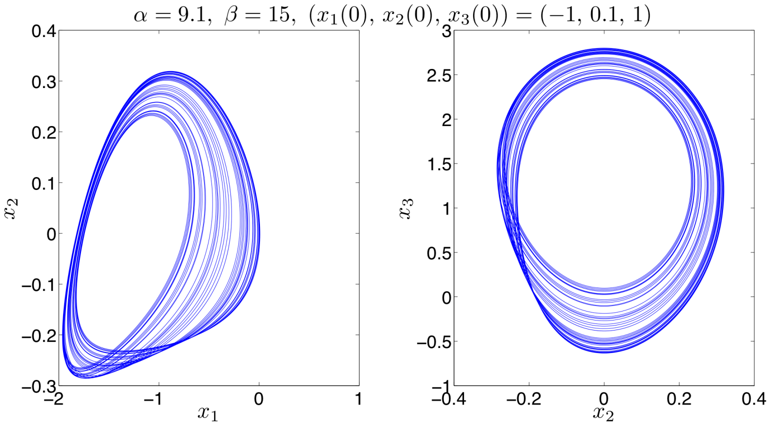


Figure: *Stable single-scroll chaotic attractor mostly located at* $x_1 < -1$ *and* $x_3 > 1$.

Numerical simulations for $\alpha = 9.5$, $\beta = 15$

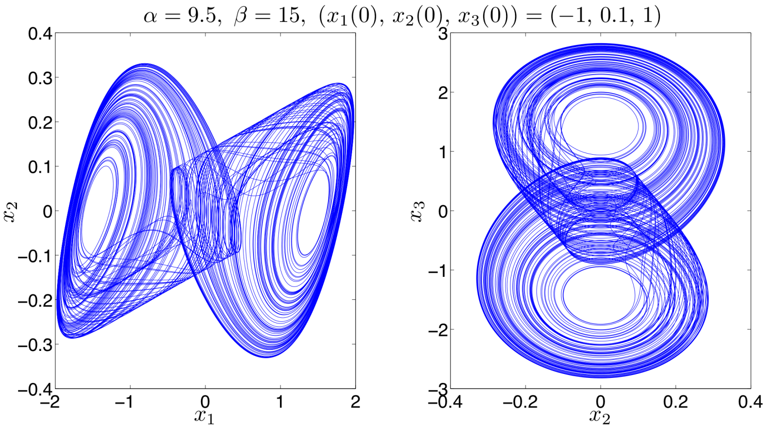


Figure: *Stable double-scroll chaotic attractor located at* $|x_1| \leqslant 2$ *and* $|x_3| \leqslant 3$.

Numerical simulations for $\alpha = 9.5$, $\beta = 15$

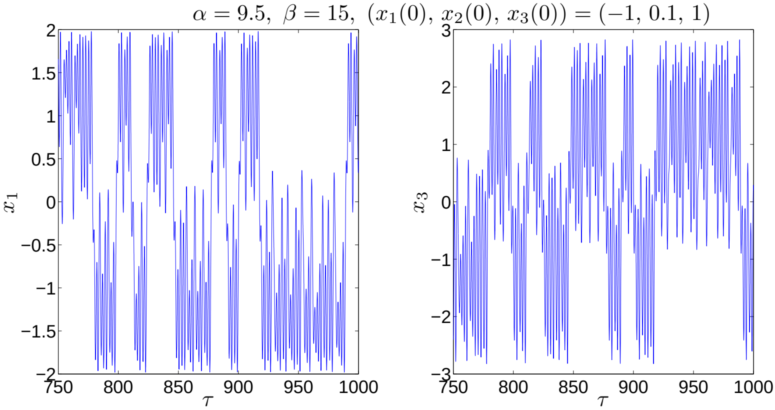


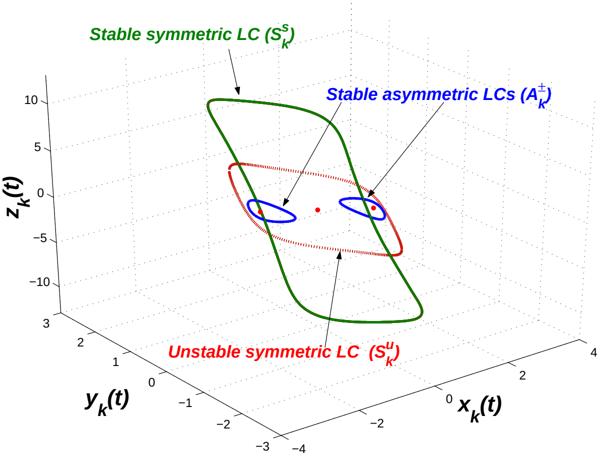
Figure: *$x_1(\tau)$ and $x_3(\tau)$ contributing to the development of the double-scroll*

Chua's circuit: Computation of limit cycles

The parameters α and β are chosen in such a way that the Chua's circuit exhibits (e.g. $\alpha = 8$ and $\beta = 15$):

- Three unstable equilibrium points (denoted by P^+, P^-, P^0 and corresponding to $x = \pm 1.5$ and to $x = 0$ respectively).
- Two stable asymmetric limit cycles (denoted by A^+ and A^-) mainly lying in the regions $x > 1$ and $x < -1$ respectively.
- One stable symmetric limit cycle (denoted by S^s).
- One unstable symmetric limit cycle (denoted by S^u).

Chua's circuit: Complex dynamic behavior



Chua's circuit: Lur'e model

$$\begin{cases} \frac{dx}{dt} & = & -\alpha x + \alpha y - \alpha n(x) \\ \frac{dy}{dt} & = & x - y + z \\ \frac{dz}{dt} & = & -\beta y \end{cases}$$

State equations can be cast in a Lur'e model

$$L(D)x(t) = -\alpha n[x(t)]$$

$$L(s) = \frac{s^3 + (1+\alpha)s^2 + \beta s + \alpha\beta}{s^2 + s + \beta}$$

$$n(x) = -\frac{8}{7}x + \frac{4}{63}x^3$$

Efficient HB implementations

1. Consider the time samples vectors

$$y(t) = L(D)x(t)$$

$$\underline{y} = [y(t_0),\ y(t_1), ..., y(t_{2N}),\ y(t_{2N+1})]'$$

$$\underline{x} = [x(t_0),\ x(t_1), ..., x(t_{2N}),\ x(t_{2N+1})]'$$

$$\underline{s} = [s(t_0),\ s(t_1), ..., s(t_{2N}),\ s(t_{2N+1})]'$$

$$t_p = \frac{T}{2N+1}p \qquad\qquad p = 1, \ldots, 2N+1$$

Efficient HB implementations

2. Impose that the HB equation be satisfied for $t = t_p$

$$y(t_p) + \boldsymbol{n}[x(t_p)] = s(t_p), \qquad p = 1, \ldots, 2N+1$$

that in vector notation yields

$$\underline{y} + \boldsymbol{n}[\underline{x}] = \underline{s}$$

with

$$\boldsymbol{n}[\underline{x}] = [\ \boldsymbol{n}[x(t_1)],\ \boldsymbol{n}[x(t_2)], ..., \boldsymbol{n}[x(t_{2N+1})]\ ]'$$

Efficient HB implementations

$$\underline{x} = \Gamma^{-1}\underline{X}, \qquad \underline{X} = [A_0,\ A_1, ..., A_N,\ B_1, ..., B_N]'$$

$$\Gamma^{-1} = \begin{bmatrix} 1 & \gamma^c_{1,1} & \gamma^s_{1,1} & \cdots & \gamma^c_{1,N} & \gamma^s_{1,N} \\ 1 & \gamma^c_{2,1} & \gamma^s_{2,1} & \cdots & \gamma^c_{2,N} & \gamma^s_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \gamma^c_{2N+1,1} & \gamma^s_{2N+1,1} & \cdots & \gamma^c_{2N+1,N} & \gamma^s_{2N+1,N} \end{bmatrix}$$

$$\begin{aligned} \gamma^c_{p,q} &= \cos(q\omega t_p) = \cos\left(\frac{q2\pi p}{2N+1}\right) \\ \gamma^s_{p,q} &= \sin(q\omega t_p) = \sin\left(\frac{q2\pi p}{2N+1}\right) \end{aligned}$$

Efficient HB implementations

$$\underline{y} = \Gamma^{-1}\,\Omega(\omega)\,\underline{X}$$

$$\Omega(\omega) = \begin{bmatrix} L(0) & 0 & 0 & \dots & 0 & 0 \\ 0 & R_1 & I_1 & \dots & 0 & 0 \\ 0 & -I_1 & R_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R_N & I_N \\ 0 & 0 & 0 & \dots & -I_N & R_N \end{bmatrix}$$

$$R_k = \boldsymbol{Re}\{L(jk\omega)\}, \quad I_k = \boldsymbol{Im}\{L(jk\omega)\}$$

$$\boldsymbol{n}[\underline{x}] = \boldsymbol{n}[\Gamma^{-1}\underline{X}]$$

Efficient HB implementations

$$\underline{y} + \boldsymbol{n}[\underline{x}] = \underline{s}$$

$$\Gamma^{-1}\,\Omega(\omega)\,X + \boldsymbol{n}[\Gamma^{-1}\underline{X}] = \underline{s}$$

$$\Omega(\omega)\,\underline{X} + \Gamma\,\boldsymbol{n}[\Gamma^{-1}\underline{X}] = \Gamma\,\underline{s}$$

The $2N+1$ equations in the $2N+1$ unknowns $\underline{X}$ can be solved without performing any integrals.

Limit cycle stability

- Limit cycles may present the same stability characteristics of equilibrium points: they may be stable, unstable or behave as saddles.
- The stability of limit cycles is studied through the *Poincarè map*, that reduces the stability property of a limit cycle to those of a nonlinear discrete system.
- The stability can also be studied through spectral techniques.

Periodic limit cycle: final remarks

1. The describing function technique is very effective for detecting the existence of limit cycles (either stable or unstable) and also for a preliminary study of their bifurcations.
2. The harmonic balance technique allows to determine with a good accuracy the main limit cycle characteristics.
3. HB based technique can be exploited for computing FMs, even in large-scale systems (*Gilli et al.*).
4. Once the limit cycle has been detected through a spectral technique, the Floquet's multipliers can also be computed via a time-domain technique. The application to large arrays of nonlinear oscillators (Chua's circuits) has allowed to determine all the significant limit cycle bifurcations (*Gilli et al.*).