Nonlinear Oscillators: from circuit models to applications

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Outline

- Introduction
  - Nonlinear Information Processing and Neuromorphic Systems
  - Devices (Memristor technology and Nonlinear Oscillatory Circuits)
  - Physical Phenomena (Syncronization)
- Applications (associative memories and pattern recognition systems)
- Enginnering tools for nonlinear oscillations
- Conclusions

Neuromorphic Computing Systems

- The **Human Brain Project** in EU plans to use a supercomputer to recreate everything known about the human brain — a hugely ambitious goal!
- Leading neuroscientists in the US are now focussed on understanding how the brain works through the **Brain Activity Map** (BAM) project, but it's difficult to peer deeply enough into a brain to map the activity of every neuron. Because zebrafish embryos are transparent, the task is easier.
- Understand how neurons that make up the brain carry out their functions.

Neuromorphic Computing Systems

So the race is on to develop a different kind of chip that more accurately mimics the way the brain works. So-called **neuromorphic chips must be built from devices that behave like neurons** – in other words they transmit and respond to information sent in spikes rather than in a continously varying voltage.

One reason the brain is so **power efficient** is that neural spikes charge only a small fraction of a neuron as they travel. By contrast, conventional chips keep each and every transmission line at a certain voltage all the time.

Clearly, recent advancements in **memristor technology** and **spintronics** are making possible entirely new ways to design chips. However, **there is a long way to go** before synthetic systems can begin to match the capability of natural ones.

“more Moore” and “Beyond Moore”

	Logic		Memory
	High performance	*Low standby power*	
F_{min} (nm)	4.5	8	10
n_{2D} (cm^{-2})	6×10^{11} $(1/8F^2)$	2×10^{11} $(1/8F^2)$	2.5×10^{11} $(1/4F^2)$
V (Volts)	0.65	0.7	5 (read) 15 (write)
E_{bit} (J)	2.93×10^{-18}	7.41×10^{-18}	$\sim10^{-13}$ (read)* $\sim10^{-12}$ (write)*
P_{leak} (W)	2.34×10^{-9}	3.30×10^{-13}	low
T_{3D}	9F	9F	6F
n_{3D} (cm^{-3})	1.5×10^{17} $(1/72F^3)$	2.7×10^{16} $(1/72F^3)$	4.2×10^{16} $(1/24F^3)$

*random access operation

“Ultimate CMOS”: Limiting Density and Energetics

INVITED PAPER

PROCEEDINGS OF THE IEEE | *Vol. 100, May 13th, 2012*

Science and Engineering Beyond Moore’s Law

Data Transfer issue - Energy Consumption

Rough calculation:

- In 22 nm, swapping 1 bit in a transistor has an energy cost:

~ 1 attojoule (10^{-18} J)

- Moving a 1-bit data on the silicon cost:

~1 picojoule/mm (10^{-12} J/mm)

- Moving a data 10^9 per second (1 GHz) in silicon has a cost:

1 pJ/mm x 10^9 s^{-1} = ~1 milliwatt/mm

- 64 bit bus @ 1 GHz:

~64 milliwatts/mm (with 100% activity)

- For 1 cm of 64 bit bus @ 1 GHz : **0,64 W/cm**

On modern chips, there are about km of wires on chip, even with low toggle rate, this lead to several Watt/cm^2

Removing the memory hierarchy

- What if we have small, fast, low power, persistent storage cell?

- This will drive re-thinking the complete **memory hierarchy** and **system architecture**.
- **Coexistence of data and computational properties in a single device**.

- There are new technologies that have that potential: MRAM, RRAM, **Memristors**, ...

Information Processing Devices

Device	Entity	Properties		
		Control Variable	State Variable	Output Variable
FET – Novel Materials (III-V, Ge, carbon-based, etc.)	Electron	Charge	Charge	Charge
SpinFET	Electron	Charge	Spin	Charge
Spin-Torque	Electron	Spin	Spin	Charge
Spin-Wave	Electron	Spin Waves	Spin	Charge Photon
Tunneling Transistor	Electron	Charge	Charge	Charge
Molecular switch	Electron or Atoms	Charge	Charge	Charge
NEMS	Atoms	Charge	Position	Charge
Atomic Switch	Atoms	Charge	Position	Electron
Memristor	Atoms	Charge	Charge,	Electron
Magnetic Cellular Automata	FM Domain	Magnetic dipole	Spin	FM Domain
Moving Domain Wall	FM Domain	Magnetic Dipole	Spin	FM Domain
Multi-Ferroic Tunnel Junction	FM Domain	Spin	Charge	Electron
Optical or Plasmonics	Atoms or Electrons	Charge	Optical Density	Photons
Thermal Transistor	Phonons	Thermal Energy	Temperature	Phonons

Taxonomy for Candidate Information Processing Devices

Breakthrough in Memristor Technology

- **non-volatile memories $\rightarrow$ low–power, high–density**
- **neuromorphic systems $\rightarrow$ Memristor mimics biological synapse**
  - As in a living creature the weight of a synapse is adapted by the ionic flow through it, so the conductance of a memristor is adjusted by the flux across or the charge through it depending on its controlling source.
- **novel computer architectures $\rightarrow$ memory and process coexist**
  - Memristor will play a fundamental role in the realization of novel neuromorphic computing architectures merging memory and computation. This fundamental step will begin to bridge the main divide between biological computation and traditional computation, because memristor permits to bring data close to computation (the way biological systems do) and they use very little power to store that information.

Fundamentals in Memristor

- non-volatile memories → low-power, high-density
- neuromorphic systems → memristors mimics synapses
- computer architectures → memory and processing coexist

Important issues:

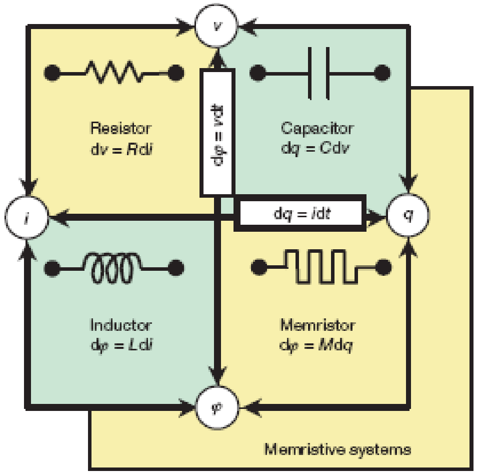
- full understanding of nonlinear dynamics
- modeling

Memristor - L. Chua (1971)

IEEE TRANSACTIONS ON CIRCUIT THEORY, VOL. CT-18, NO. 5, SEPTEMBER 1971

Memristor—The Missing Circuit Element

LEON O. CHUA, SENIOR MEMBER, IEEE



$$f_R(v(t), i(t)) = 0$$

$$f_C(v(t), q(t)) = 0$$

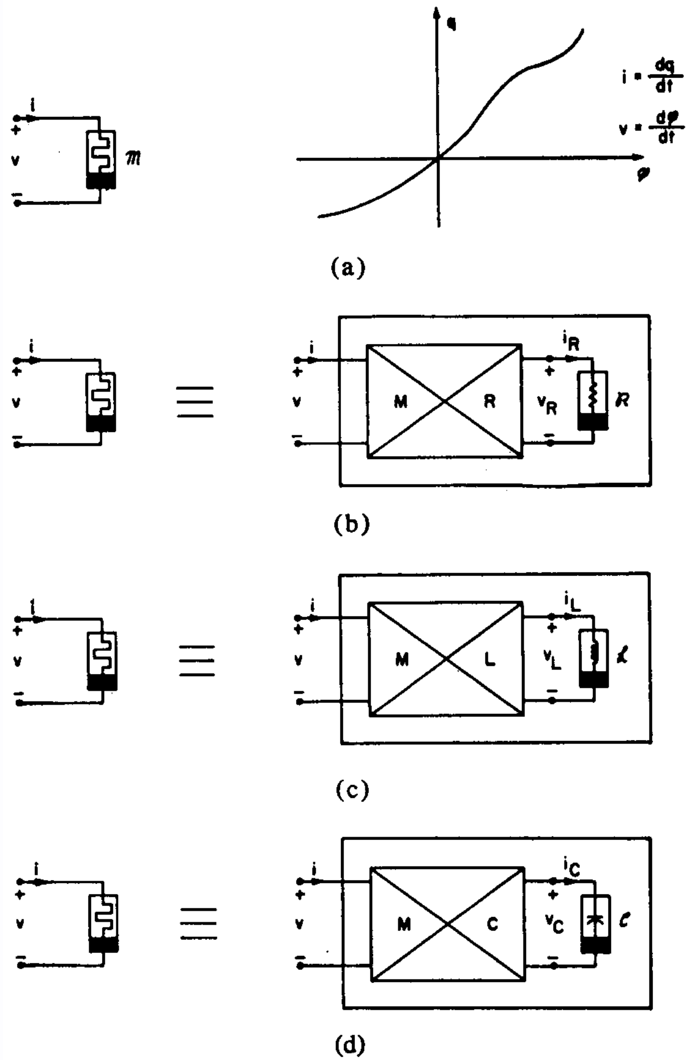
$$f_L(\varphi(t), i(t)) = 0$$

$$f_M(\varphi(t), q(t)) = 0$$

Charge-controlled "ideal" memristor

$$\varphi(t) = f(q(t)) \Rightarrow v(t) = M(q(t))\, i(t)$$

(memristance) $M(q(t)) = \dfrac{d\,f(q)}{dq}$



Memristor - L. Chua and S. Kang (1976)

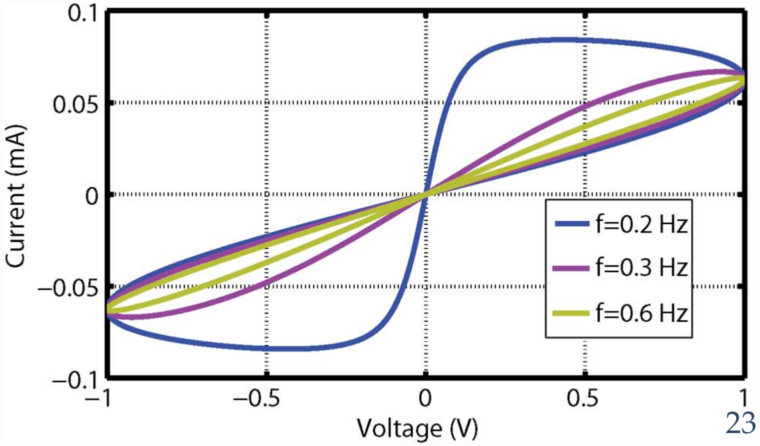
$$v(t) = M(\mathbf{w}(t))\, i(t), \quad \mathbf{w} \in R^n$$

$$\frac{d\,\mathbf{w}(t)}{dt} = h(\mathbf{w}(t), i(t), t)$$

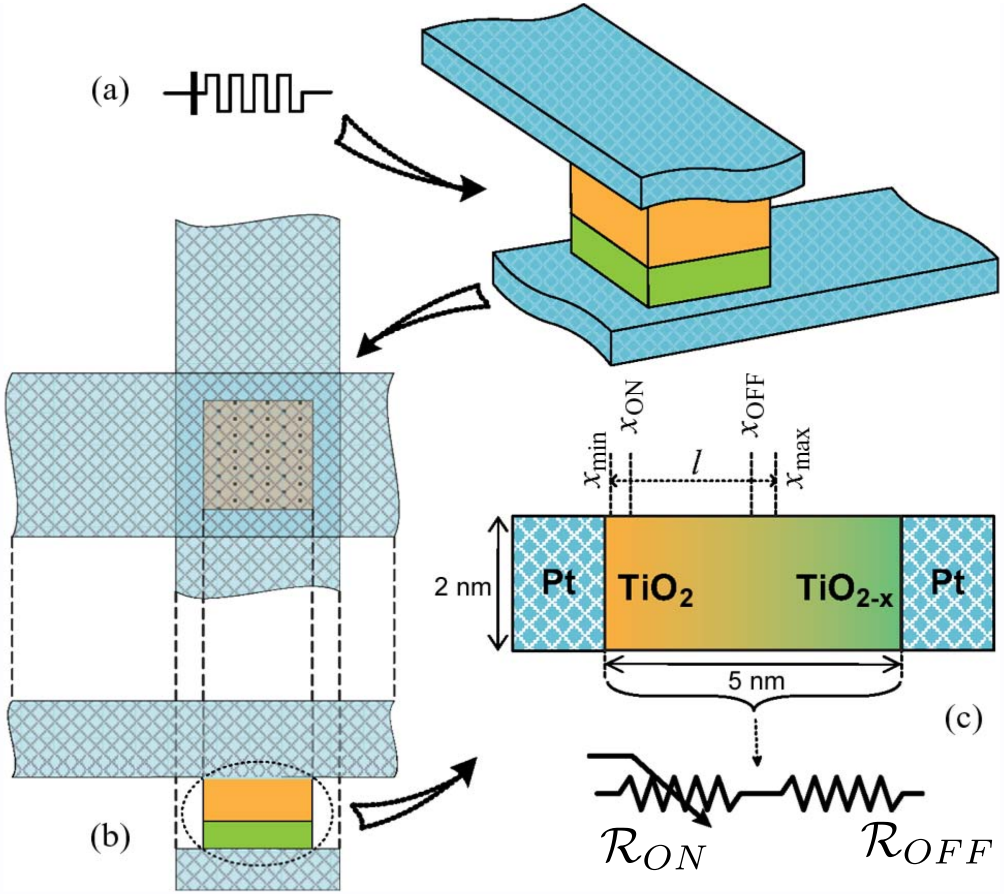
Recently, "Memristors" and "Memristive Devices" have been used interchangeably

Main properties:

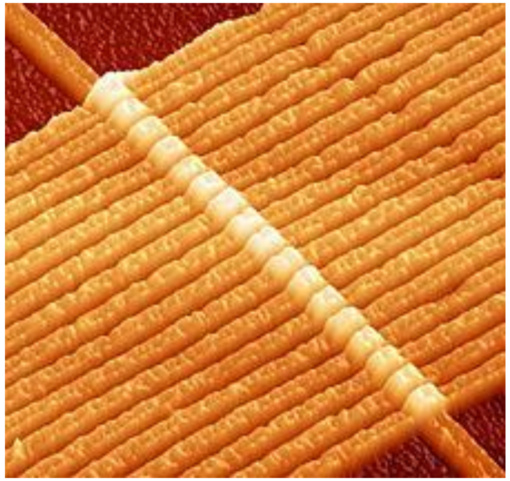
- ☑ passivity criterion $\Rightarrow M(\mathbf{w}(t)) \geq 0$
- ☑ non-volatile memory property $\Rightarrow h(\mathbf{w}(t), 0, t) = 0, \quad \forall t$
- ☑ v-i pinched hysteresis loop (Lissajous figure) for any periodic source. The pinched hysteresis loop shrinks continuously as the frequency increases

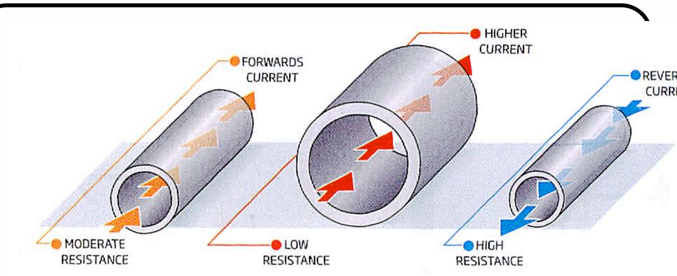


Memristor - HP Labs (2008)



$$v(t) = \left(\mathcal{R}_{\mathrm{ON}} \frac{w(t)}{D} + \mathcal{R}_{\mathrm{OFF}} \left(1 - \frac{w(t)}{D} \right) \right) i(t)$$





Mappings:

- water $\Longleftrightarrow$ current
- pipe diameter $\Longleftrightarrow$ film resistance

Memristor-based nonlinear oscillators

Memristor: model comparison

Memristor Model Comparison

Alon Ascoli, Fernando Corinto, Vanessa Senger, and Ronald Tetzlaff

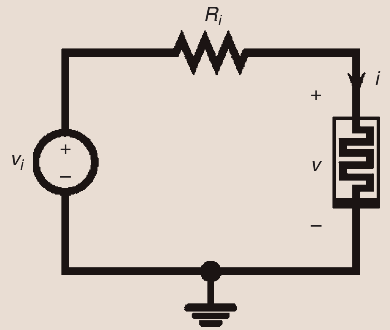


Figure 1. Memristor-based circuit used in the optimization strategy of Section III.

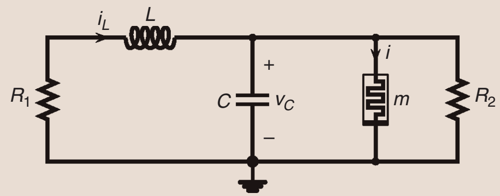


Figure 6. Memristor-based second-order oscillator.

Reference model

$$\frac{dw}{dt} = f_{\text{off}} \sinh\left(\frac{|i|}{i_{\text{off}}}\right) \exp\left(-\exp\left(\frac{w - a_{\text{off}}}{w_c} - \frac{|i|}{b}\right) - \frac{w}{w_c}\right)$$

for $i > 0$, while it is

$$\frac{dw}{dt} = -f_{\text{on}} \sinh\left(\frac{|i|}{i_{\text{on}}}\right) \exp\left(-\exp\left(\frac{a_{\text{on}} - w}{w_c} - \frac{|i|}{b}\right) - \frac{w}{w_c}\right)$$

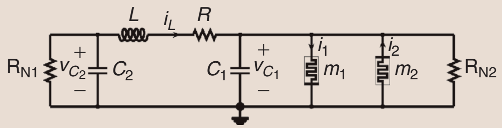


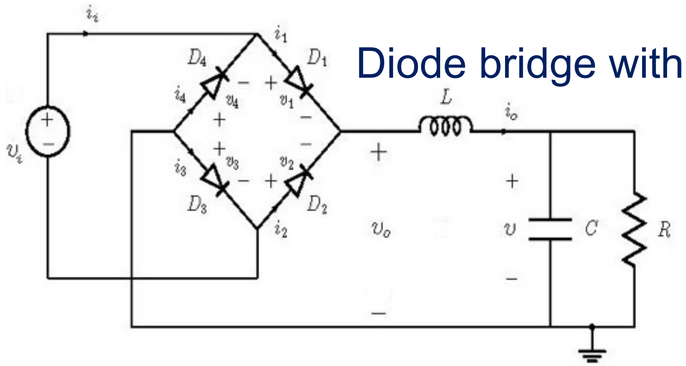
Figure 12. Memristor-based third-order oscillator.

Table 4.
Comparison among the memristor models. For sake of brevity we use the acronym BM to indicate Biolek's Memristor.

	BM	BCM	Team
Test 1	-	✓	-
Test 2-a	✓	✓	✓
Test 2-b	-	-	✓
Test 2-c	✓	✓	-
Test 3	#	#	#

Memristor-based nonlinear oscillators

Numerical simulations

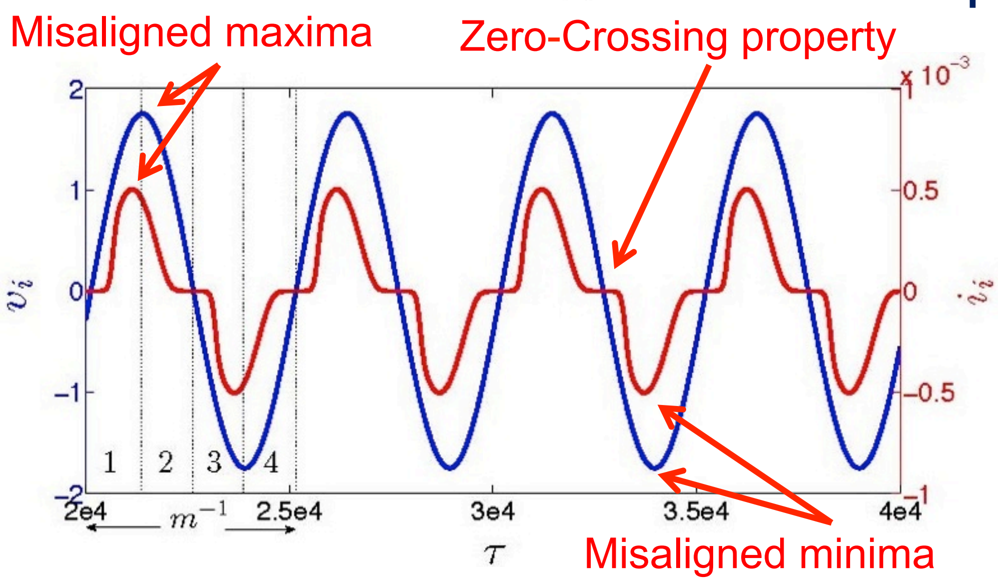


Diode bridge with LCR filter.

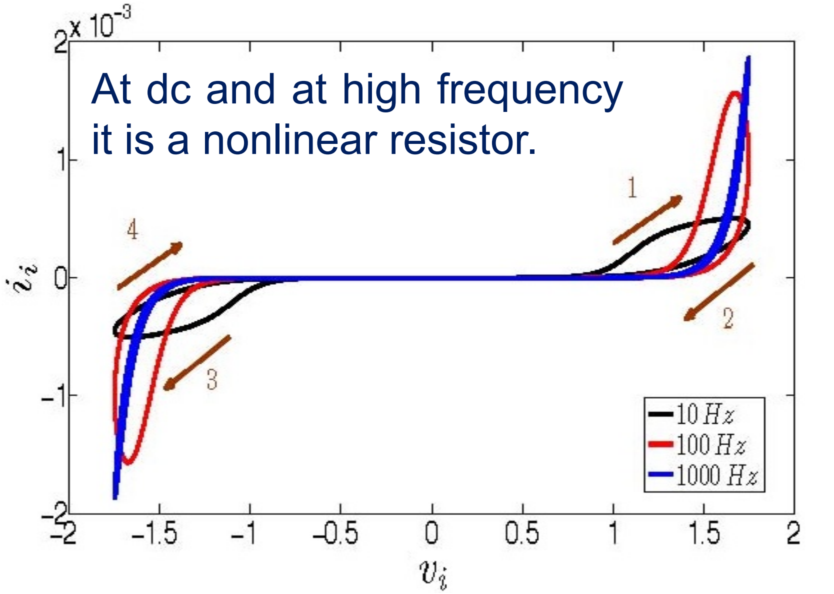
$R / \mathrm{k\Omega}$	$C / \mu\mathrm{F}$	$L / \mu\mathrm{H}$	I_S / nA	n
1.5	4	2.5	2.7	1.8

initial conditions: $v(0) = 10\ \mathrm{mV}$, $i_o(0) = 10\ \mathrm{mA}$

input: $v_i(t) = v_{io} \sin(2\pi f t)$ with $v_{io} = 1.75\ \mathrm{V}$



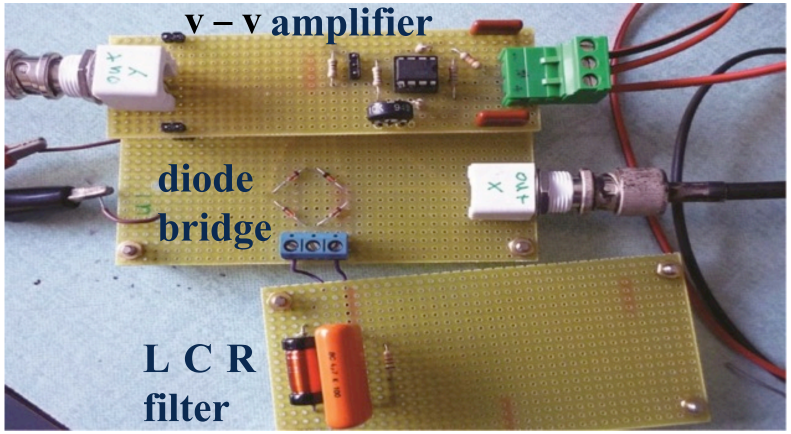
$v_i(t)$ and $i_i(t)$ for $f = 10\ \mathrm{Hz}$ ($m^{-1} = f^{-1} t_0^{-1}$).



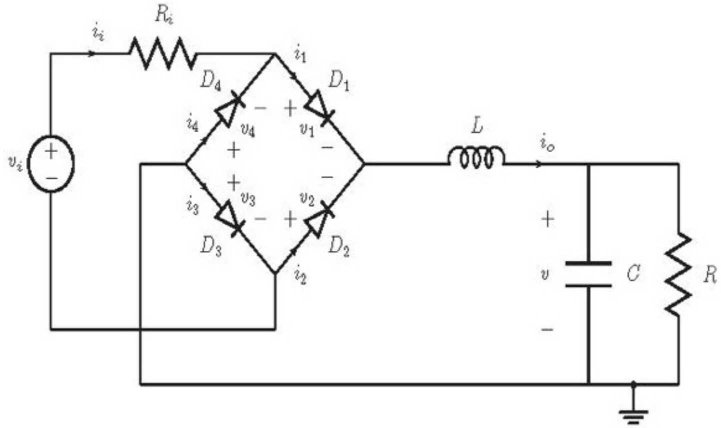
f-dependence of non-self-crossing type II loop.

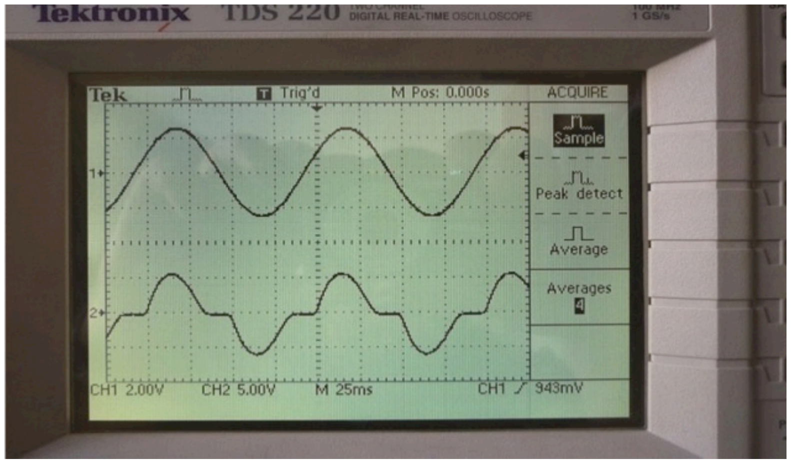
Memristor-based nonlinear oscillators

Experimental demonstration

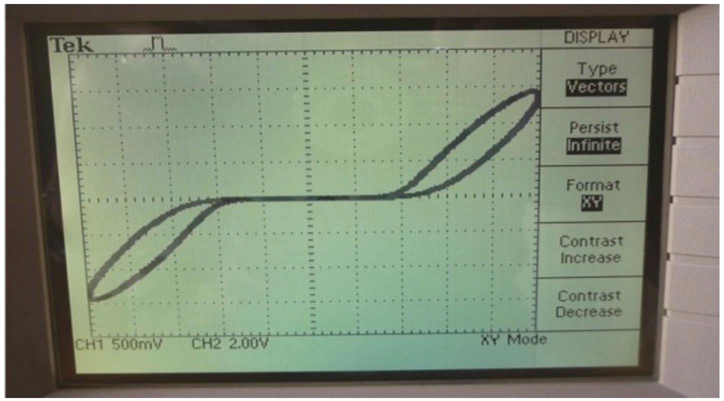


- R_i added to measure i_i in conjunction with $v-v$ amplifier of gain G
- discrete realization employs diodes D1N4148, $R = 1.5\ \mathrm{k\Omega}$, $C = 4.7\ \mu F$, $L = 220\ \mu H$





$v_y(t)$



$v_x(t)$

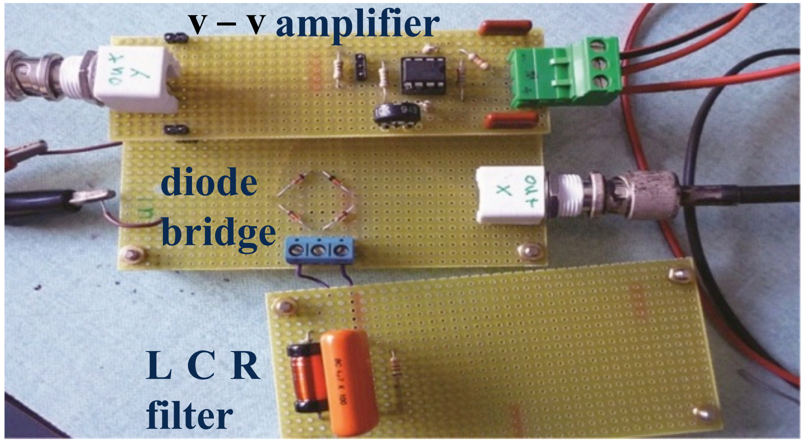
Top: $v_x(t) = v_{io}\sin(2\pi ft)$, $v_{io} = 2.5\ \mathrm{V}$ and $f = 10\ \mathrm{Hz}$

Bottom: $v_y(t) = GR_i i_i(t)$ with $R_i = 1\ \mathrm{k\Omega}$ and $G = 10$

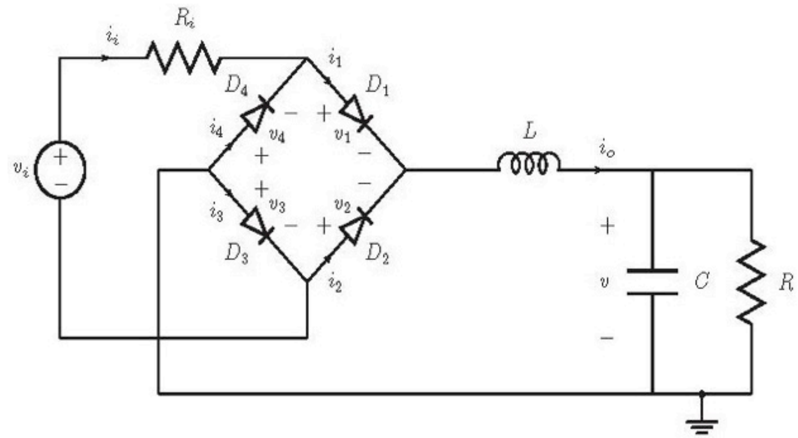
Pinched hysteresis loop

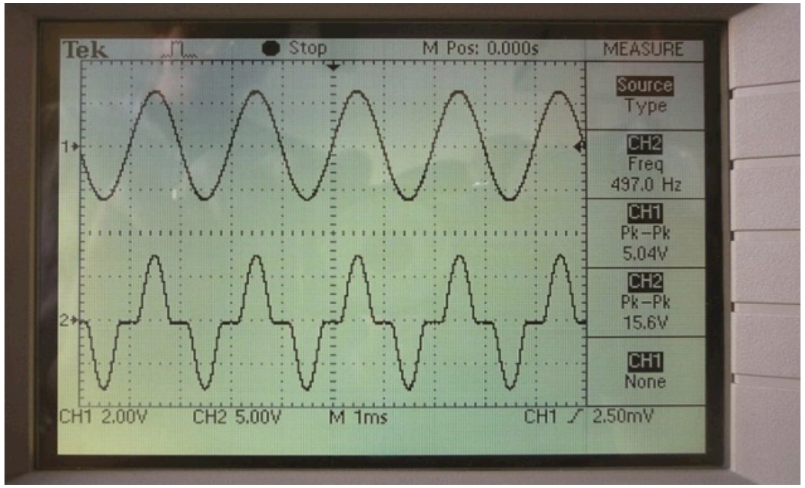
Memristor-based nonlinear oscillators

Decrease in the area of the lobes

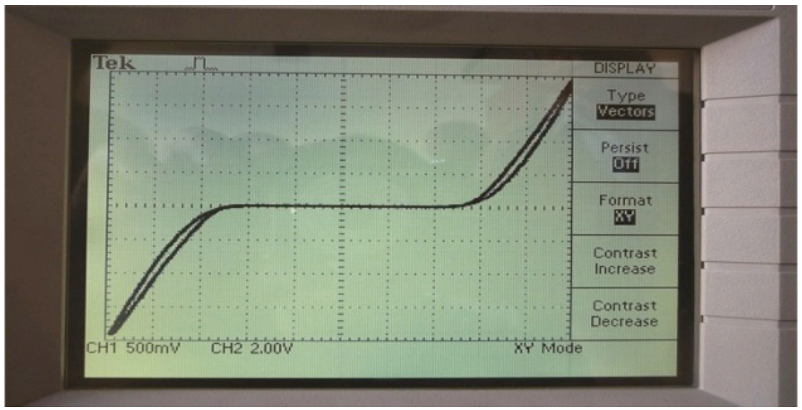


discrete realization components: diodes D1N4148, $R = 1.5\ \mathrm{k}\Omega$, $C = 4.7\ \mu F$, $L = 220\ \mu H$





$v_y(t)$



$v_x(t)$

Top: $v_x(t) = v_{io}\sin(2\pi f t)$, $v_{io} = 2.5$ V and $f = 500$ Hz

Bottom: $v_y(t) = G R_i i_i(t)$ with $R_i = 1\ \mathrm{k}\Omega$ and $G = 10$

Shrinking of the area of the lobes

Nonlinear dynamics

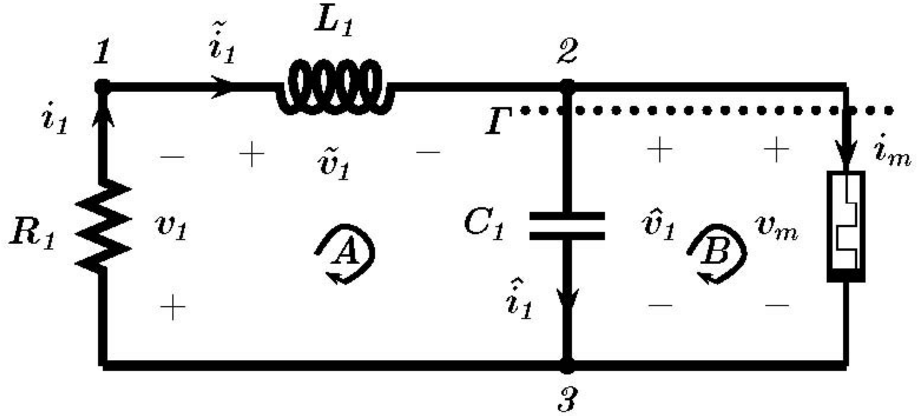
Memristor-based nonlinear oscillators

In Circuit Theory physical quantities of interest are i and v.

However the ideal memristor is univocally described by the $q-\varphi$ relation:

→ *circuits based upon it should be analyzed in terms of charge and flux*.

Example: derive state eqs. by integration of KVLs and KCLs

$0 >$ 

$$\begin{cases} \dfrac{dx_1}{dt} = \alpha x_2 - \alpha q_m(x_1) \\ \dfrac{dx_2}{dt} = -\xi x_1 + \beta x_2 \end{cases}$$

$x_1 = \hat{\varphi}_1,\ x_2 = \tilde{q}_1$: states

$$\begin{cases} \alpha = C_1^{-1}, \\ \beta = -R_1 L_1^{-1} \\ \xi = L_1^{-1} \end{cases}$$: parameters

Memristor nonlinearity (hp.: $b > a > 0$):

$$q_m(x_1) = bx_1 + (a-b)\,n(x_1), \quad n(x_1) = \frac{1}{2}\left(|\,x_1+1\,| - |\,x_1-1\,|\right)$$

Memristor-based nonlinear oscillators

The system is characterized by Lur'e model $L(D)x_1 = -q_m(x_1), \;\; L(D) = \dfrac{D^2 - \beta D + \alpha\xi}{\alpha(D-\beta)}$

It exhibits ("Nonlinear dynamics of memristor oscillators", Corinto et al. 2012)

(a) only one equilibrium, i.e. $\mathbf{x_0} = (0,0)$,

if $a > \xi\beta^{-1}$ or $b < \xi\beta^{-1}$

(b) an infinite number of equilibria, i.e.

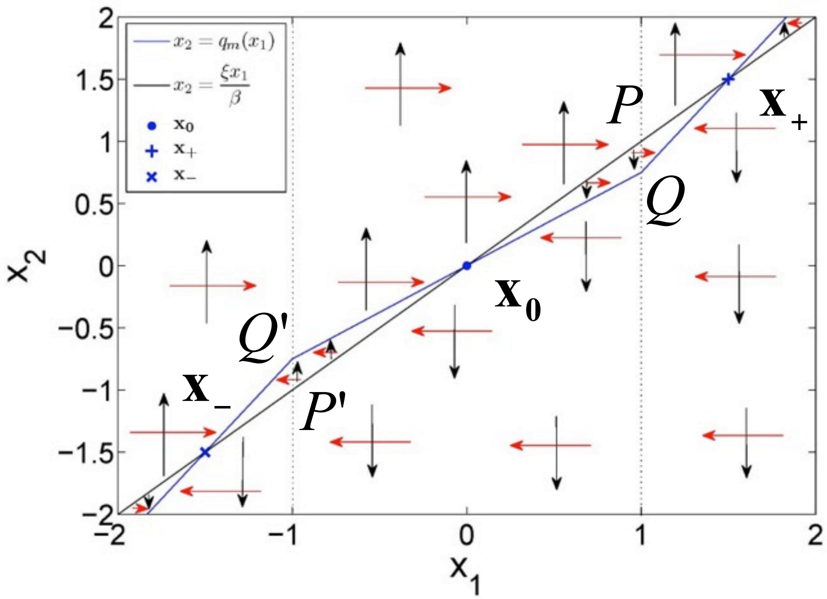
$$\mathbf{x}^* \in \{(x_1, x_2) : x_2 = ax_1, \forall x_1 \in [-1,+1]\}$$

if $a = \xi\beta^{-1}$

(c) 3 equilibria, i.e.

$$\mathbf{x_0}, \mathbf{x}_+ = \left(\beta\frac{b-a}{\beta b - \xi}, \xi\frac{b-a}{\beta b-\xi}\right), \mathbf{x}_- = -\mathbf{x}_+$$

if $a < \xi\beta^{-1}$ and $b > \xi\beta^{-1}$

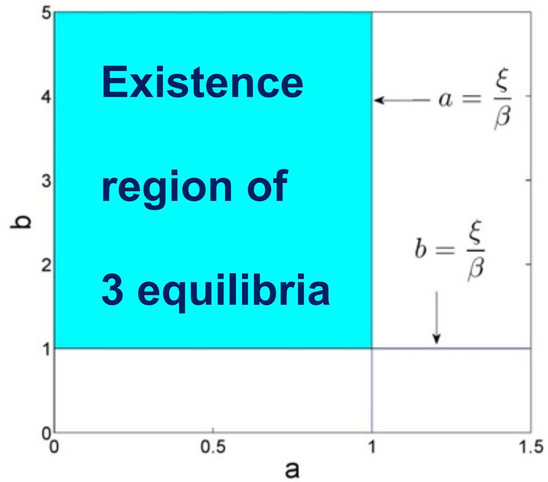


Flux lines and equilibria in case (c)

Topological constraints: $\begin{cases} \xi\beta^{-1} < x_{2P} < x_{2+} \\ x_{2Q} < a \end{cases}$

Memristor-based nonlinear oscillators

Local stability of equilibria



Jacobian of the system:

$$J = \begin{pmatrix} -\alpha \frac{dq_m(x_1)}{dx_1} & \alpha \\ -\xi & \beta \end{pmatrix}$$

where

$$q_m(x_1) = b + \frac{a-b}{2}\left(\mathrm{sgn}(x_1+1) - \mathrm{sgn}(x_1-1)\right)$$

The eigenvalues of an equilibrium : $\lambda_{+,-} = \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}$

Local stability the origin $\mathbf{x_0} = (0,0)$:

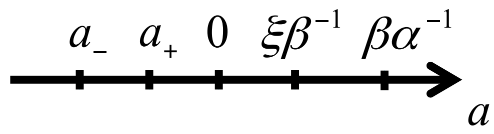
- determinant $\Delta_0 = \alpha(\xi - a\beta) > 0$ since $a < \xi\beta^{-1}$
- stability depends on sign of trace $\tau_0 = \beta - a\alpha$: if $\boxed{a < \beta\alpha^{-1}}$ then $\mathbf{x_0}$ is LU
- Spiral/nodal behavior depends on sign of $4\Delta - \tau^2$

Memristor-based nonlinear oscillators

Condition for a Hopf bifurcation

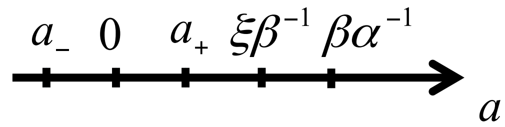
Using all the previous results, we have:

- if $\xi\beta^{-1} < 0.25\beta\alpha^{-1} < \beta\alpha^{-1}$ then $\mathbf{x}_0$ is an UN $\forall a:\ 0 < a < \xi\beta^{-1}$

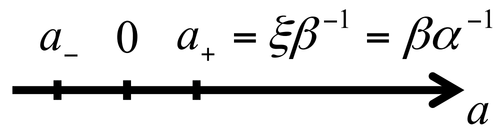


- if $0.25\beta\alpha^{-1} < \xi\beta^{-1} < \beta\alpha^{-1}$ then $\mathbf{x}_0$ is an UF for $0 < a < a_+$

  an UN for $a_+ < a < \xi\beta^{-1}$



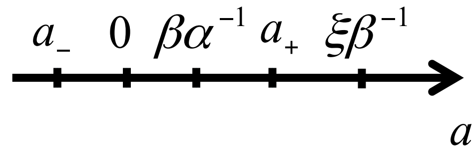
- if $\xi\beta^{-1} = \beta\alpha^{-1}$ (deg. case) then $\mathbf{x}_0$ is an UF $\forall a:\ 0 < a < \xi\beta^{-1}$



- if $\xi\beta^{-1} > \beta\alpha^{-1}$ then $\mathbf{x}_0$ is an UF for $0 < a < \alpha\beta^{-1}$

  is a SF for $\alpha\beta^{-1} < a < a_+$

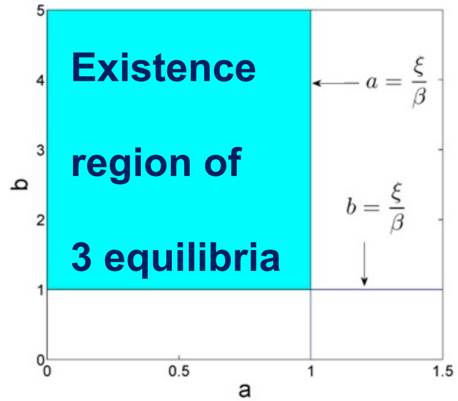
  is a SN for $a_+ < a < \xi\beta^{-1}$



Remark: in this case a *Hopf supercritical bifurcation* occurs for $\boxed{a = \beta\alpha^{-1}}$

Memristor-based nonlinear oscillators

Local stability of the other two equilibria



Local stability of $\mathbf{x}_+ = (x_{1+}, x_{2+})$:

- determinant $\Delta_+ = \alpha(\xi - b\beta) < 0$ since $b > \xi\beta^{-1}$
- Thus the equilibrium is a saddle $\forall b : b > \xi\beta^{-1}$
- Speed of dynamics along saddle manifolds depends on trace*.

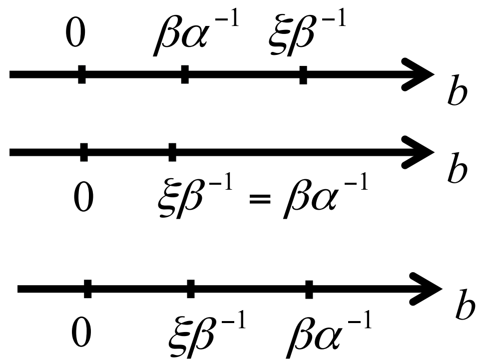
In particular, since $\tau_+ = \beta - b\alpha < 0$ for $b > \beta\alpha^{-1}$, then

a) if $\xi\beta^{-1} > \beta\alpha^{-1}$ then $\tau_+ < 0 \quad \forall b : b > \xi\beta^{-1}$

b) if $\xi\beta^{-1} = \beta\alpha^{-1}$ (deg. case) then $\tau_+ < 0 \ \forall b : \ b > \xi\beta^{-1}$

c) if $\xi\beta^{-1} < \beta\alpha^{-1}$ then $\tau_+ > 0$ for $\xi\beta^{-1} < b < \beta\alpha^{-1}$

$\tau_+ < 0$ for $b > \beta\alpha^{-1}$

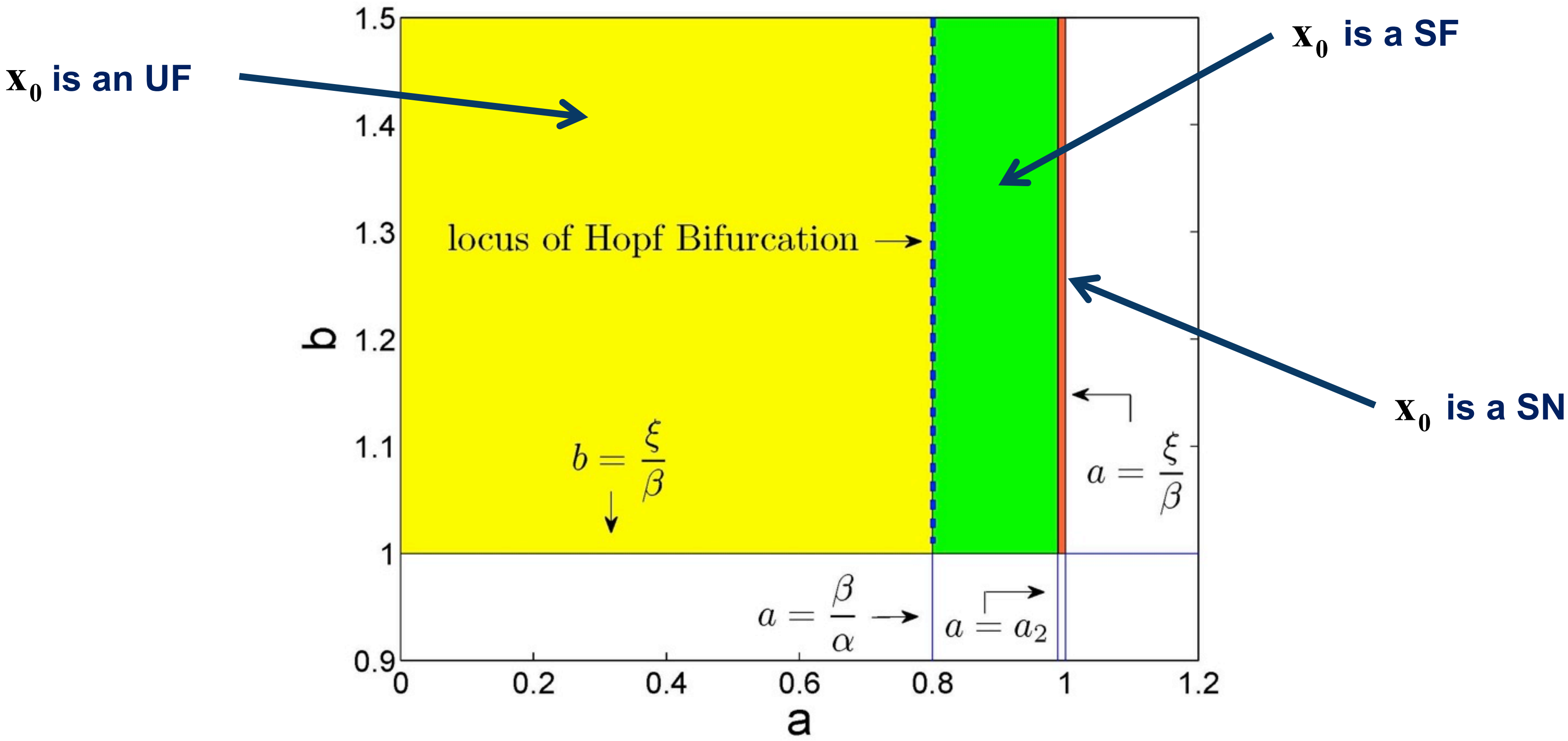


* if $\tau_+ < 0$ dynamics are faster on W_+^S than on $W_+^U \left(|\lambda_-| > |\lambda_+|, \lambda_{+,-} = \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta} \right)$

Remark: same conclusions may be drawn regarding stability of $\mathbf{x}_-$

Memristor-based nonlinear oscillators

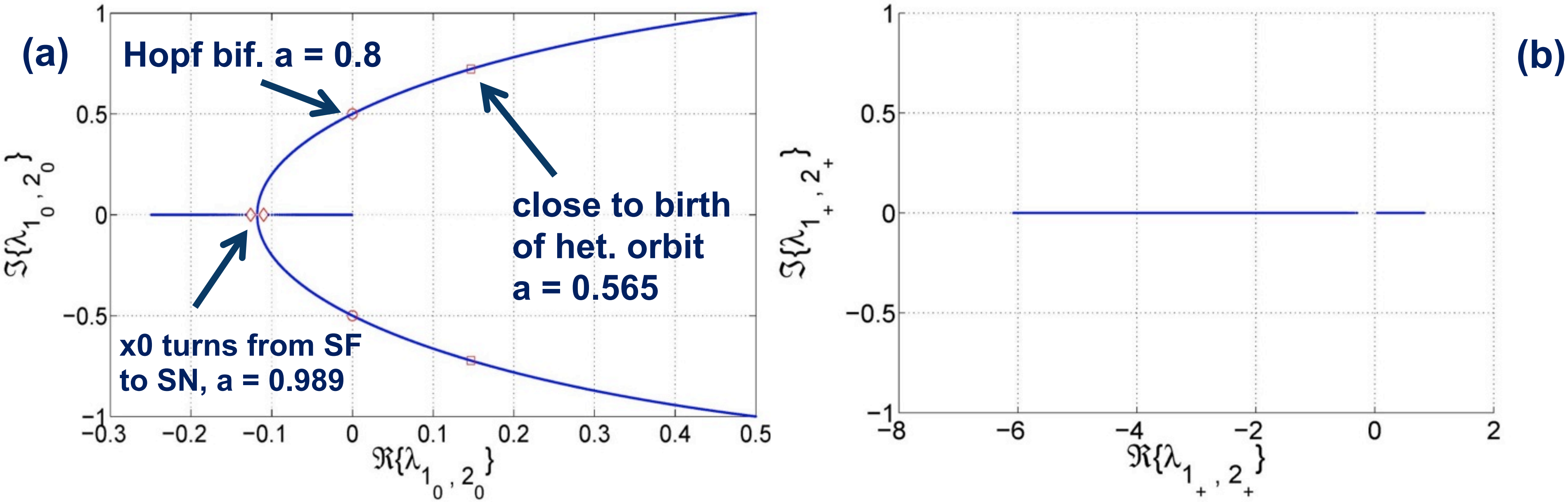
Dynamics on a-b plane for $a < \xi\beta^{-1}$, $b > \xi\beta^{-1}$ and $\xi\beta^{-1} > \beta\alpha^{-1}$.



$\mathbf{x}_+$ and $\mathbf{x}_-$ are saddles with dynamics on W^S faster than on W^U

Memristor-based nonlinear oscillators

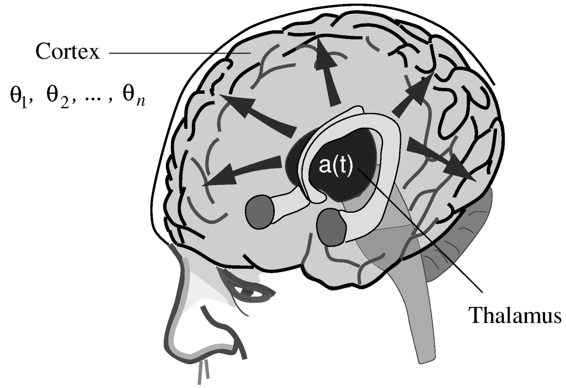
Eigenvalues of equilibria for $a < \xi\beta^{-1}$, $b > \xi\beta^{-1}$ and $\xi\beta^{-1} > \beta\alpha^{-1}$.

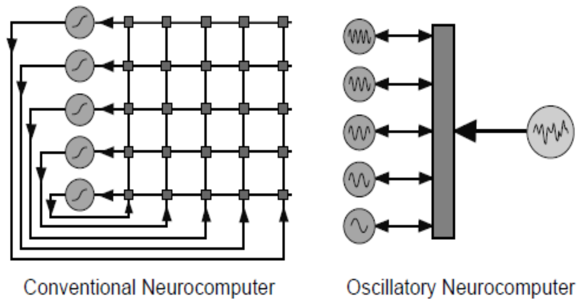


(a) eigenvalues of $\mathbf{x_0}$ for $0 < a < \xi\beta^{-1}$ and **(b)** eigenvalues of $\mathbf{x}_+$ for $b > \xi\beta^{-1}$ ($\xi = 1,\ \beta = 1,\ \alpha = 1.25$)

Oscillatory model of neurocomputing

- Oscillations experimentally observed in visual cortex after stimulus
- Synchronized oscillations observed in parts of the brain not geometrically close
- Synchronized oscillations is linked to association
- Can we build an image recognition system from coupled oscillators?



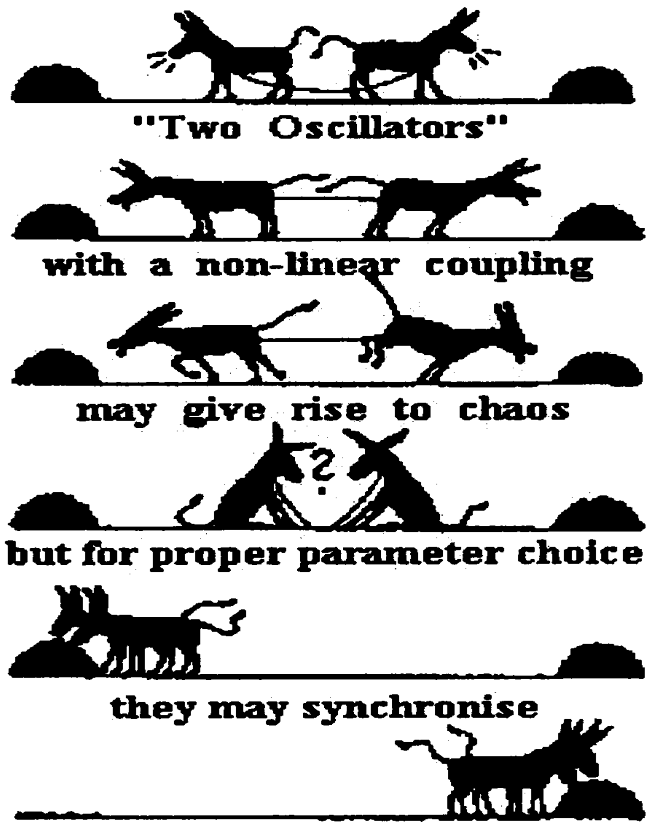


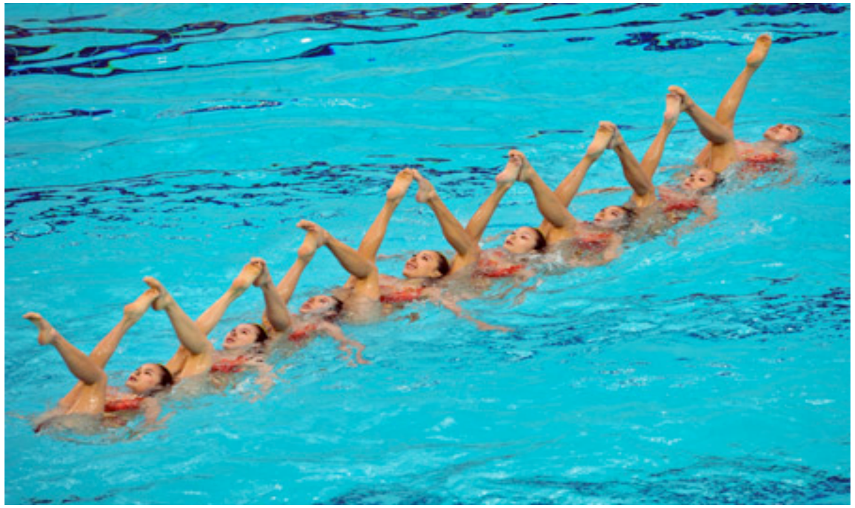
Hoppensteadt and Izhikevich, Phys Rev L, VOLUME 82, NUMBER 14, April 5, 1999

What is synchronization?

- Synchronize: to agree in time, to happen at the same time, to represent or arrange (events) to indicate coincidence or coexistence
- It is an important concept in: Physics, Biology, Telecommunication, Computer science, Cryptography, Multimedia, Photography, Music (rhythm)
- Synchronicity is a word coined by the Swiss psychologist Carl Jung to describe the “temporally coincident occurrences of acausal events.”

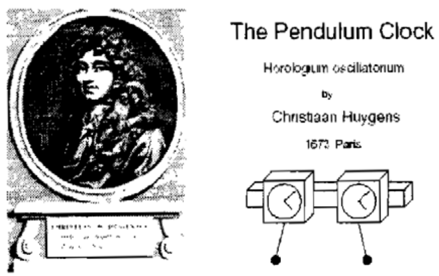
What is synchronization?





A historical perspective
Christiaan Huygens (1658)

Synchronization of Pendulum Clocks



"It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously.

Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."

A historical perspective
Engelbert Kaempfer (1680)

Synchronization in a large population of oscillating systems

Engelbert Kaempfer (1680)

The glowworms represent another shew, which settle on some Trees, like a fiery cloud, with this surprising circumstance, that a whole swarm of these insects, having taken possession of one Tree, and spread themselves over its branches, sometimes hide their Light all at once, and a moment after make it appear again with the utmost regularity and exactness

This very early observation reports on synchronization in a large population of oscillating systems. The same physical mechanism that makes the insects to keep in sync is responsible for the emergence of synchronous clapping in a large audience or onset of rhythms in neuronal populations.

A historical perspective

- Sleep-Wake rhythms: biological systems can adjust their rhythms to external signals. Under natural conditions, biological clocks tune their rhythms (i.e. synchronize) in accordance with the 24-hour period of the Earth's daily cycle (First observed by J.J. Dortous de Mairan, 1729)

- Synchronization of triode oscillators (Appleton, van der Pol, van der Mark, 1922-1928)

The concept of “Synchronization”

- In a classical context, synchronization (from Greek: syn = the same, common and: chronos = time) means adjustment of rhythms of self-sustained periodic oscillators due to their weak interaction (coupling); this adjustment can be described in terms of phase locking and frequency entrainment [(1)].

(1) If you have two vibrating objects with the same natural frequency or corresponding harmonic, they will both have a forced vibration effect on each other. This process, given time, normally leads to a condition where both objects synchronize. Of interest, both oscillators do not, necessarily, must have exactly the same natural frequency. If there is enough "coupling" between the oscillators, they will sometime "lock-in" with one another at a slightly shifted frequency: the frequencies become equal or entrained. The onset of a certain relationship between the phases of these oscillators is often termed phase locking.

What is a self-sustained periodic oscillator ?

1. The oscillator is an active system. It contains an internal source of energy that is transformed into oscillatory behavior. Being isolated, it continues to generate the same rhythm until the source of energy expires. It is described as an autonomous dynamical system.

2. The form of the oscillation is determined by the parameters of the system and does not depend on initial conditions.

3. The oscillation is stable to (small) perturbations.

The above properties are characteristic of nonlinear oscillators

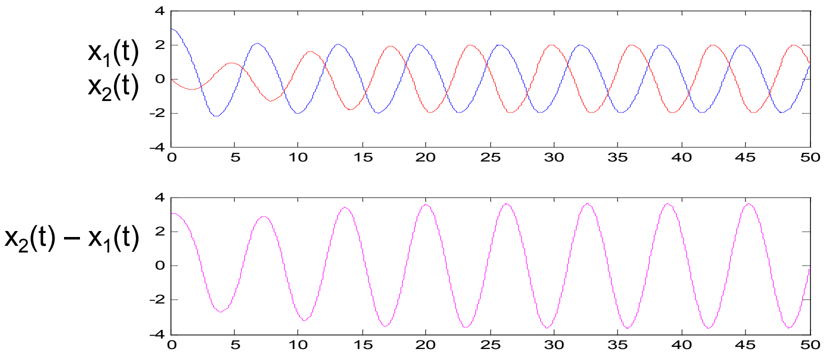
Electronic nonlinear circuits

Example: Two identical coupled Van der Pol oscillators

$$\omega_1 = \omega_2 \ , \ \lambda_1 = \lambda_2$$

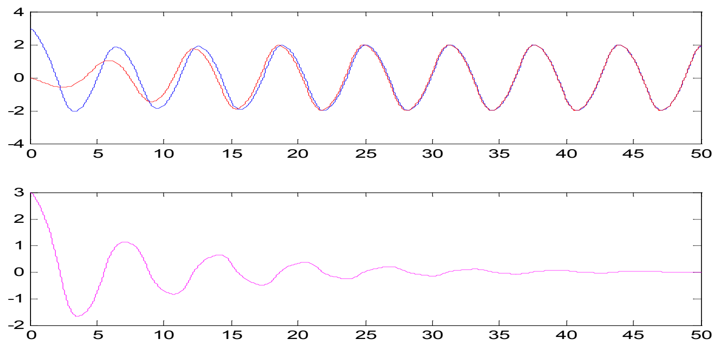
$$\begin{cases} \dfrac{dx_1}{dt} = \omega_1 y_1 + d\left(x_2 - x_1\right) \\ \dfrac{dy_1}{dt} = -\omega_1 x_1 + \lambda_1\left(\rho^2 - x_1^2 - y_1^2\right) y_1 \end{cases} \qquad \begin{cases} \dfrac{dx_2}{dt} = \omega_2 y_2 + d\left(x_1 - x_2\right) \\ \dfrac{dy_2}{dt} = -\omega_2 x_2 + \lambda_2\left(\rho^2 - x_2^2 - y_2^2\right) y_2 \end{cases}$$

Uncoupled: $d = 0$



phase difference remains

Coupled: $d = 0.1$



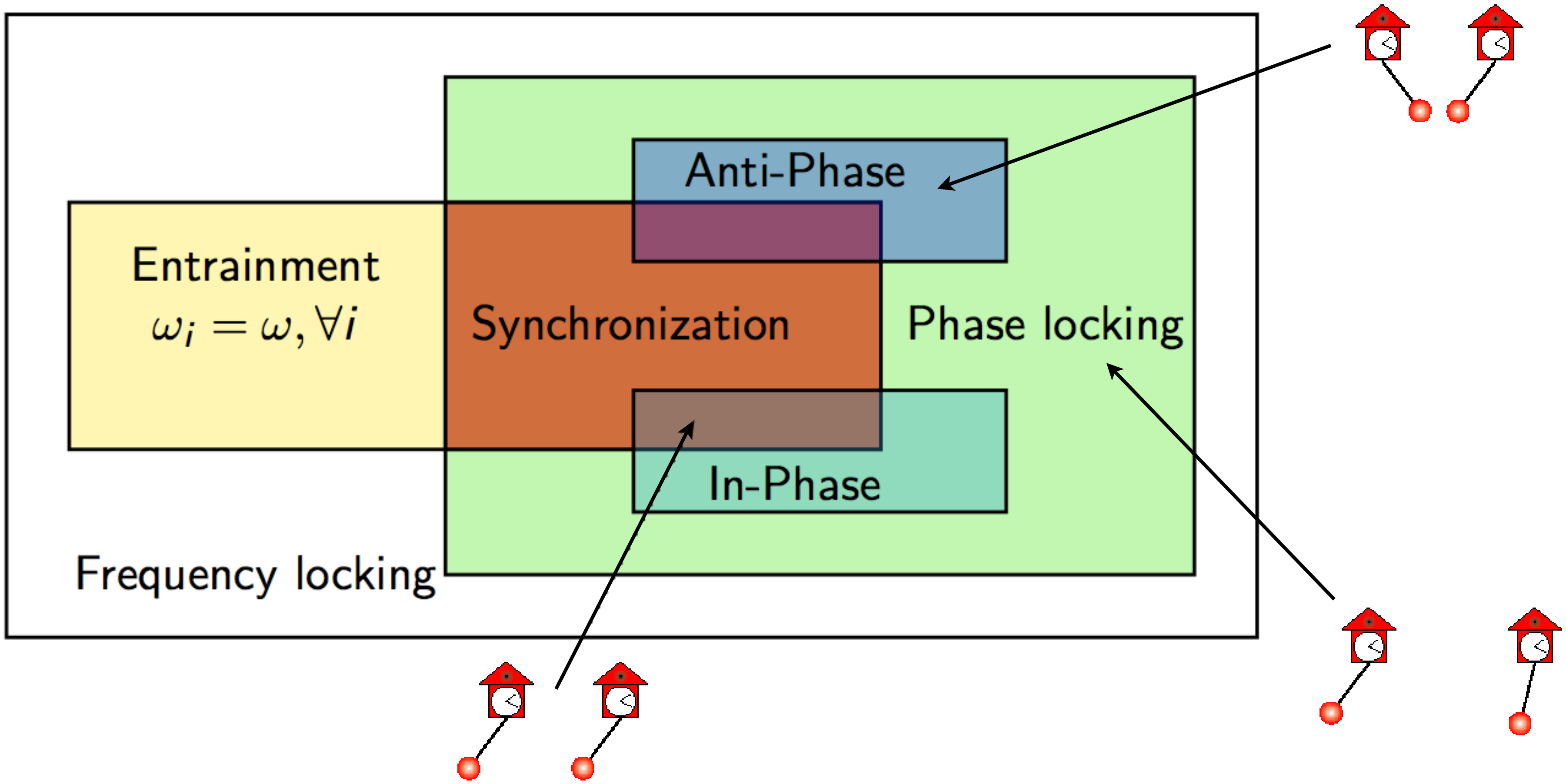
phase difference vanishes

Take home messagge

- Synchronization properties are influenced by the general properties of the oscillatory network: complex systems can be more or less prone to synchronize due to their specific features.

- Synchronization requires knowledge of both **nonlinear dynamics** and of **complex systems**.

The dynamics of coupled periodic oscillators: strong synchronization

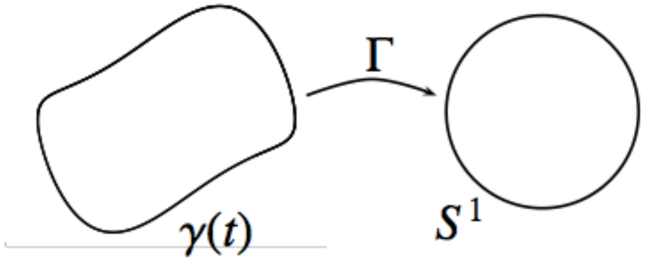


Izhikevich and Kuramoto (2006)

Oscillatory networks

Single oscillator

$\dot{X}_i = F_i(X_i) \quad X_i \in \boldsymbol{R}^m, \quad F_i : \; \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m, \quad (i = 1, 2, ..., n)$
has at least one hyperbolic T_i-periodic solution $\gamma_i(t) : \; \boldsymbol{R} \rightarrow \boldsymbol{R}^m$



$\theta_i(t) = \omega_i \; t, \; \theta_i \in S^1 = [0, \, 2\pi[\, , \; \omega_i = 2\frac{\pi}{T_i}$

Weakly Connected Oscillatory Networks ($\varepsilon \ll 1$)

$\dot{X}_i = F_i(X_i) \; + \; \varepsilon \; G_i(\boldsymbol{X}), \quad \boldsymbol{X} = [X_1', \ldots X_n']', \quad G_i : \; \boldsymbol{R}^{m \times n} \rightarrow \boldsymbol{R}^m$

$$\theta_i(t) = \omega_i \; t + \phi_i(\epsilon t)$$

Oscillatory networks:
Global dynamic behaviour

Weakly Connected Oscillatory Networks ($\varepsilon \ll 1$)

$$\dot{X}_i = F_i(X_i) + \varepsilon\, G_i(\boldsymbol{X}), \quad \boldsymbol{X} = [X_1', \ldots X_n']', \quad G_i : \boldsymbol{R}^{m \times n} \to \boldsymbol{R}^m$$

$$\theta_i(t) = \omega_i\, t + \phi_i(\epsilon t)$$

- **Time–domain techniques** do not allow to identify all the limit cycles (either stable or unstable).
  - It would require to consider *infinitely many* initial conditions.
  - Unstable limit cycles cannot be detected through simulation.
- By means of **Spectral techniques** (*Describing Function* and *Harmonic Balance*), the computation of all the limit cycles is reduced to a non-differential algebraic problem.
- Such methods are not suitable for characterizing the global dynamic behavior of complex networks with a large number of attractors.

Oscillatory networks: Malkin Theorem

Weakly Connected Oscillatory Networks ($\varepsilon \ll 1$)

$$\dot{X}_i = F_i(X_i) + \varepsilon\, G_i(\boldsymbol{X}), \quad \boldsymbol{X} = [X_1', \ldots X_n']', \quad G_i : \boldsymbol{R}^{m\times n} \to \boldsymbol{R}^m$$

$$\theta_i(t) = \omega_i\, t + \phi_i(\epsilon t)$$

Phase deviation equation

$$\dot{\phi}_i = \frac{\omega}{T}\int_0^T Q_i'(t)\, G_i\left[\gamma\left(t + \frac{\phi - \phi_i}{\omega}\right)\right] dt,$$

$$T = m.c.m.(T_1, \ldots, T_n)$$

$$\gamma\left(t + \frac{\phi - \phi_i}{\omega}\right) = \left[\gamma_1'\left(t + \frac{\phi_1 - \phi_i}{\omega_1}\right), \ldots, \gamma_n'\left(t + \frac{\phi_n - \phi_i}{\omega_n}\right)\right]'$$

$$\dot{Q}_i(t) = -[DF_i(\gamma_i(t))]'Q_i(t), \quad Q_i'(0)F_i(\gamma_i(0)) = 1$$

Joint application of the DF and MT

1. The periodic trajectories $\gamma_i(t)$ of the uncoupled oscillators are approximated through the *describing function technique*.
2. Once the approximation of $\gamma_i(t)$ is known, a first harmonic approximation of $Q_i(t)$ is computed, by exploiting the linear adjoint problem and the normalization condition.
3. The approximated phase deviation equation is derived by analytically computing the integral expression given by the *Malkin's Theorem*.

The phase equation is analyzed in order to determine the total number of stationary solutions (equilibrium points) and their stability properties. They correspond to the total number of limit cycles of the original weakly connected network.

Applications

- Synchronous states can be exploited for dynamic pattern recognition and to realize associative and dynamic memories. By means of a simple learning algorithm, the phase-deviation equation is designed in such a way that given sets of patterns can be stored and recalled. In particular, two models of WCONs have been proposed as examples of associative and dynamic memories.
- Spiral waves are the most universal form of patterns arising in dissipative media of oscillatory and excitable nature. By focusing on oscillatory networks, whose cells admit of a Lur'e description and are linearly connected through weak couplings, the occurrence of spiral waves has been studied.

Oscillatory associative memories

International Journal of Bifurcation and Chaos, Vol. 17, No. 12 (2007) 4365–4379

WEAKLY CONNECTED OSCILLATORY NETWORK MODELS FOR ASSOCIATIVE AND DYNAMIC MEMORIES

FERNANDO CORINTO, MICHELE BONNIN
and MARCO GILLI

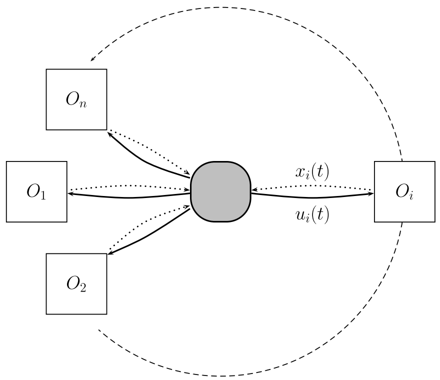
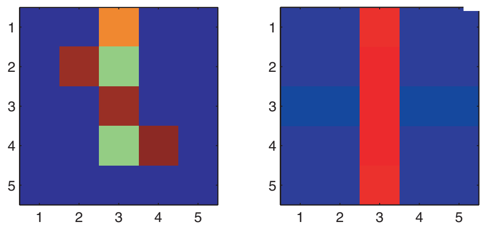
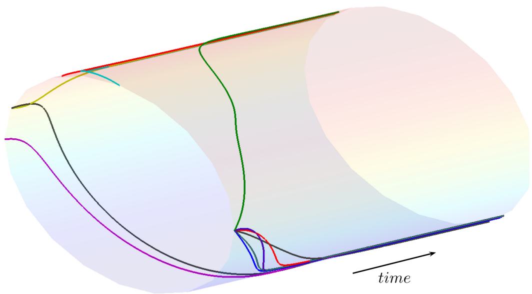
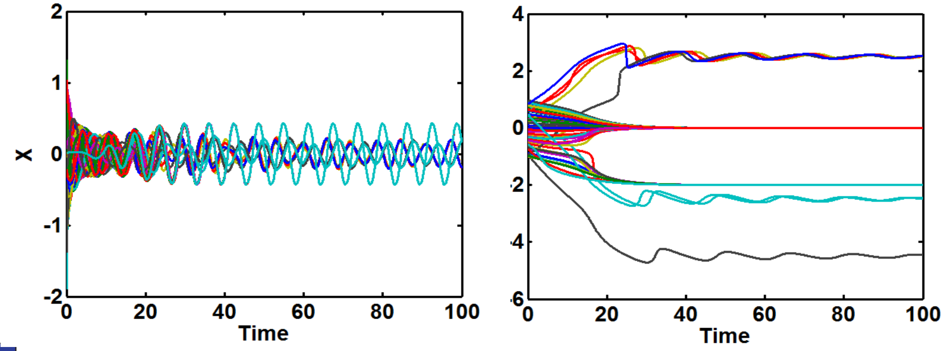
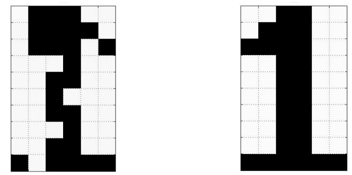


Fig. 1. Weakly connected oscillatory network having a star topology.









All oscillators are phase locked.
Degree of matching remains above a threshold.
Thus a better discrimination of matching patterns.

Oscillatory associative memories

Can oscillatory associative memories outperform "static" associative memories?

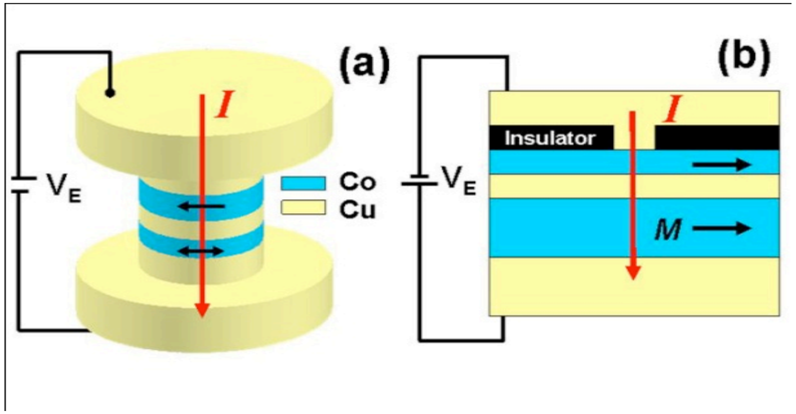
- Goal: find classes of problem solved only by oscillatory networks
- no restrictions about the architecture of the networks

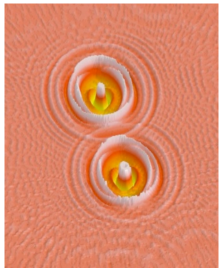
Oscillatory associative memories

Can oscillatory associative memories outperform "static" associative memories?

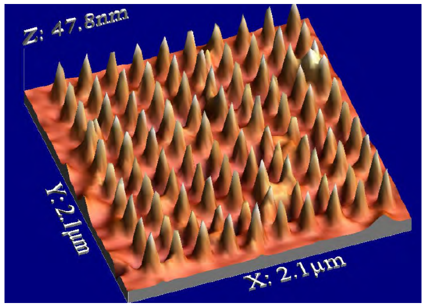
- Goal: conceive non-boolean spatio-temporal algorithms to solve a classical problem in a more efficient (in terms of speed, power, ...) way
- consider physical constrains

Spin-Torque Oscillatory arrays

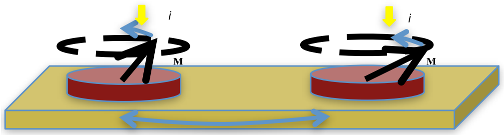




coupled STOs

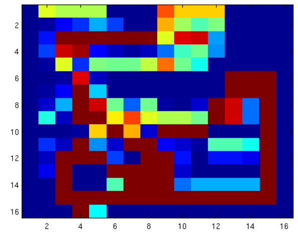


array of STOs

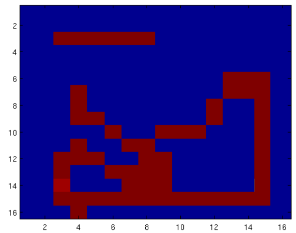


pattern recognition tasks

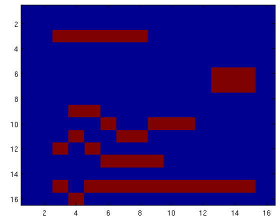
(a) Input



(b) Output

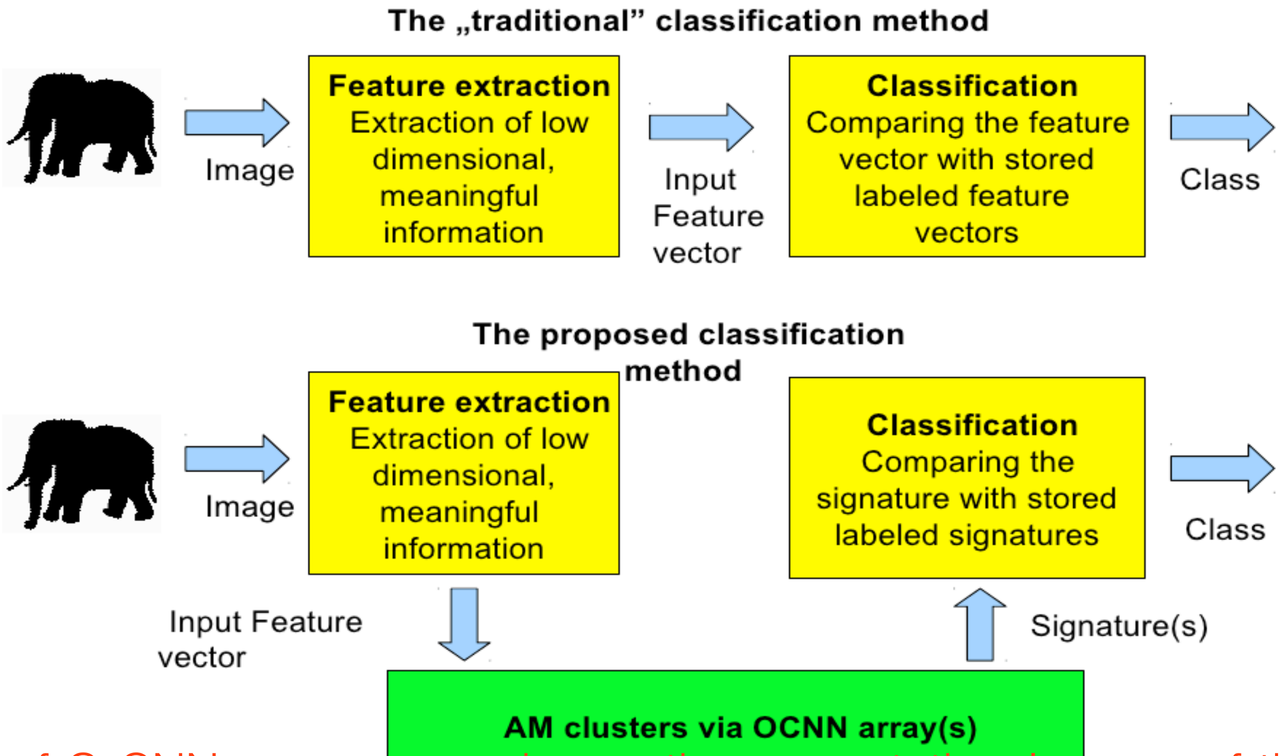


(c) Thresholded output



(d) Horizontal edge detection

Spin-Torque Oscillatory arrays



The addition of O-CNN arrays can enhance the computational power of the architecture and increase the detection rate.
The OCNN array can transform the input feature vector in a way which helps classification.

Conclusions and Perspectives

- Simulation with real-life data: Images taken by a mobile robot has been used for classification with similar results.

- Boundary conditions /lateral input/: with side input, changing the boundary conditions the properties of the array can be changed, this can be used to increase the computational strength (programmability) of the array

- OCNN array with different spin oscillators: The usage of two different dynamics in one network would increase the possible outcomes of one OCNN array

- Transient based computation: using the evolution of the phase shift to determine extra properties about the input vectors

- Synchronization requires knowledge of both **nonlinear dynamics** and of complex systems.

Nonlinear analysis tools

- Differential or integral equations represent suitable mathematical models of physical systems.
- Approximate analytical tools are required for studying (analysis and design) nonlinear dynamical systems describing electrical circuits, mechanical and biological systems, ...
  - Tools for detecting oscillations