Backward Euler method and solution of nonlinear equations

Consider an initial-value problem:

$$y' = f(x, y), \quad y(0) = y_0 \quad (*)$$

We look for the solution of this problem for a series of steps $x_n = nh$ for $n \geq 1$, for some step size h. Denote the exact solution by $y_n = y(x_n)$ and the approximate solution at x_n by Y_n.

Using the backward difference approximation to $y'(x_{n+1})$, we obtain

$$y'(x_{n+1}) = \frac{y_{n+1} - y_n}{h} + O(h)$$

and thus the difference approximation to ODE $(*)$ at x_{n+1} becomes

$$Y_{n+1} = Y_n + h\, f(x_{n+1}, Y_{n+1}) \quad (**)$$

This scheme is called the Backward Euler method. This method is implicit: to find Y_{n+1}, the nonlinear equation $(**)$ has to be solved.

Stability

To find the region of absolute stability of the Backward Euler method, we consider the model problem $(*)$ with $f(x, y) = \lambda y$ with the initial data $y(0) = 1$. The exact solution $y = e^{\lambda x}$ decays to zero if $Re(\lambda) < 0$ and $x \to \infty$. The difference scheme $(**)$ gives:

$$Y_{n+1} = Y_n + h\lambda Y_{n+1}$$

and then

$$Y_{n+1} = \frac{Y_n}{1 - h\lambda}, \quad Y_0 = 1$$

The solution of the above is

$$Y_n = \frac{1}{(1 - h\lambda)^n}$$

The region of absolute stability is the set of points $z = h\lambda$ in the complex plane, for which $|Y_n| \to 0$ for $x_n \to \infty$. This set is given by $|1 - h\lambda| > 1$ which defines the points z at distance greater than 1 from the point $(1, 0)$ in the complex plane (= the complex plane minus the closed unit circle centred at $(1, 0)$). Hence, the region of absolute stability covers the region $Re(\lambda) < 0$, and the Backward Euler method is A-stable.

Newton's method

To solve $(**)$ for a nonlinear function f, the Newton's method can be used. Reformulate the nonlinear equation $(**)$ to

$$F(u) = 0$$

with $u = Y_{n+1}$ and $F(u) = u - Y_n - h\, f(x_{n+1}, u)$. Choose an initial guess u_0 and solve $F(u) = 0$ iteratively using

$$u_k = u_{k-1} - \frac{F(u_{k-1})}{F'(u_{k-1})}$$

where u_k is the solution for Y_{n+1} at iteration k.

Fixed point iteration

To solve $(**)$, the fixed point iteration can also be used. Reformulate the nonlinear equation $(**)$ to

$$u = G(u)$$

with $G(u) = Y_n + h\, f(x_{n+1}, u)$. Starting with an initial guess u_0, find u_k iteratively using

$$u_k = G(u_{k-1})$$

If the real-valued function G satisfies

$$|G(w) - G(v)| \leq K|w - v|$$

for any two real w, v and a constant $K < 1$, then there exists a unique fixed point u such that $G(u) = u$, and the fixed point iteration with any initial guess u_0 converges to u.