Nonlinear dynamics and chaos with practical applications

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PLVS RATIO QVAM VIS

Introduction to Dynamical Systems

Nothing in Nature is random... A thing appears random only through the incompleteness of our knowledge.

Spinoza

Chaos vs. Randomness

- Do not confuse chaotic with random:

Random:

– irreproducible and unpredictable

Chaotic:

– deterministic - same initial conditions lead to same final state… **but the final state is very different for small changes to initial conditions**

– ***difficult or impossible to make long-term predictions***

Fundamental properties of deterministic chaos

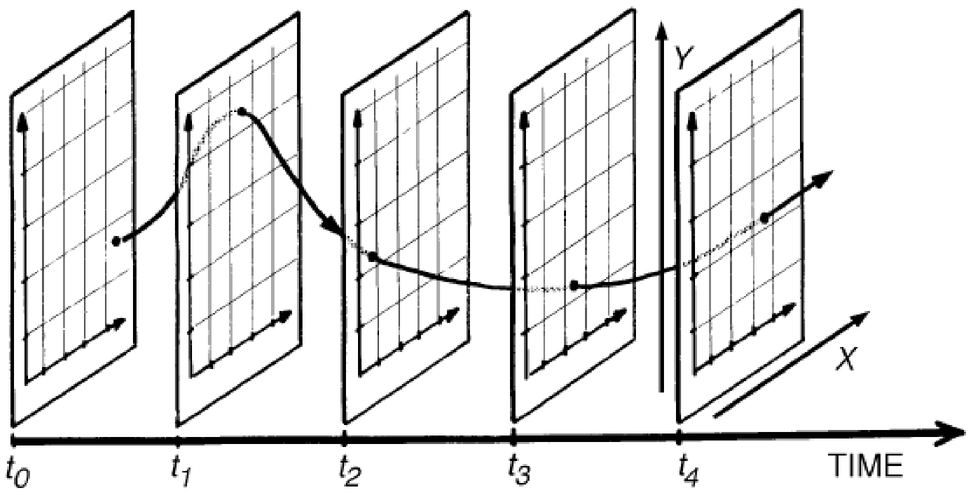
- Sensitive dependence on initial conditions
- Existence of attractors – chaotic limit sets
- Existence of a countable infinity of unstable periodic orbits within the attractor
- Existence of dense orbit in the attractor
- Fractal structure in the state space

MODELING AND DYNAMICAL SYSTEMS

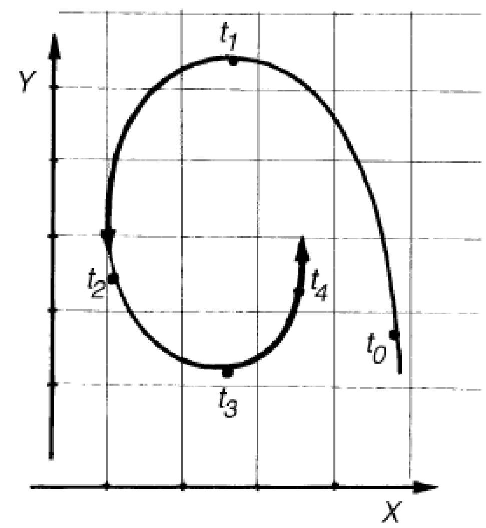
- **Modeling is fundamental to analysis and understanding of physical, biological, and social systems.**
  - *Basic assumption*: internal states described by a few observables.
  - *Mathematical idealization*: process leads to geometric characterization of idealized states (state-space model).
  - *Conventional interpretation*: assumed correspondence between actual states and geometric model points.
- **Dynamical systems are a primary paradigm for modeling.**
  - A dynamical system is one in which a set of internal parameters (called *states*) obeys a set of temporal rules.
  - Study of dynamical systems divides into applied dynamics, mathematical dynamics, experimental dynamics.

STATE VARIABLE REPRESENTATIONS

- *Time series* — **scalar variable versus time**
  - Traditional approach, used especially in statistical contexts
- *Phase space* — **state variables with time as a parameter**
  - Geometric perspective provides several benefits



Time series



Phase space (2-D)

CONTINUOUS (ANALOG) DYNAMICAL SYSTEMS

- State depends continuously on time t.
- Governing rule is usually an ordinary or partial differential eqn.:

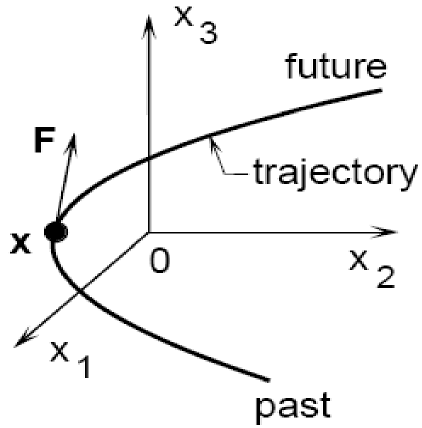
$$\left.\begin{array}{ll} d\mathbf{x}/dt =: \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) & (\textit{autonomous or unforced}) \\ \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) & (\textit{nonautonomous or forced}) \end{array}\right\} \text{ODE}$$

$$\mathbf{G}(\mathbf{x}, \mathbf{u}_{\mathbf{x}}, \mathbf{u}_{\mathbf{x}^{(2)}}, \ldots, \mathbf{u}_{\mathbf{x}^{(n)}}) = \mathbf{0} \qquad \text{PDE}$$

F: vector field (smooth)

$\dot{\mathbf{x}}$: velocity

F tangent to trajectory at x



Representative third-order ODE

DISCRETE (DIGITAL) DYNAMICAL SYSTEMS

- State depends on discrete set of times t_i.
- Governing rule is usually a difference equation (DE):

$$\mathbf{x}_{n+1} = \mathbf{\Phi}(\mathbf{x}_n) \qquad \text{(autonomous)}$$

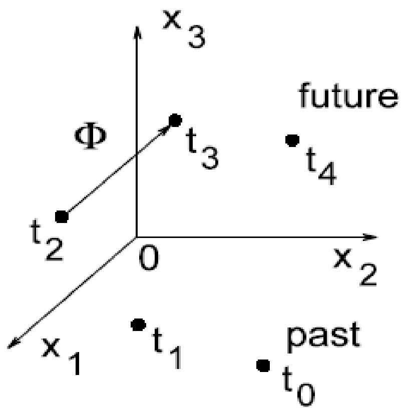
$$\mathbf{x}_{n+1} = \mathbf{\Phi}(\mathbf{x}_n, t_n) \qquad \text{(nonautonomous)}$$

with $\mathbf{x}_n := \mathbf{x}(t_n)$.

$\mathbf{\Phi}$: state transition map

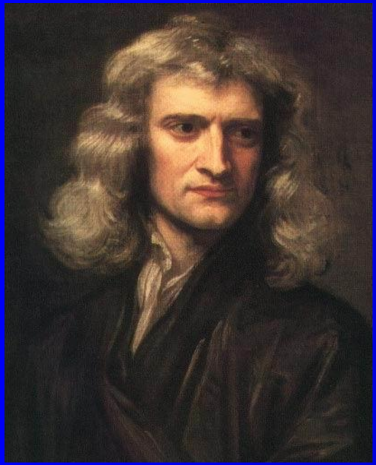
$\mathbf{x}_n$: present state

$\mathbf{x}_{n+1}$: next state



Representative third-order DE

Background



Newton, Liebniz

Deterministic Cause, Effect

Calculus, Continuity

“Very Similar” Cause
“Very Similar” Effect

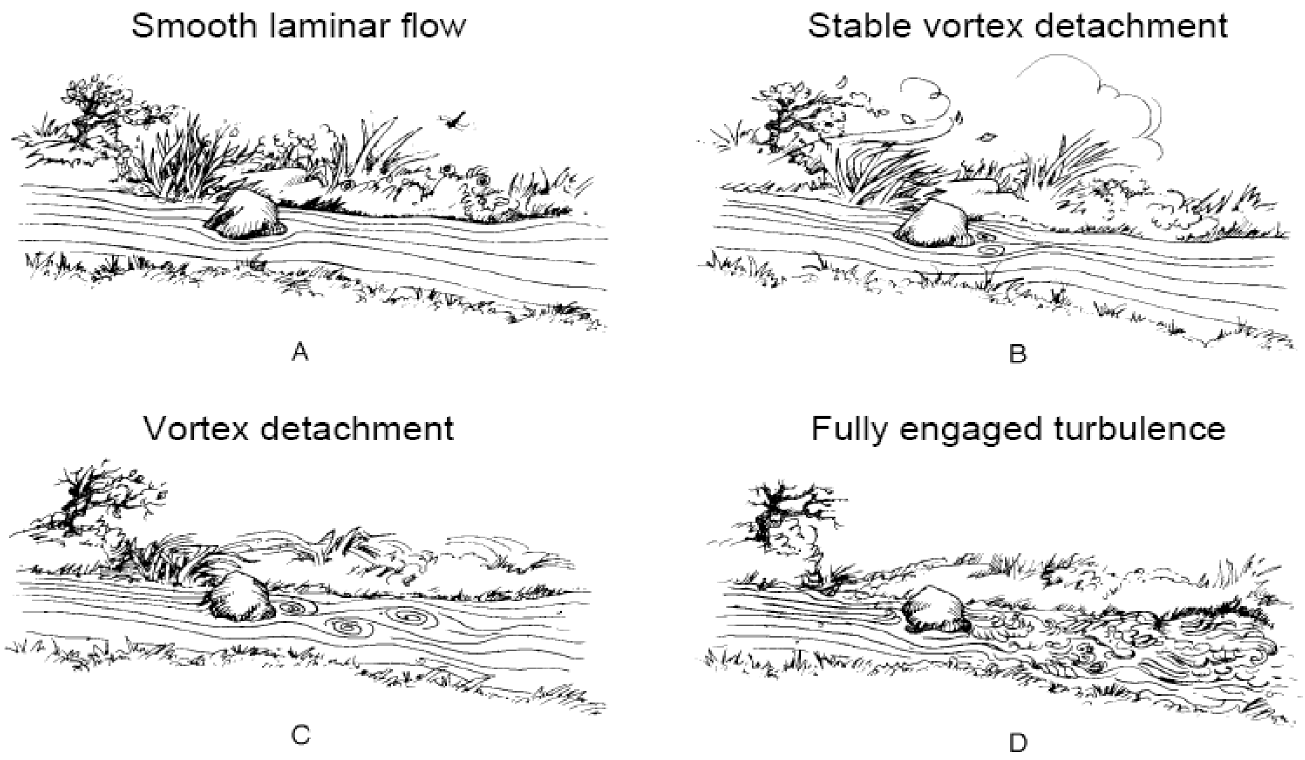
Beginner's guide to chaos

Over the last decade, physicists, biologists, astronomers and economists have created a new way of understanding the growth of complexity in nature. This new science, called chaos, offers a way of seeing order and pattern where formerly only the random, erratic, the unpredictable — in short, the chaotic — had been observed.

James Gleick*

What is chaos

- Chaos is bounded, random-like behavior in a deterministic dynamical system — that is, “noise” with an underlying order.
- *Example:* progression from order to disorder in a flowing stream



Chaos properties

- **Characteristic features**
  - Essentially continuous, possibly banded Fourier spectrum.
  - *Sensitivity to initial conditions*: nearby orbits diverge very rapidly.
  - Ergodicity and mixing of the orbits
    - *ergodicity* — each orbit visits entire chaotic region infinitely often
    - *mixing* — any region of initial states quickly dispersed throughout chaotic region
- **Important observations**
  - System can be continuous or discrete, but must be nonlinear.
  - System can be forced or unforced, dissipative or lossless.
  - Continuous system must be third-order or higher; discrete or PDE system can be any order.

Chaos is *aperiodic long-term behavior* in a *deterministic* system that *exhibits sensitive dependence on initial conditions*

- The simplest example: One-dimensional *non-linear* maps
  - the logistic map

- Phenomenology:
  - Initial conditions, fixed points and linear stability
  - Bifurcation analysis, period doubling
  - Bifurcation diagrams
  - Chaos

- Analysis:
  - Lyapunov exponents
  - Stretching and folding
- Conclusions

One-dimensional maps

- One-dimensional maps, definition:
  - a set V (e.g. real numbers between 0 and 1) $0 \leq x_n \leq 1$
  - a map of the kind $f: V \rightarrow V$

$$x_{n+1} = f(x_n)$$

- Linear maps: $x_{n+1} = a x_n + b$
  - a and b are constants
  - linear maps are invertible with no ambiguity → no chaos

- Non-linear maps: The logistic map $0 \leq x \leq 1$

$$f(x) = \mu x(1-x)$$

One-dimensional maps

- Non-linear maps: The logistic map $0 \leq x \leq 1$

$$f(x) = \mu x(1-x)$$

with $0 \leq \mu \leq 4$

- Motivation: Discretization of the *logistic equation* for the dynamics of a biological population *x*

$$\dot{x} = bx - cx^2$$

b: birth rate (assumed constant)

cx: death rate depends on population (competition for food, …)

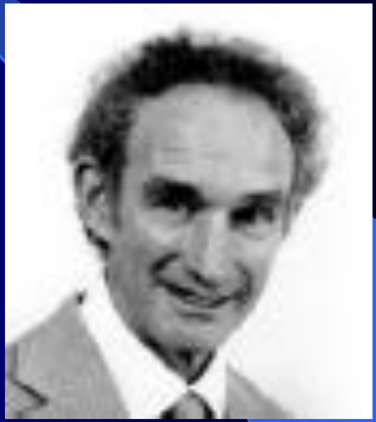
How do we explore the logistic map?

Simple Model, Complex Dynamics

Robert M. May

Simple mathematical models with very complicated dynamics,

Nature 261 (1976) 459-467.



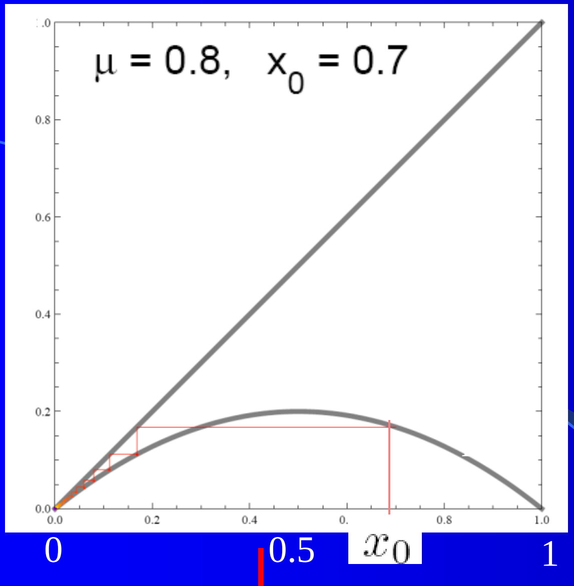
General Paradigm

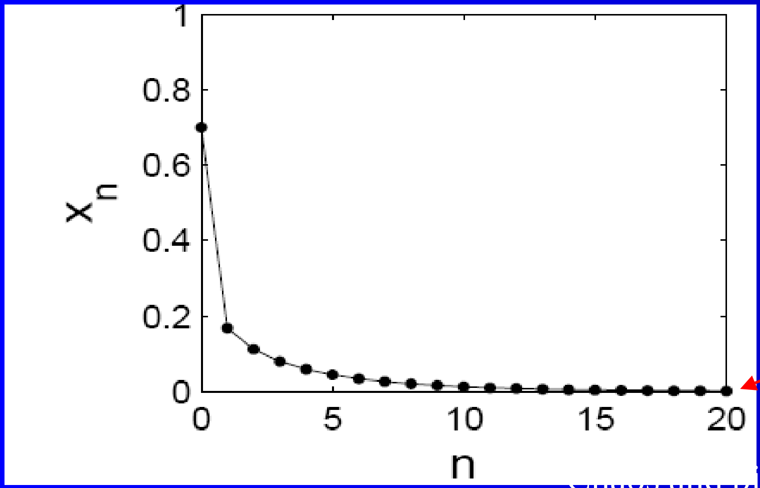
for

Emergence of Chaos

Distinguish Deterministic Chaos from Stochastic Flux ?

Geometric representation $f(x) = \mu x(1-x)$ $0 \leq x \leq 1$





Evolution of a map: $\mu = 0.8, \quad x_0 = 0.7$

1) Choose initial conditions
2) Proceed vertically until you hit *f(x)*
3) Proceed horizontally until you hit *y=x*
4) Repeat *2)*
5) Repeat *3)*

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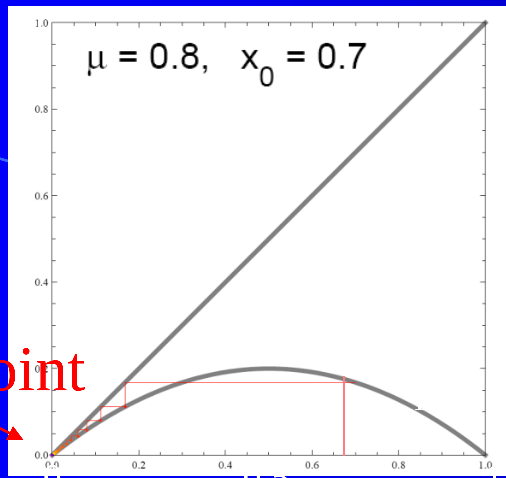
.
.

Evolution of the logistic map

fixed point ?

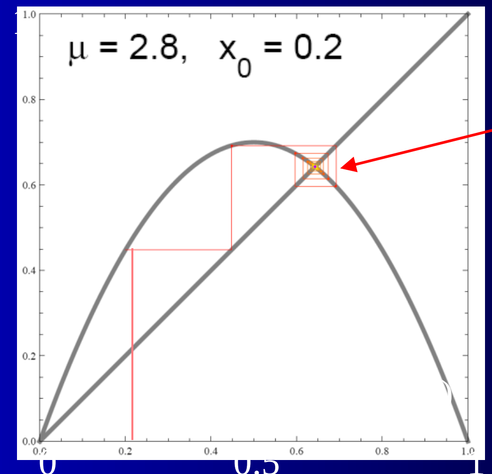
- Phenomenology of the logistic map $f(x) = \mu x(1-x)$ $0 \le x \le 1$

a)



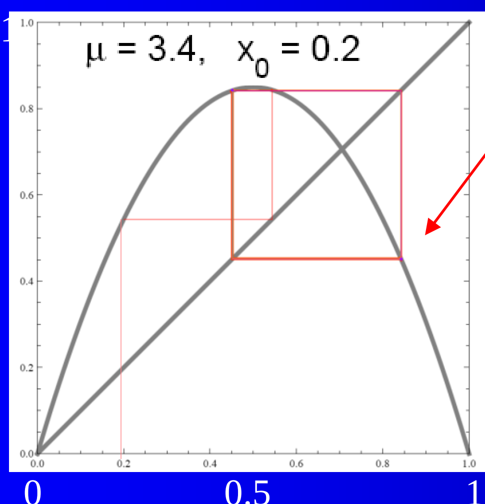
fixed point

b)



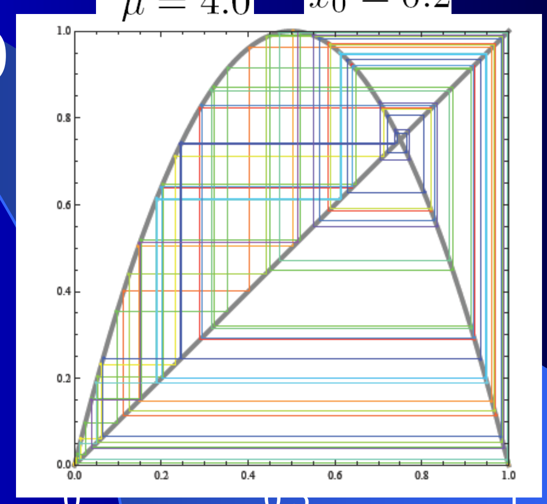
fixed point

c)



2-cycle?

d) $\mu = 4.0$ $x_0 = 0.2$



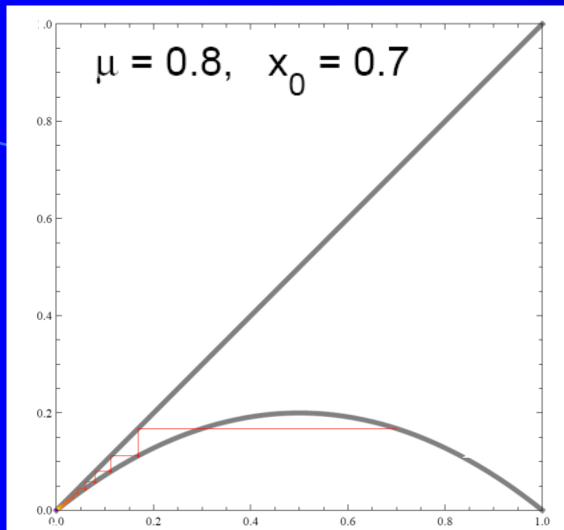
chaos?

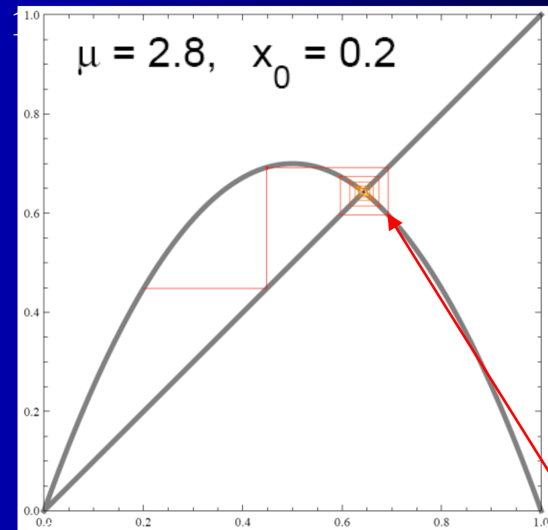
What's going on? *Analyze first a) → b then b) → c) , …*

- Geometrical representation

$$f(x) = \mu x(1-x)$$

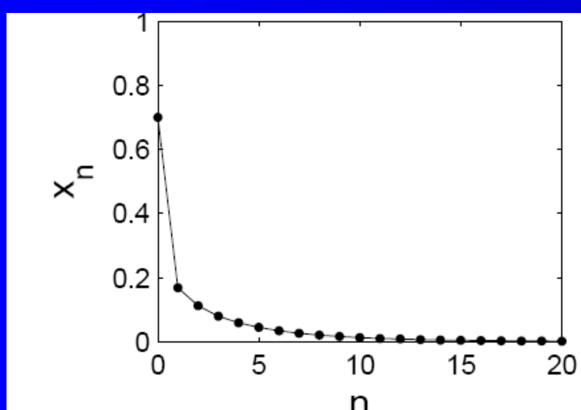
$$0 \le x \le 1$$

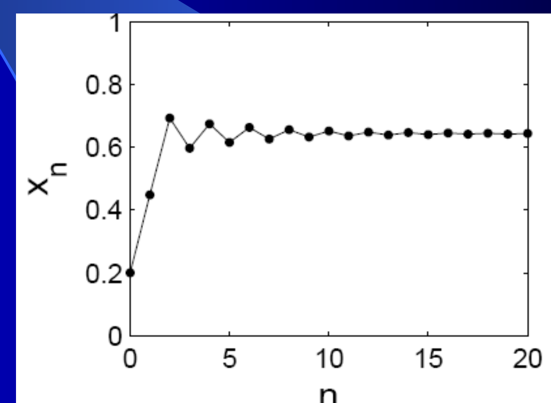




- Evolution of the logistic map

fixed point





How do we analyze the existence/stability of a fixed point?

- **Fixed points**

- Condition for existence: $x_f = f(x_f)$
- Logistic map: $x_f(1 - \mu + \mu x_f) = 0$ $\quad x_f = 0, \quad 1 - \frac{1}{\mu}$ $\quad \mu \geq 1$
- Notice: since $0 \leq x \leq 1$ the second fixed point exists only for $\mu \geq 1$

- **Stability**

  - Define the distance of x_n from the fixed point x_f

$$\delta_n = x_n - x_f$$

  - Consider a neighborhood of x_f

Taylor expansion

$$|\delta_{n+1}| = |x_{n+1} - x_f| = |f(x_f + \delta_n) - x_f| \simeq \left| f'(x_f) \right| |\delta_n|$$

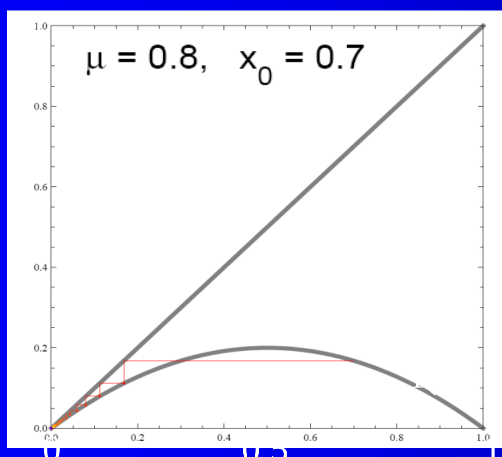
  - The requirement $|\delta_{n+1}| < |\delta_n|$ implies

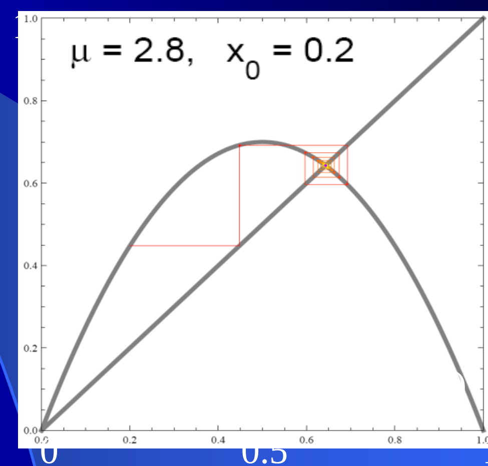
$$\left| \frac{df}{dx} \right|_{x=x_f} < 1$$

→ Logistic map?

- **Stability and the Logistic Map** $f(x) = \mu x(1-x)$

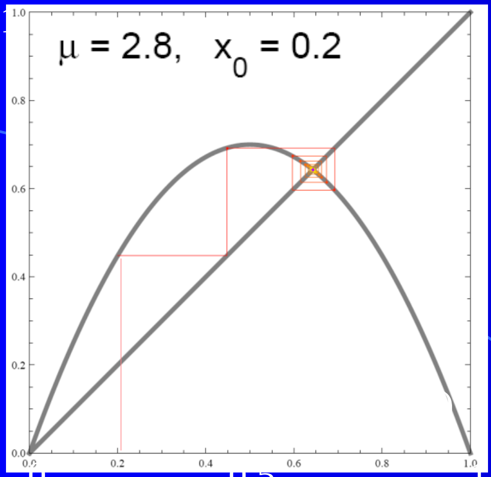
  - Stability condition: $\dfrac{df}{dx} = \mu(1-2x) < 1$
  - First fixed point: stable (*attractor*) for $\mu < 1$
  - Second fixed point: stable (*attractor*) for $1 < \mu < 3$
  - No coexistence of 2 stable fixed points for these parameters (transcritical biforcation)

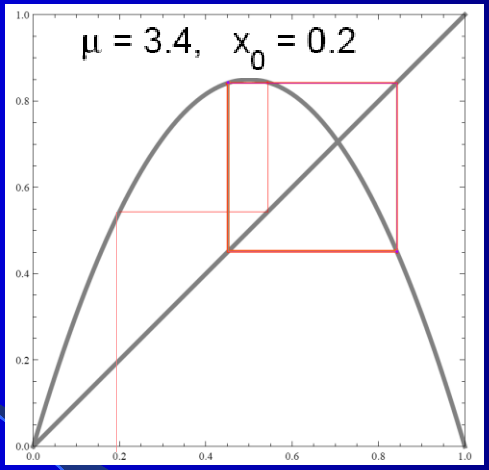




What about $3 \leq \mu \leq 4$?

- Period doubling

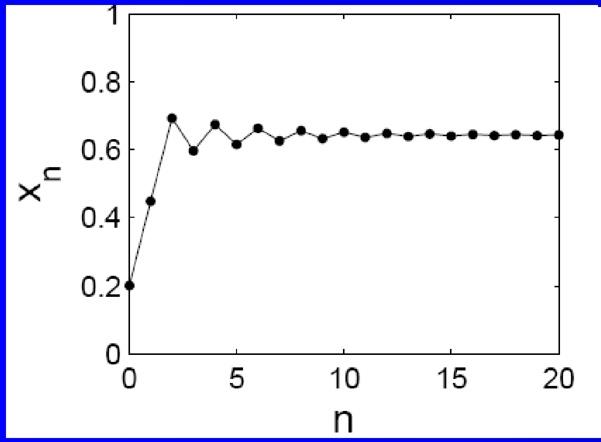


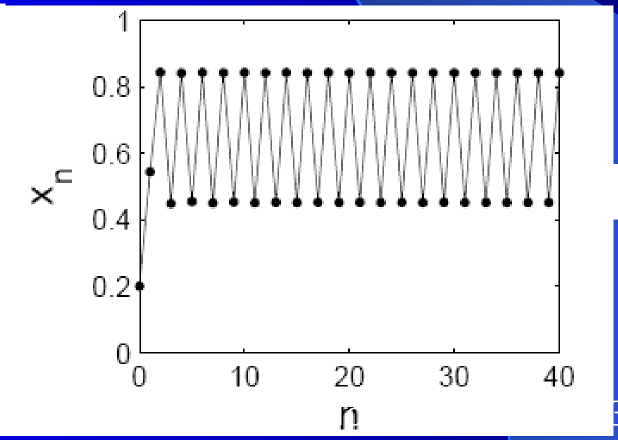


Observations:

1) The map oscillates between two values of x

- Evolution of the logistic map

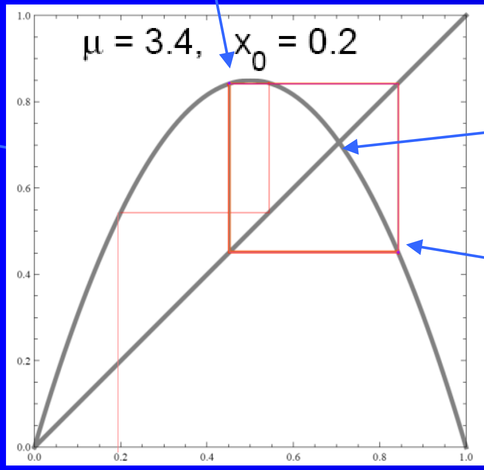




2) Period doubling:

$$x_{n+2} = x_n$$

Period doubling



- At $\mu = 3$ the fixed point x^* becomes unstable since

$$\left|\frac{df}{dx}\right|_{x=x_f} > 1$$

-Observation: an attracting 2-cycle starts → *(flip)-bifurcation*

The points are found solving the equations

and thus:
$$x_2 = \mu x_1(1-x_1)$$
$$x_1 = \mu x_2(1-x_2)$$

Why do these points appear?

$$x_{1,2} = (1+\mu \pm \sqrt{\mu^2 - 2\mu - 3})/2\mu$$

These points form a 2-cycle for $f(x)$

However, the relation $x_{n+2} = x_n$ suggests they are fixed points for the iterated map $f^2(x)$

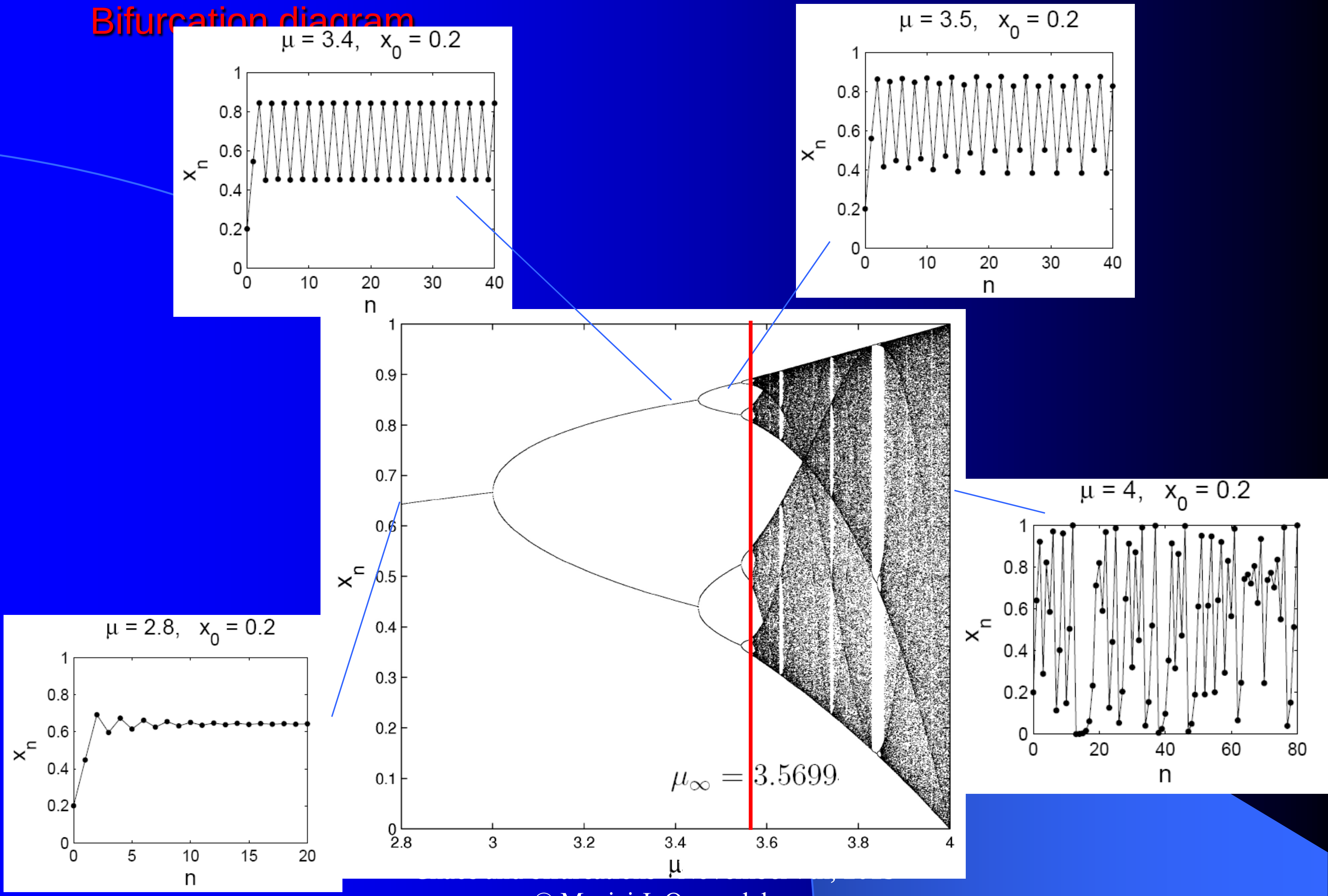
Stability analysis for $f^2(x)$: $-1 < \mu^2(1-2x_1)(1-2x_2) < 1$

and thus: $3 < \mu < 1+\sqrt{6} = 3.44949$

- For $\mu = 1+\sqrt{6}$, loss of stability and bifurcation to a 4-cycle

Plot of fixed point μ vs

- Bifurcation diagram



- Bifurcation diagram

Plot of fixed points vs μ

Observations:

1) Infinite series of period doublings at *pitchfork-like (flip) bifurcations*
2) After a point

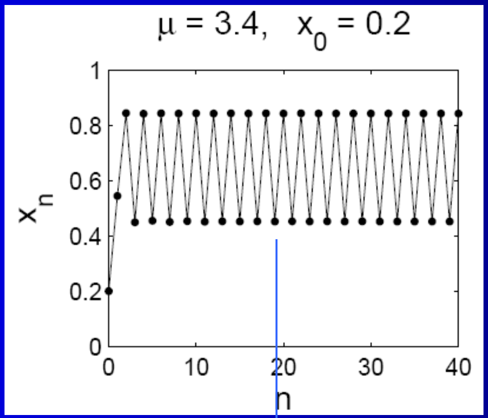
$$\mu_\infty = 3.5699$$

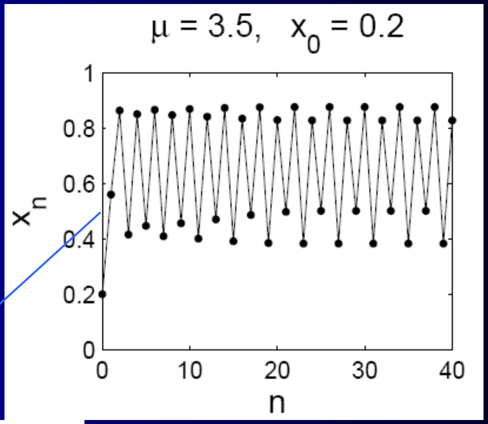
chaos seems to appear

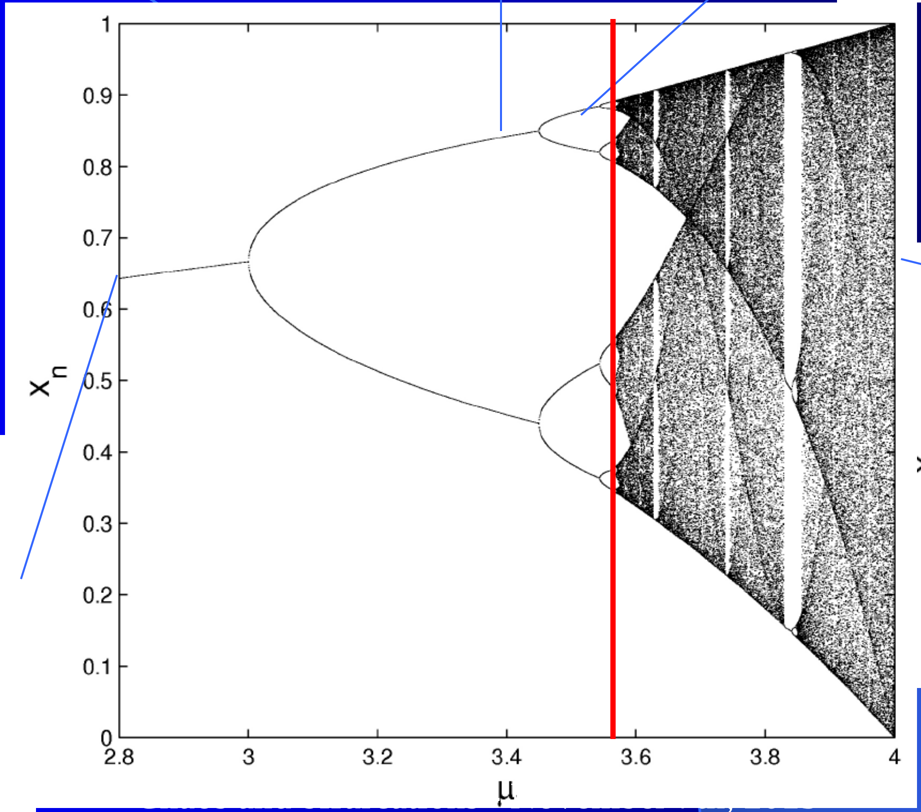
3) Regions where stable periodic cycles exist occur for

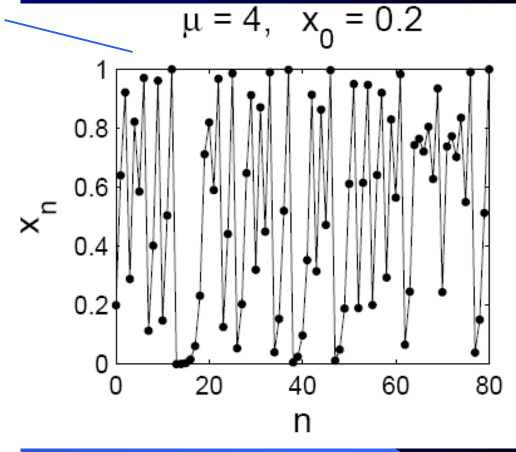
$$\mu \geq \mu_\infty$$

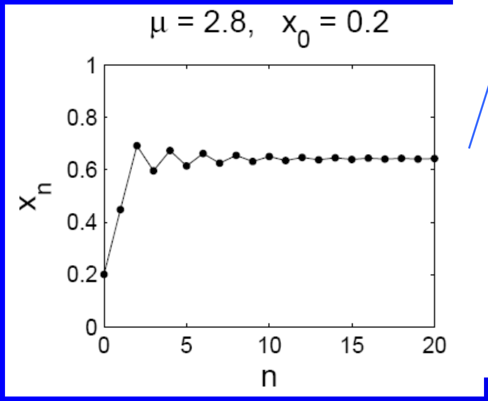
What is general?











Metric Universalities

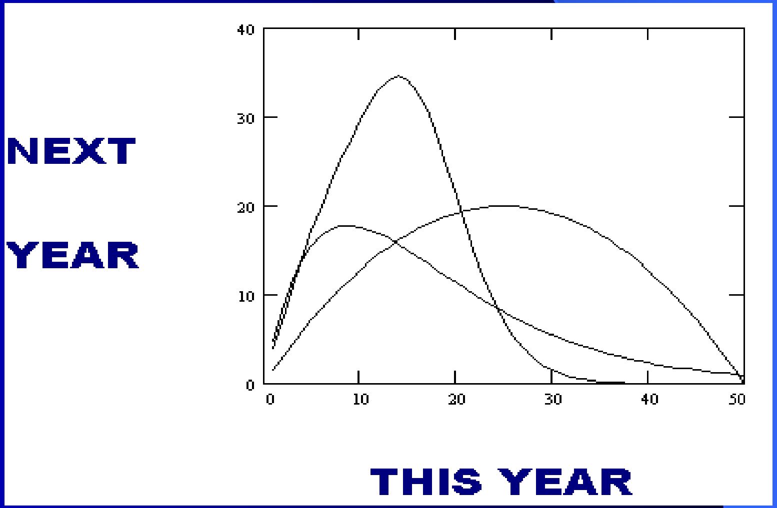
M. Feigenbaum

Feigenbaum Number

Feigenbaum Ratios

Periodicity
Sequence

Quantitative
Identicality



- Bifurcation diagram

General points:

1) Period doubling is a quite general route to chaos (other possibilities, e.g. *intermittency*)

2) Period doublings exhibit *universal properties*, e.g. they are characterized by certain numbers that do not depend on the nature of the map. For example, the ratio of the spacings between consecutive values of μ at the bifurcation points approaches the universal "Feigenbaum" constant. The latter occurs for all maps that have a quadratic maximum

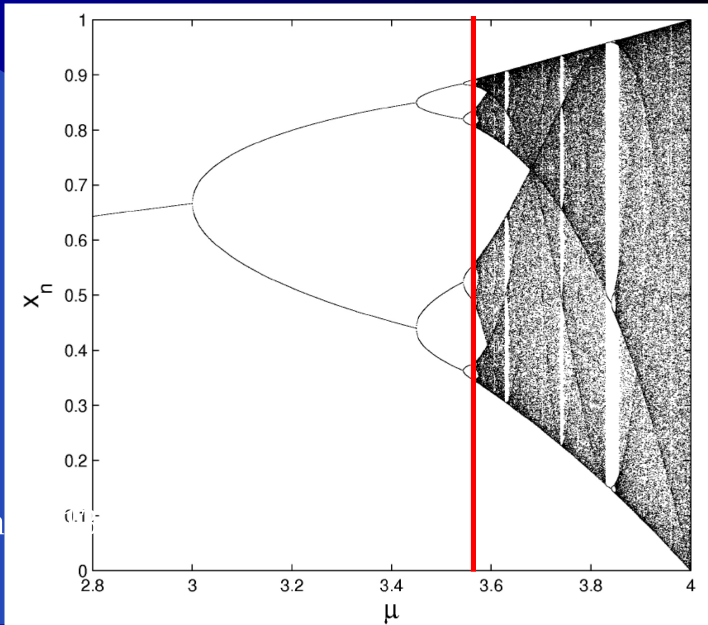
$$\delta = \lim_{k\to\infty}\left(\frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}\right) = 4.669201609\ldots$$

3) Thus, we can *predict* where the cascade of period doublings ends, and *something else* starts
4) The *something else* looks chaotic, however, can we quantify how chaotic really is?

- How do we characterize/quantify chaos?

Chaos: rapid divergence of nearby points in phase space

Measure of divergence: Lyapunov exponent λ



Lyapunov exponent

- One-dimensional system with initial conditions x and $x+\epsilon$ with $\epsilon \ll 1$
- After *n* iterations, their divergency is approximately $\epsilon(n) \approx \epsilon e^{n\lambda}$

  - If $\lambda < 0$ there is convergence → no chaos
  - If $\lambda > 0$ there is divergence → chaos

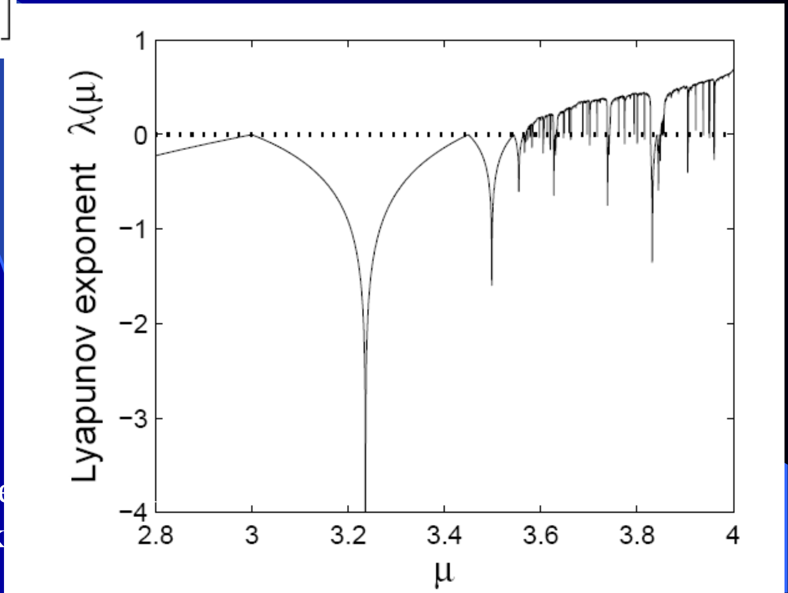
- One dimensional system $x_{n+1} = f(x_n) = f^n(x_0)$

After *n* steps $f^n(x_0+\epsilon) - f^n(x_0) \approx \epsilon e^{n\lambda}$

Thus: $\lambda \approx \frac{1}{n} \ln \left[\frac{f^n(x_0+\epsilon) - f^n(x_0)}{\epsilon} \right] \approx \frac{1}{n} \ln \left[\left. \frac{df^n(x)}{dx} \right|_{x=x_0} \right]$

(chain rule) $= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$

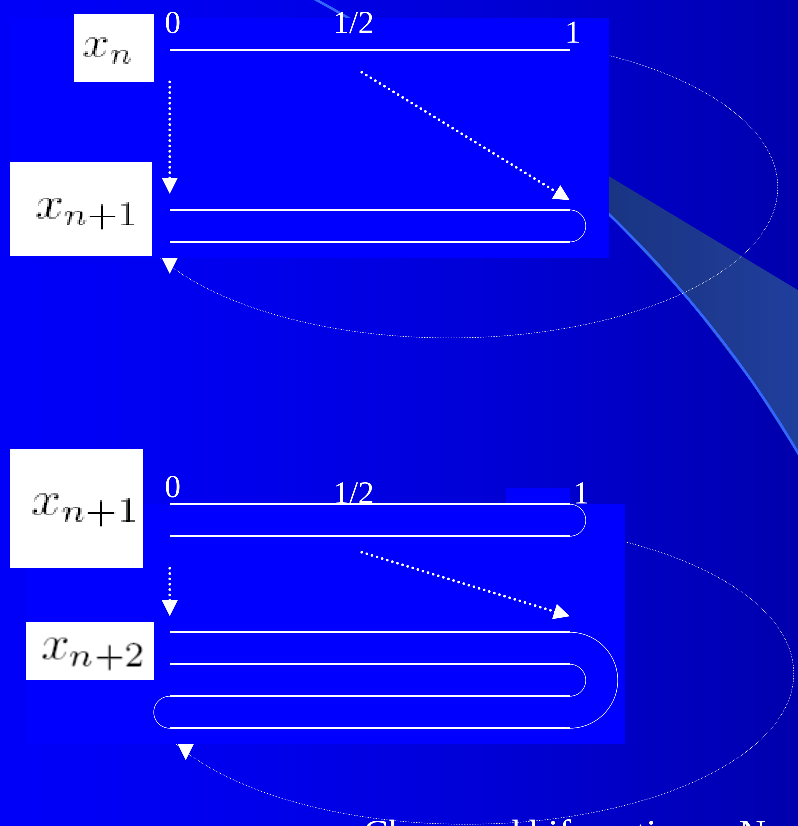
Logistic map



Stretching and folding

- Beginning of the lecture: “Chaos: is *aperiodic long-term behavior* in a *deterministic* system that *exhibits sensitive dependence on initial conditions* ”

- However, in general it is necessary to have a mechanism to keep chaotic trajectories within a finite volume of phase-space, despite the expoential divergence of neighboring states



- “stretching” (divergence) for (0,1/2)

  “folding” (confinement) for (0,1/2)

- - “stretching+folding” is responsible for loss of information on initial conditions as the iteration number (time) increases
  - for 1D maps, non-linearity makes “time”-inversion ambiguous → loss of information

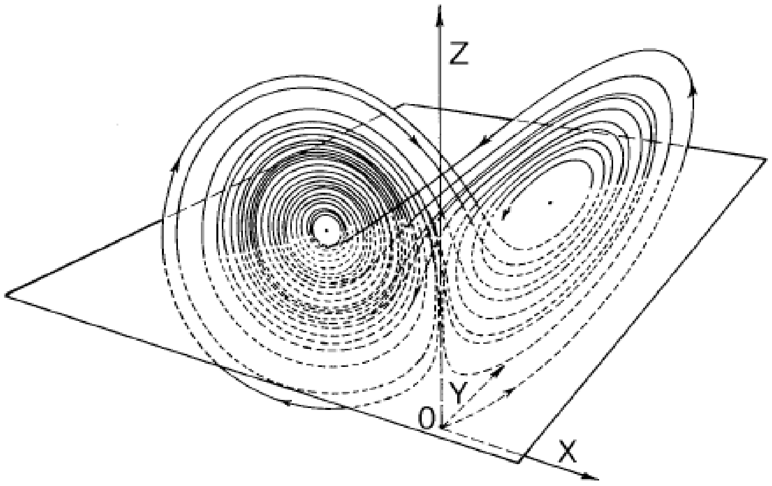
Conclusions

- Chaos
  - the logistic map
- Phenomenology:
  - Initial conditions, fixed points and linear stability
  - Bifurcation analysis, period doubling
  - Bifurcation diagrams
  - Chaos
- Analysis:
  - Lyapunov exponents
  - Stretching and folding
- Conclusions

Lorenz system

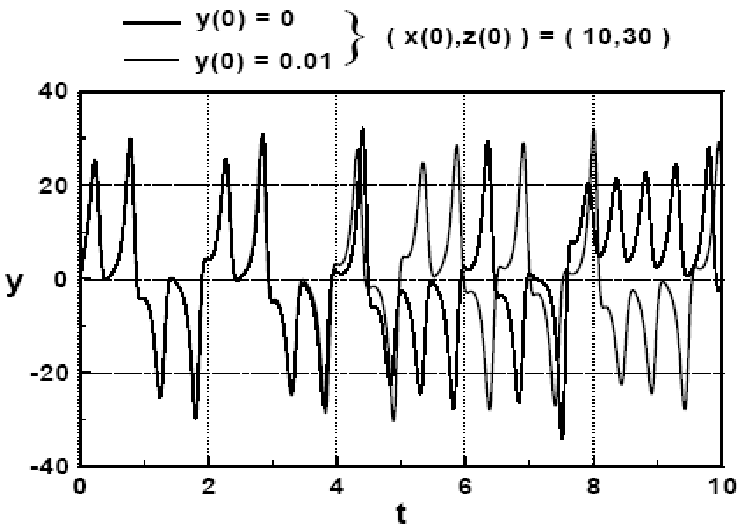
- **One of the first strange attractors discovered in the natural sciences (Lorenz, 1963).**
  - Third-order autonomous dynamical system modeling thermal convection and flow in viscous fluid or atmosphere (σ, B, R are physical parameters).

$$\dot{x} = \sigma(y - x), \quad \dot{y} = R\,x - y - x\,z, \quad \dot{z} = -B\,z + x\,y$$



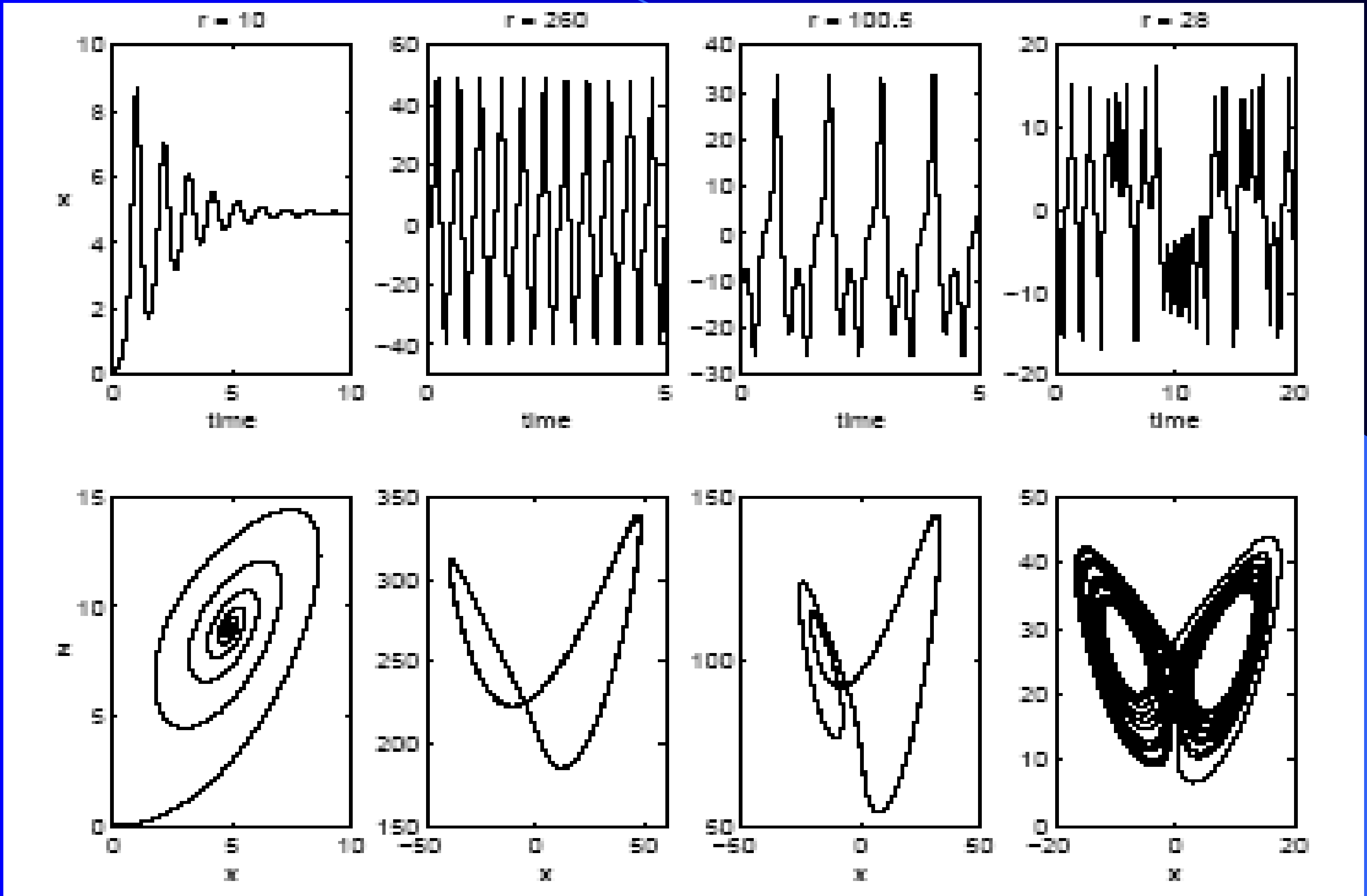
Lorenz Attractor

$(\sigma = 10,\ B = \frac{8}{3},\ R = 28)$



Sensitivity to Initial Conditions

Trajectories in the Lorenz system



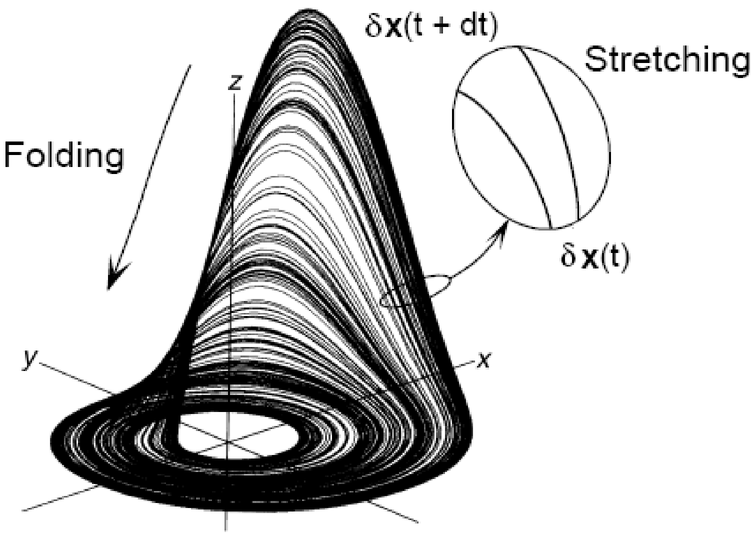
Quote from Lorenz (1993)

- "One other study left me with mixed feelings. Otto Roessler of the University of Tübingen had formulated a system of three differential equations as a model of a chemical reaction. By this time a number of systems of differential equations with chaotic solutions had been discovered, but I felt I still had the distinction of having found the simplest. Roessler changed things by coming along with an even simpler one. His record still stands."

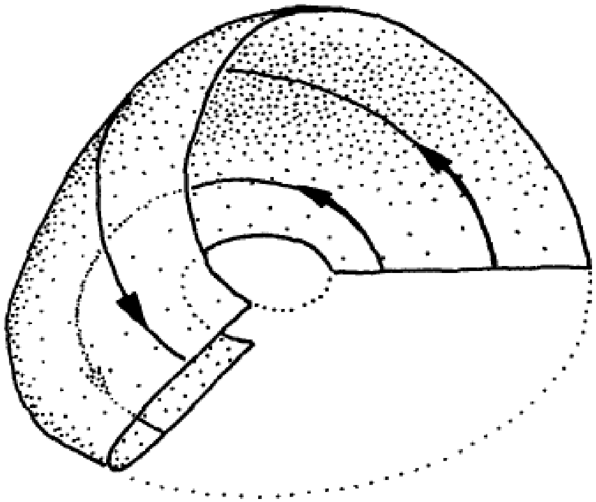
Roessler system

- Dynamical system (third-order autonomous; Rössler, 1976)

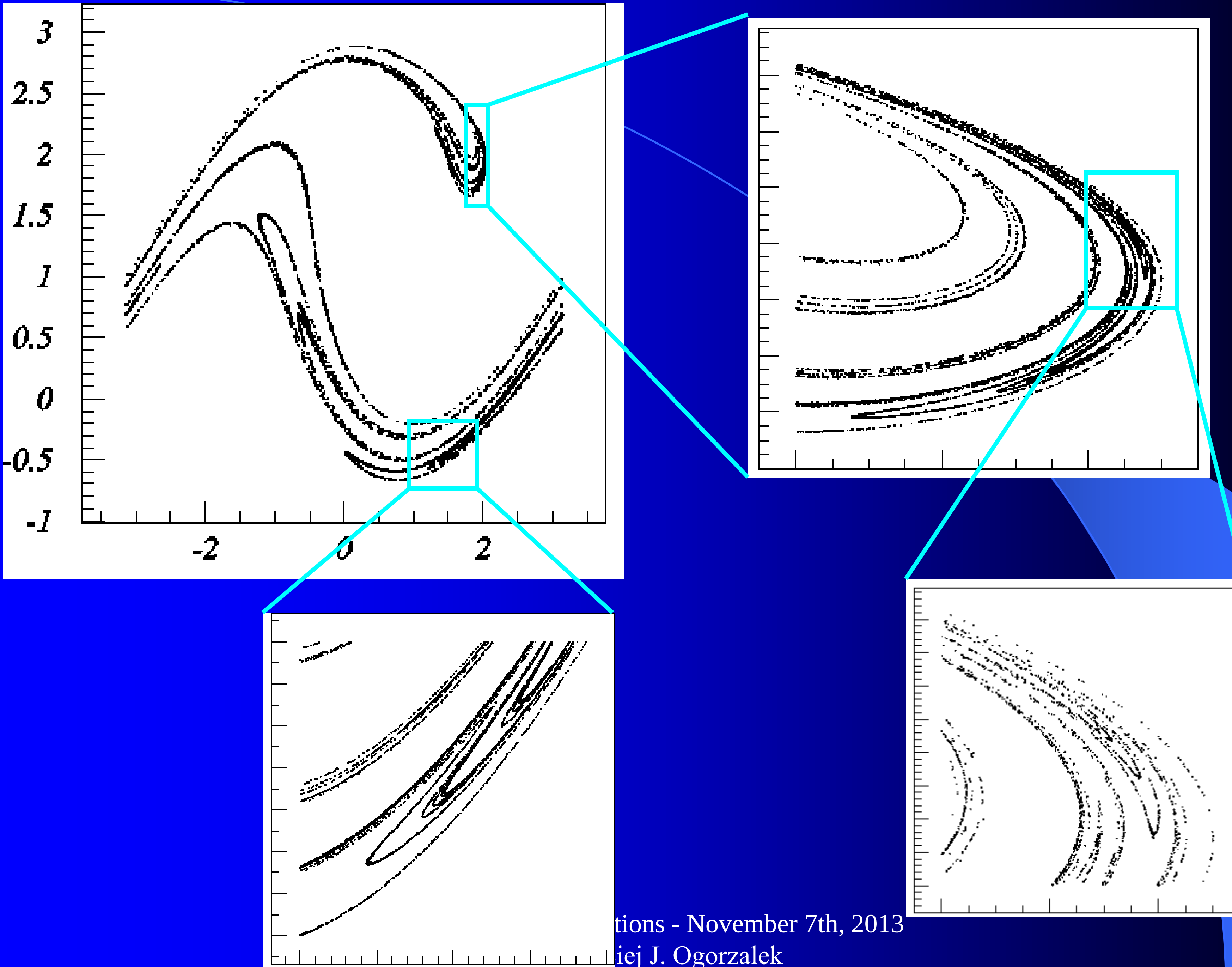
$$\dot{x} = -(y + z), \;\; \dot{y} = x + \frac{1}{5} y, \;\; \dot{z} = \frac{1}{5} + z(x - \mu)$$



$\mu = 5.7$
Initial state:
$(-1, 0, 0)$



Stretch & fold
operation


3
2.5
2
1.5
1
0.5
0
-0.5
-1
-2
0
2

Do Computers in Chaos Studies Make any Sense?

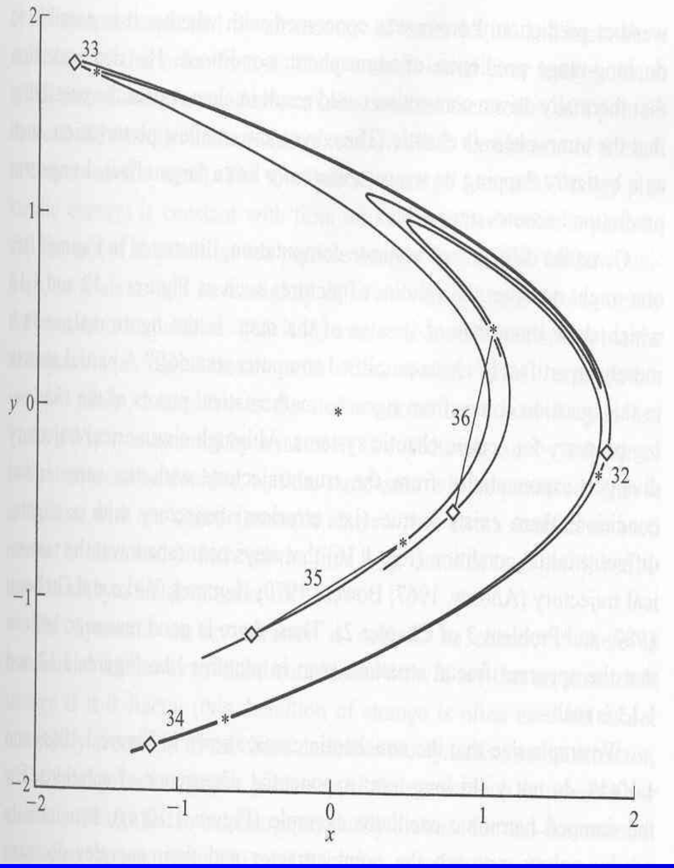
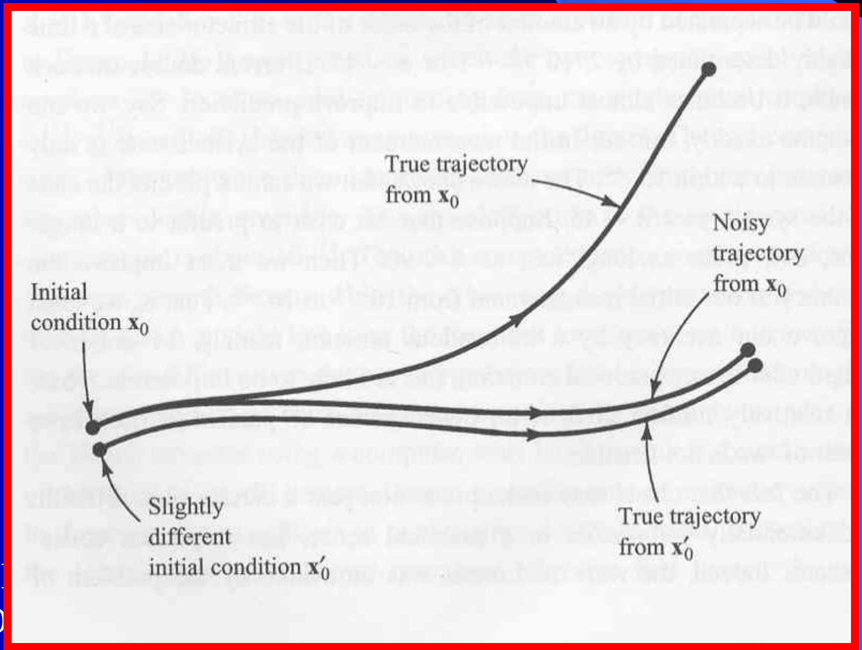


Figure 1.15 After a relatively small number of iterates, two trajectories, one computed using single precision, the other computed using double precision, both originating from the same initial condition, are far apart. (This figure courtesy of Y. Du.)

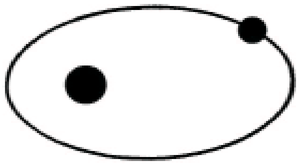
Shadowing Lemma: Although a numerically computed chaotic trajectory diverges exponentially from the true trajectory with the same initial coordinates, there exists an errorless trajectory with a slightly different initial condition that stays near ("shadows") the numerically computed one. Therefore, the fractal structure of chaotic trajectories seen in computer maps is real.



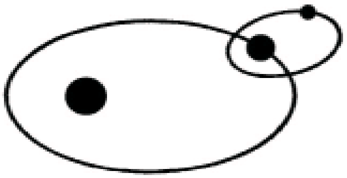
2.1 Henri Poincare

Birth of Chaos Theory

- In 1887 the King of Sweden offered a prize to the person who could answer the question "Is the solar system stable?"
- Poincare, a French mathematician, won the prize with his work on the three-body problem
- He considered, for example, just the Sun, Earth and Moon orbiting in a plane under their mutual gravitational attractions
- Like the pendulum, this system has some unstable solutions
- Introducing a Poincare section, he saw that homoclinic tangles must occur
- These would then give rise to chaos and unpredictability



Newton solved the 2-body problem

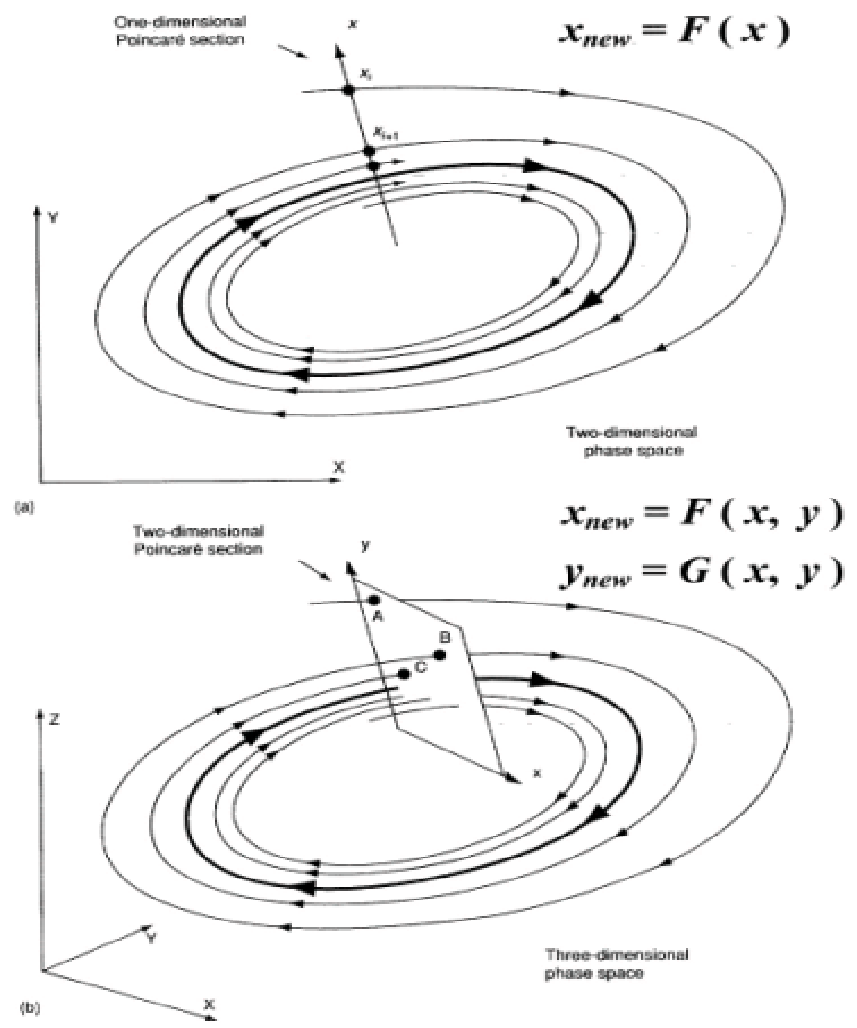


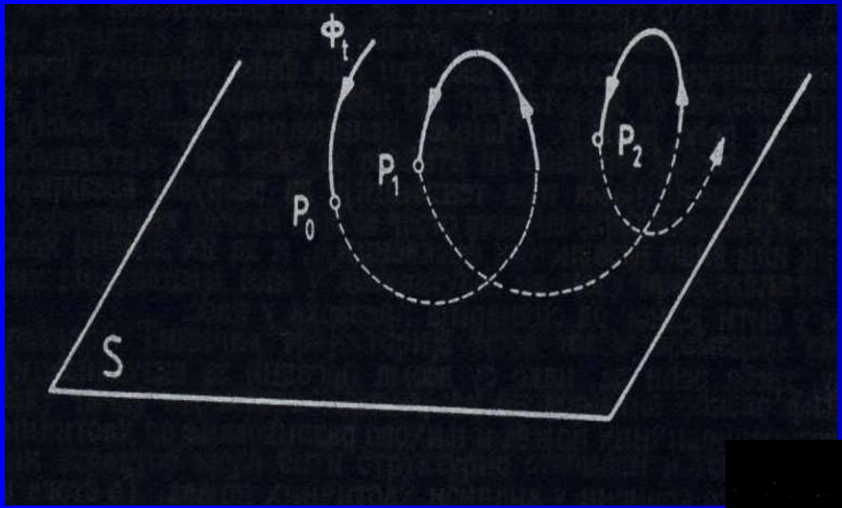
Poincaré showed that the 3-body problem is essentially 'unsolvable'

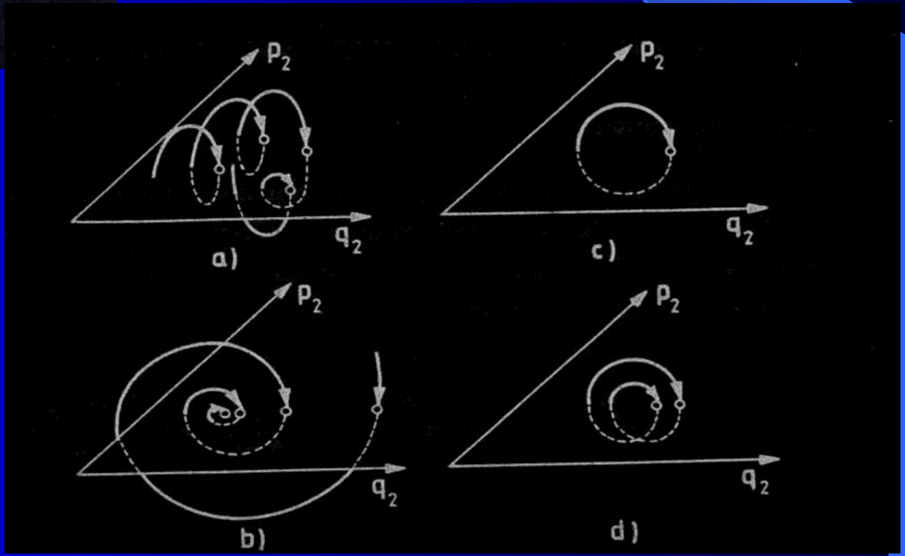
2.5 Poincare Section

- To examine chaos, Poincare used the idea of a section
- This cuts across the phase-space orbits
- The original system flows in continuous time
- On the section, we observe steps in discrete time
- The flow is replaced by what is called an iterated map
- The dimension of the phase-space is reduced by one

@




Φt
P0
P1
P2
S

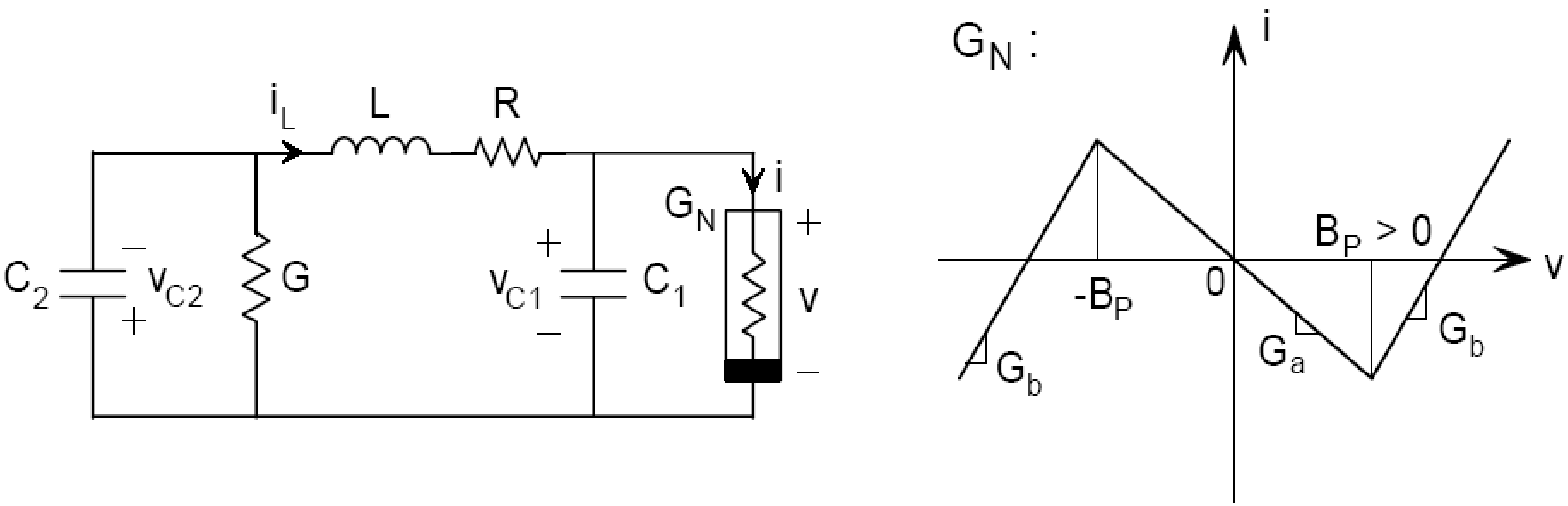

p2
q2
a)
c)
b)
d)

More examples of chaotic systems

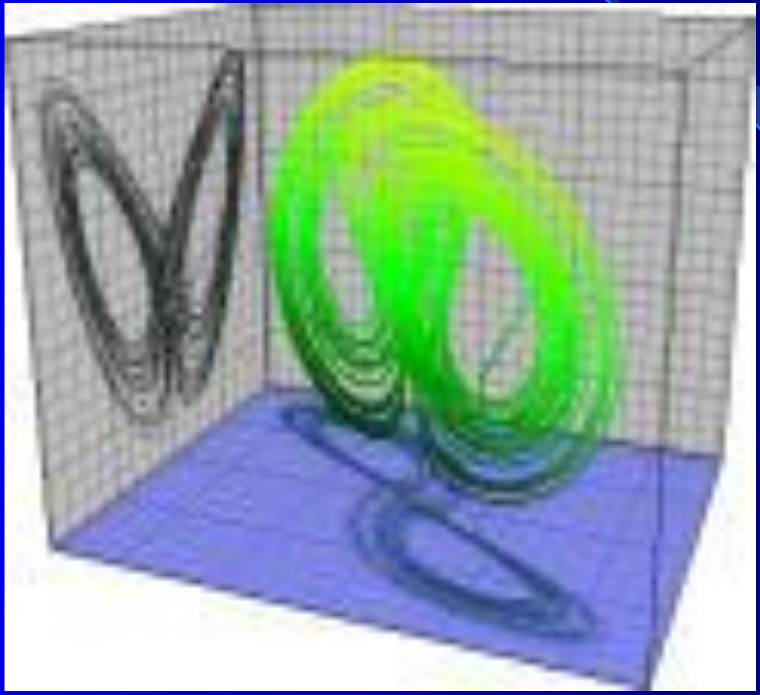
Chua's circuit

- Rich spectrum of strange attractors.
- Linear circuit plus PWL resistor makes for simple realization and replication.
- Novel flexible synthesis circuit developed for negative PWL resistor.
- Described by third-order, unforced, continuous ODE, where $g(v_{C1})$ represents PWL characteristic for $\mathbf{G}_N$:

$$\dot{v}_{C1} = \frac{1}{C_1}[i_L - g(v_{C1})], \quad \dot{v}_{C2} = \frac{1}{C_2}(i_L - Gv_{C2}), \quad \frac{di_L}{dt} = -\frac{1}{L}(v_{C1} + v_{C2} + Ri_L)$$



The Double Scroll



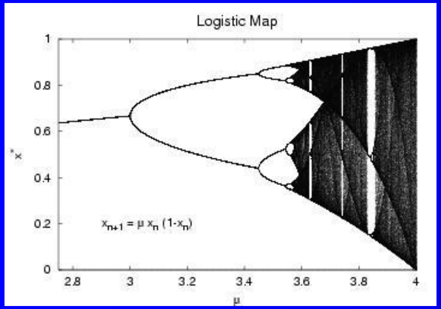
Period doubling in Chua's circuit

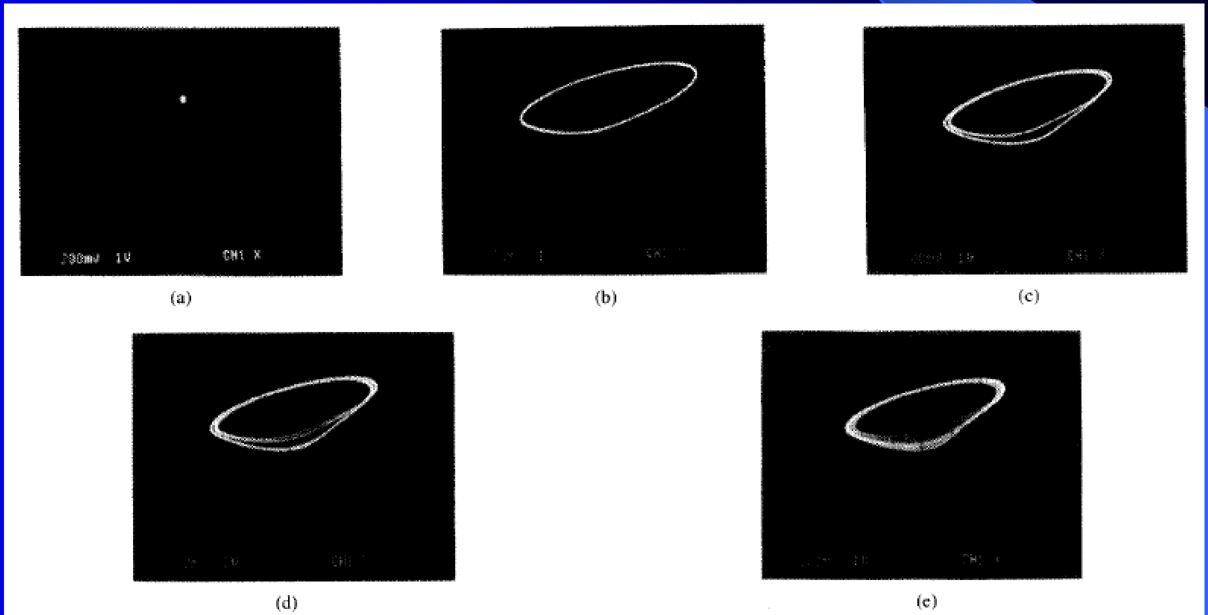
If we have those constant parameters as follows:

$$C_1 = 5.75nF, C_2 = 21.32nF, L = 12mH, r = 30.86\Omega, g_1 = -0.879mS, G_2 = -0.1124mS, V_c = 1V$$

And then change R: $R \sim (1558, 1503)\Omega$

Remember logistic map?





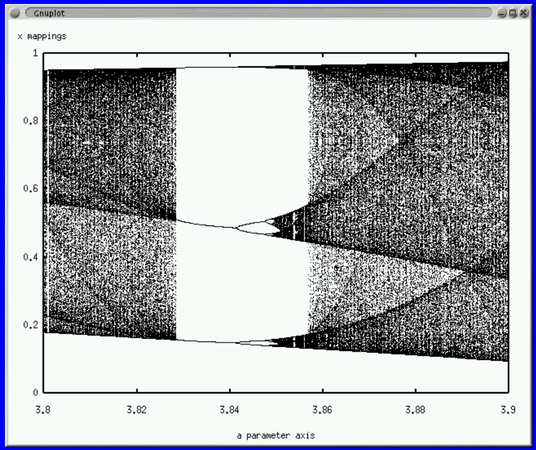
Period-3 - Intermittency

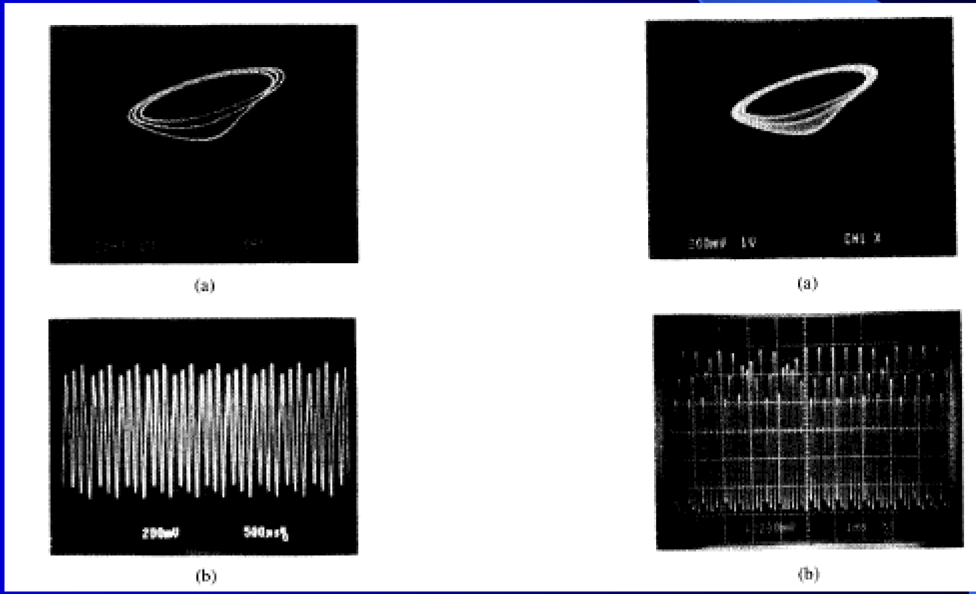
Constant parameters same as previous ones:

$$C_1 = 5.75nF, C_2 = 21.32nF, L = 12mH, r = 30.86\Omega, g_1 = -0.879mS, G_2 = -0.1124mS, V_c = 1V$$

And then change R: $R \sim (1502{,}1503)\Omega$

Still logistic map, let's zoom in part of it:



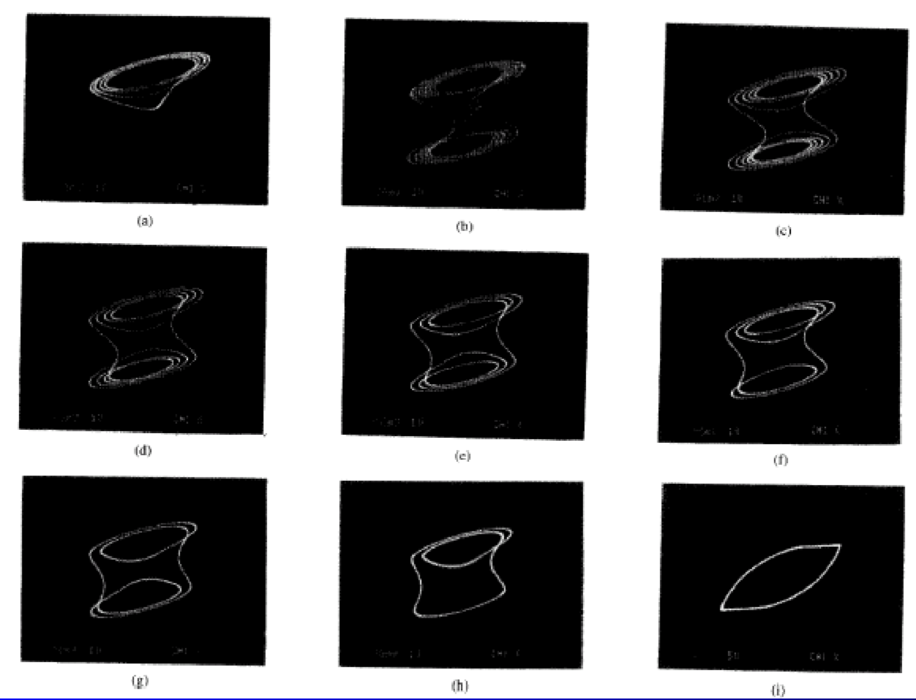


Period adding

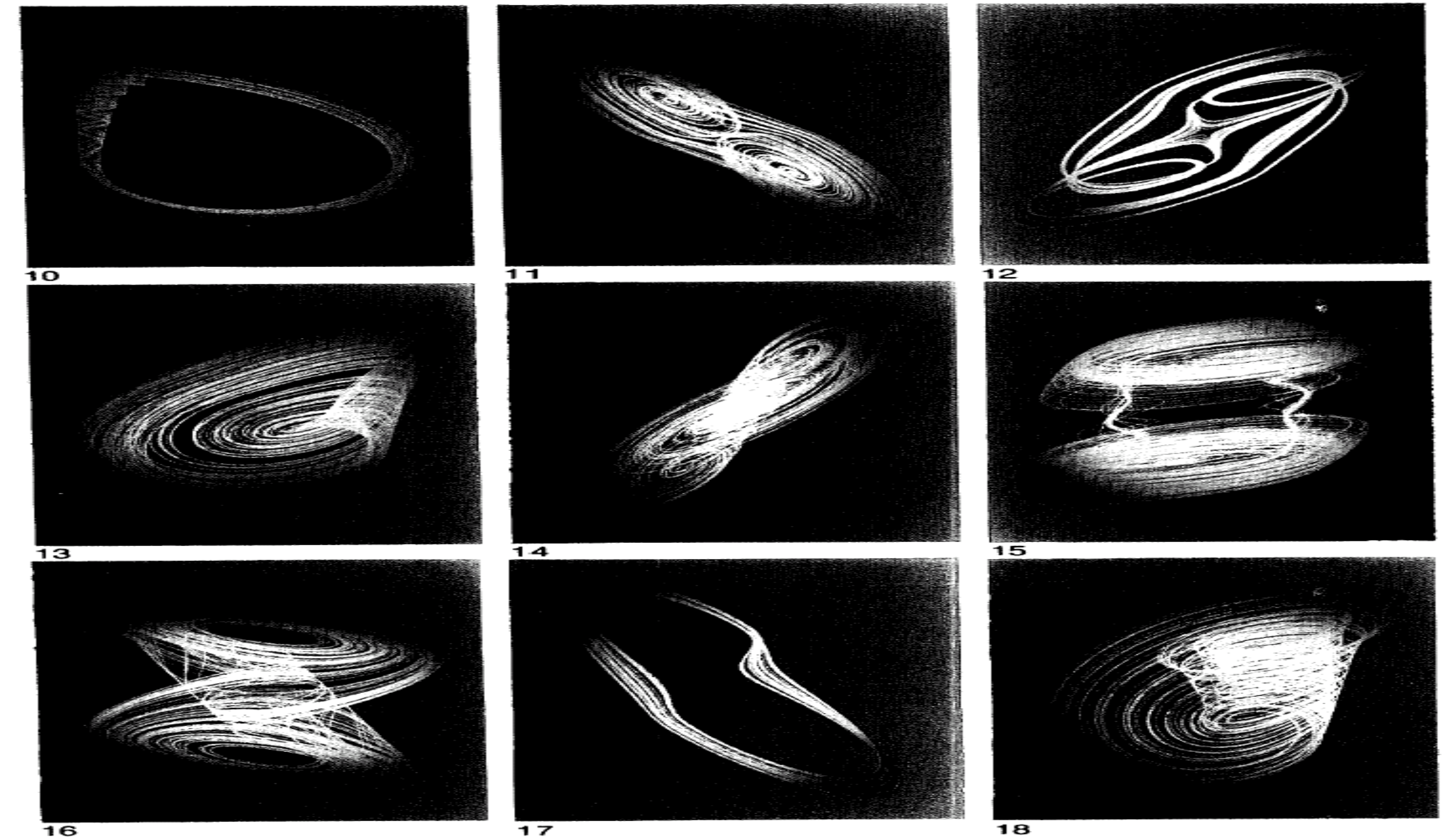
Constant parameters same as previous ones:

$$C_1 = 5.75nF, C_2 = 21.32nF, L = 12mH, r = 30.86\Omega, g_1 = -0.879mS, G_2 = -0.1124mS, V_c = 1V$$

And then change R: $R \sim (1389, 1493)\Omega$



Gallery of attractors in Chua's circuit



Chua's circuit is UNIVERSAL

CLAIM: Chua's circuit is a universal model to simulate other nonlinear systems because it is:

linearly topological conjugate to C/ ε_0 class.

What is C class?

$$\frac{d\vec{x}}{dt} = \begin{cases} A\vec{x} + b; x_1 \geq 1 \\ A_0\vec{x}, -1 \leq x_1 \leq 1 \\ A\vec{x} - b, x_1 \leq -1 \end{cases}$$

What is ε_0?

It's a set of conditions constraining C class with… a lot of math.

What is topological conjugacy?

If we have $f(x), g(y)$ and $\begin{cases} y = h(x) \\ x = h^{-1}(y) \end{cases}$

$f(x), g(y)$ are topological conjugate iff:

$$\begin{cases} g(y) = h(f(h^{-1}(y))) \\ f(x) = h^{-1}(g(h(x))) \end{cases}$$

Mechanical Example… A Pendulum

- This system demonstrates features of chaotic motion:

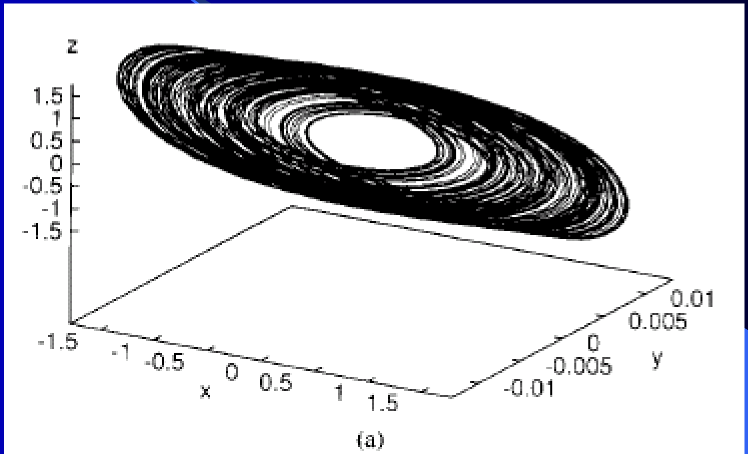
$$ml\frac{d^2\theta}{dt^2}+c\frac{d\theta}{dt}+mg\sin\theta = A\ \cos(\omega_D t+\phi)$$

- Convert equation to a ***dimensionless*** form:

$$\frac{d\omega}{dt}+q\frac{d\theta}{dt}+\sin\theta = f_0\cos(\omega_D t)$$

System by Arneodo

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = x^3 - \mu_0 x - \mu_1 y - \mu_2 z \end{cases}$$



(a)

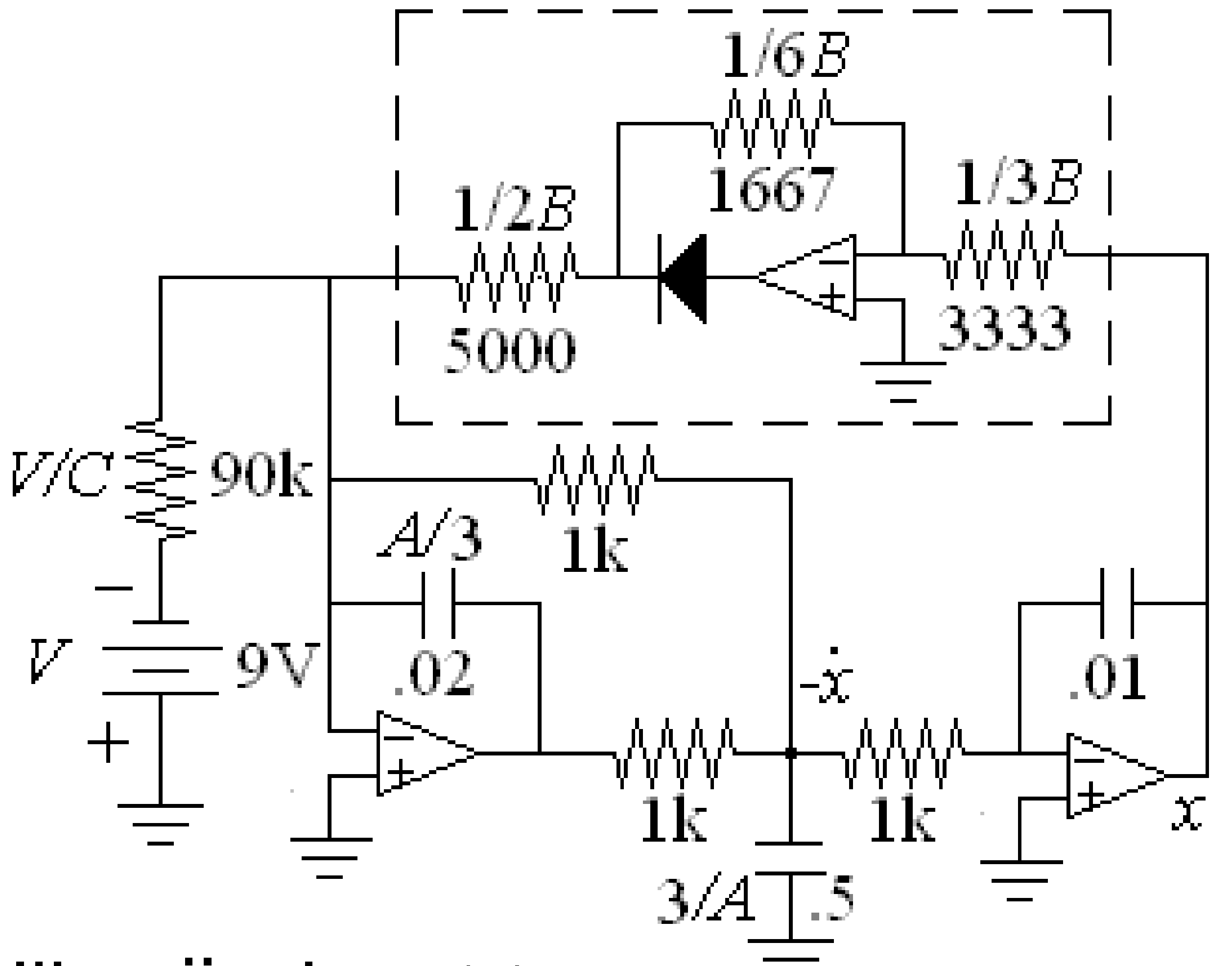
Fixed points:

$$(\sqrt{\mu_0},0,0),(0,0,0),(\sqrt{\mu_0},0,0)$$

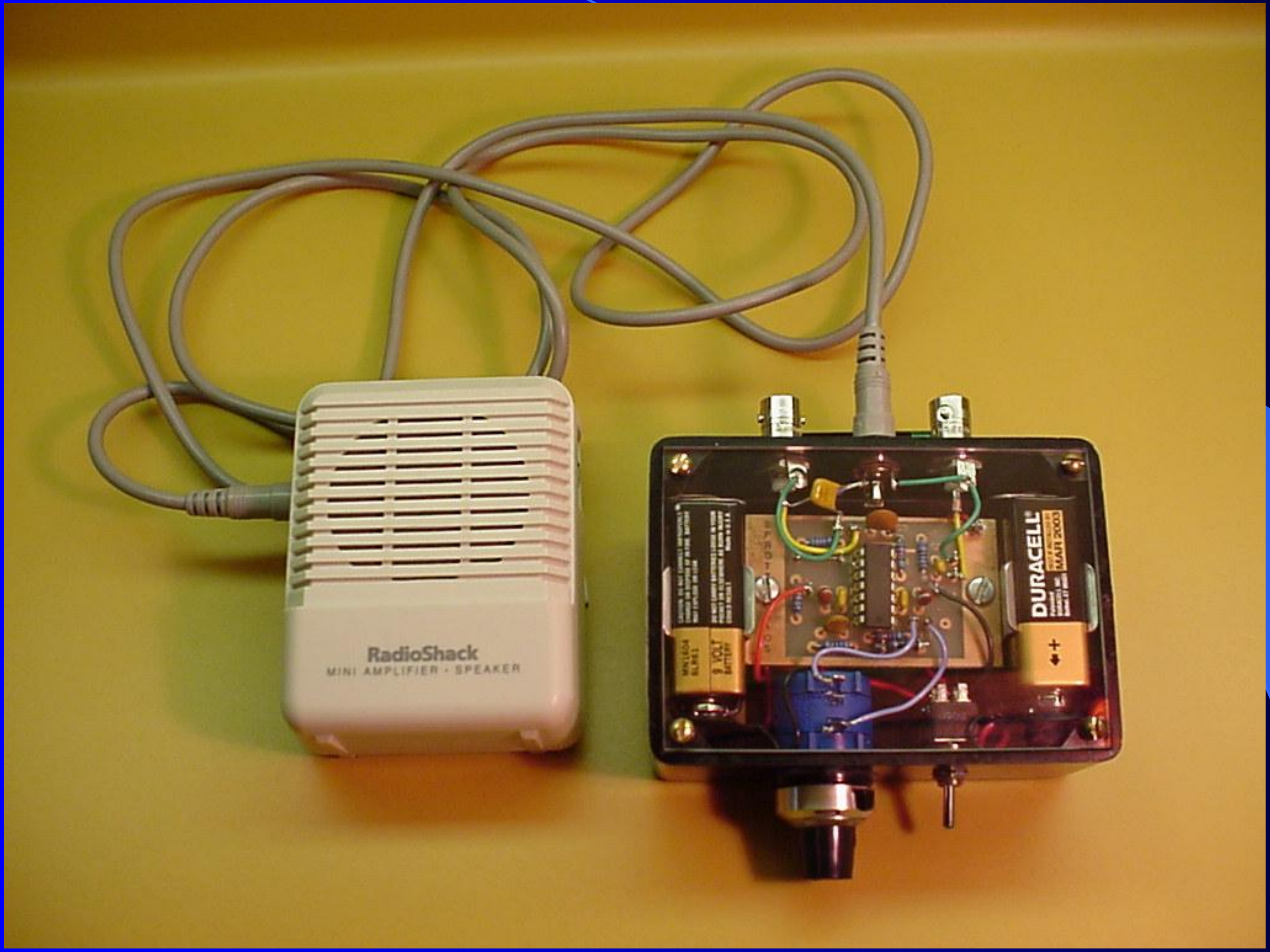
Jacobi matrix:

$$J_{1,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2\mu_0 & \mu_1 & \mu_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3\mu_0 & \mu_1 & \mu_2 \end{bmatrix}$$

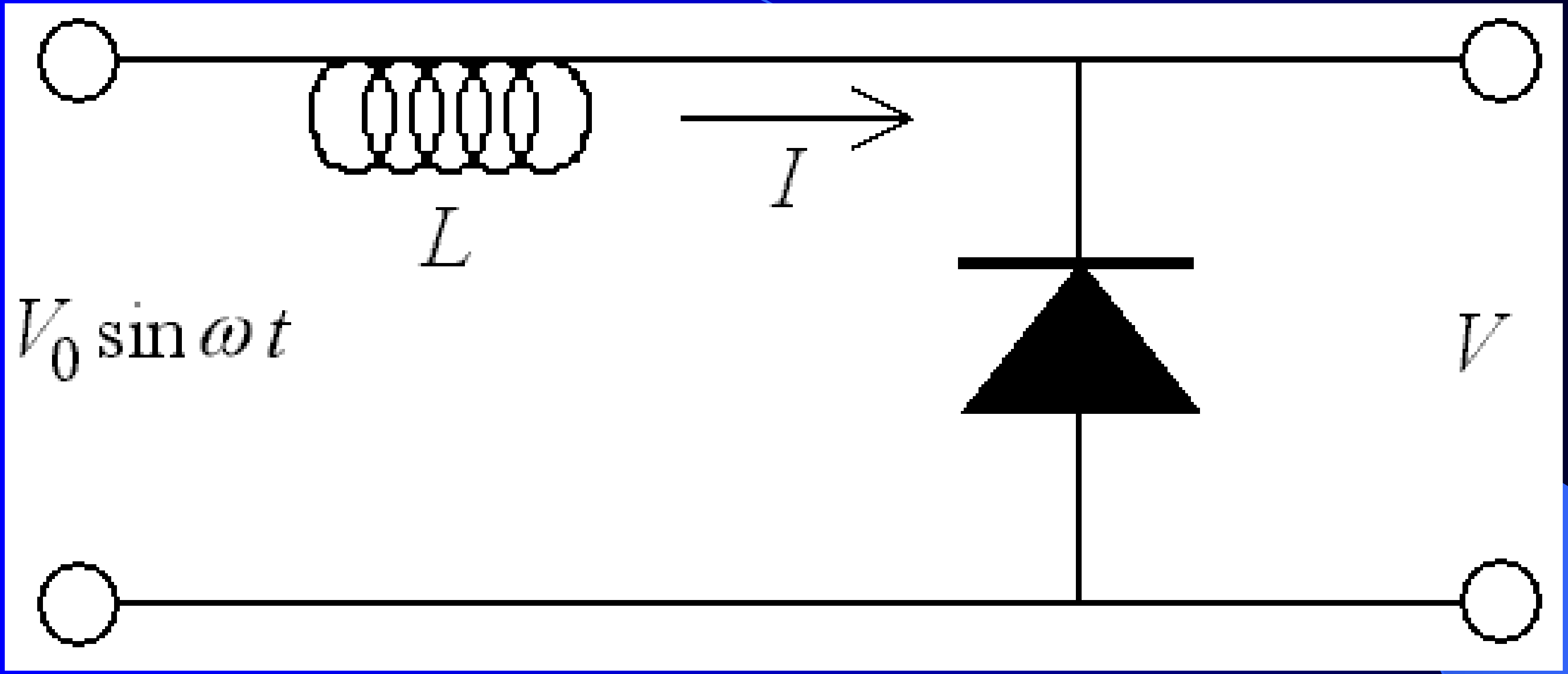
Sprott's circuit



$$\dddot{x} + A\ddot{x} + \dot{x} = B|x| - C$$


RadioShack
MINI AMPLIFIER · SPEAKER
DURACELL

Chaotic Inductor-Diode Circuit
Testa, Perez, & Jeffries (1982)



$$L\,dI/dt = V_o \sin \omega t - V$$

$$[C_o + I_o \alpha T(1-\alpha V)e^{-\alpha V}]\, dV/dt = I - I_o(1-e^{-\alpha V})$$

An Example… A Pendulum

- This system demonstrates features of chaotic motion:

$$ml\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + mg\sin\theta = A\ \cos(\omega_D t + \phi)$$

- Convert equation to a ***dimensionless*** form:

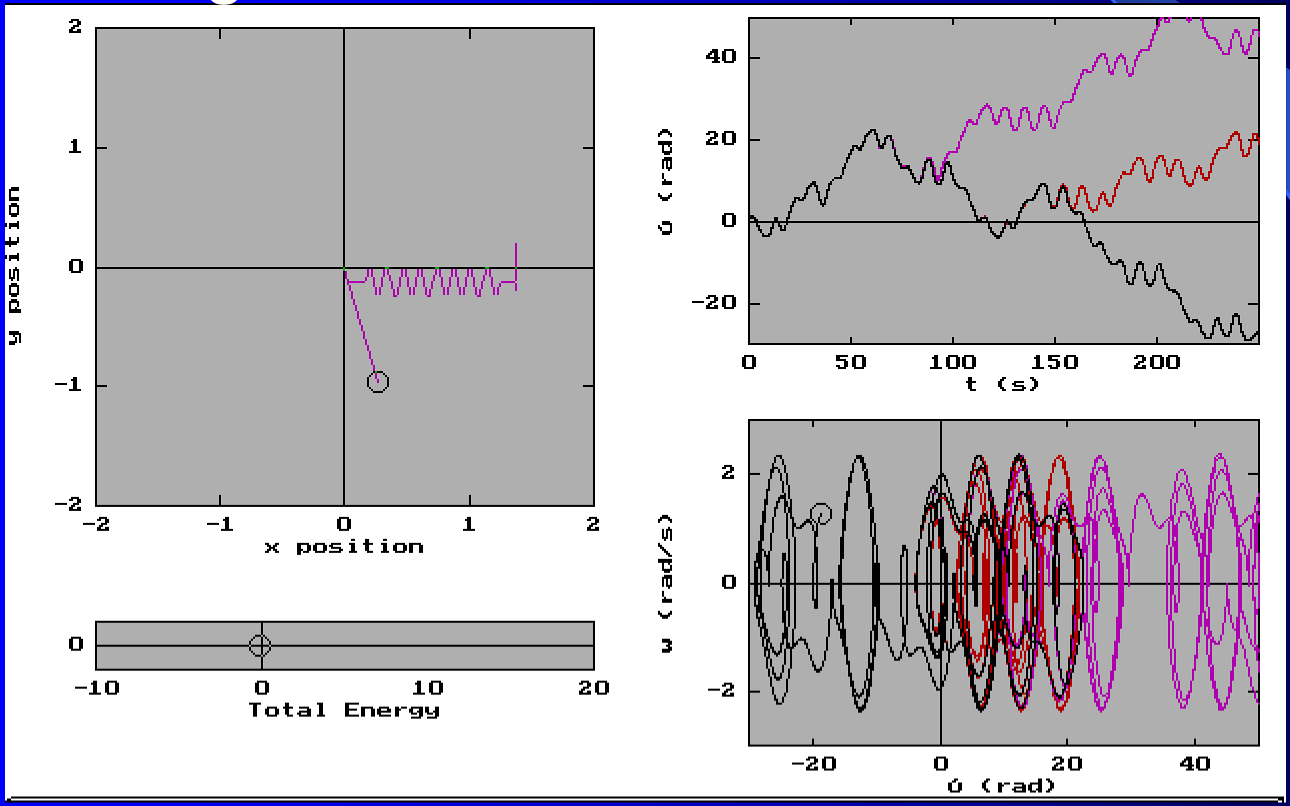
$$\frac{d\omega}{dt} + q\frac{d\theta}{dt} + \sin\theta = f_0\cos(\omega_D t)$$

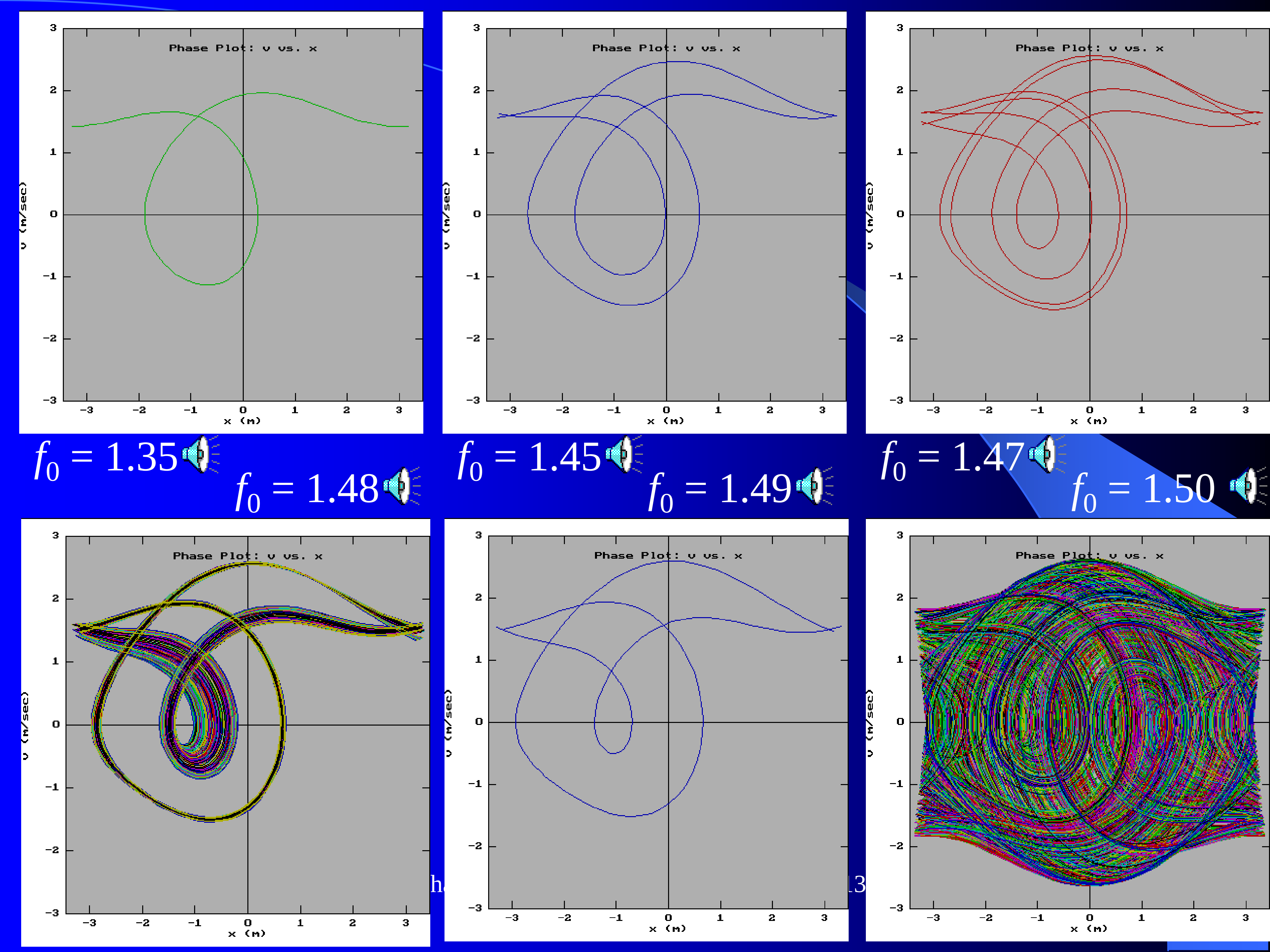
- 3 dynamic variables: ω, θ, t
- the non-linear term: $\sin\theta$
- this system is chaotic only for certain values of q, f_0, and ω_D
- In the examples:
  - $\omega_D = 2/3$, $q = 1/2$, and f_0 near 1

$$\left.\begin{aligned} x_1 &= \frac{d\theta}{dt} \\ x_2 &= \theta \\ x_3 &= \omega_D t \end{aligned}\right\} \Rightarrow \begin{cases} \dfrac{dx_1}{dt} = f_0 \cos x_3 - \sin x_2 - q x_1 \\ \dfrac{dx_2}{dt} = x_1 \\ \dfrac{dx_3}{dt} = \omega_D \end{cases}$$

Sensitive dependence on initial condition

- starting at 1, 1.001, and 1.000001 rad:





Phase Plot: v vs. x
x (m)
v (m/sec)
f_0 = 1.35
f_0 = 1.45
f_0 = 1.47
f_0 = 1.48
f_0 = 1.49
f_0 = 1.50

Bifurcations -1

Bifurcation is Latin for "forking into two", and probably derives from the fact that in some cases, such as a pitchfork bifurcation, the orbit seems to "fork" from period one to period two

A bifurcation is a qualitative change in the dynamics of a system that occurs when a control parameter is varied

A bifurcation point is the point in the parameter space where the bifurcation occurs

Typically, a bifurcation occurs when an attractor becomes unstable

Bifurcations are classified according to how stability is lost

Bifurcation types: *saddle node*, *pitchfork*, *transcritical*, *hopf*

In a map:

- real eigenvalue leaves unit circle at +1: **saddle node (or fold)**
- real eigenvalue leaves unit circle at -1: **pitchfork (or flip)**
- conjugate complex eigenvalues leave unit circle simultaneously: **hopf**

Bifurcations -2

The particular way in which stability is lost depends on the symmetry properties of the Jacobian matrix

Investigate when the eigenvalues of the Jacobian cross the imaginary axis

Consider the algebraic properties of the Jacobian matrix J:

- An antisymmetric J has pure imaginary eigenvalues
- A symmetric J has pure real eigenvalues
- Complex eigenvalues lead to a rotational flow nearby the fixed point
- Typically perturbations to the fixed point will spiral into the fixed point before the bifurcation and will spiral out to a limit cycle afterwards
- Real eigenvalues imply an absence of rotation
- Typically, perturbations about such a fixed point flow directly towards the fixed po before the bifurcation and flow directly outwards to another equilibrium afterward

Saddle-node bifurcation

A real eigenvalue passes through the origin

A saddle coalesces with a node and they are annihilated

Consider

$$\dot{x} = \mu - x^2$$

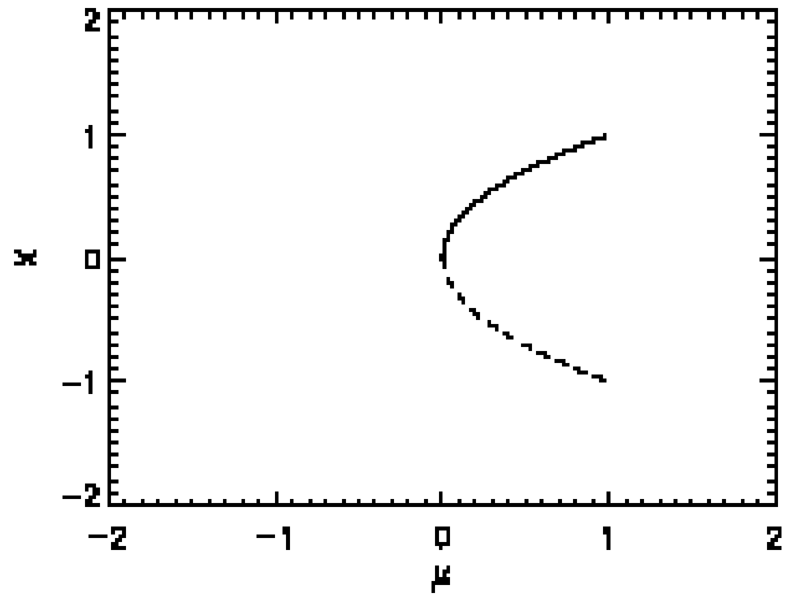
$\mu < 0 \Rightarrow$, no fixed points

$\mu > 0 \Rightarrow$, fixed points at $x_{\pm} = \pm\sqrt{\mu}$

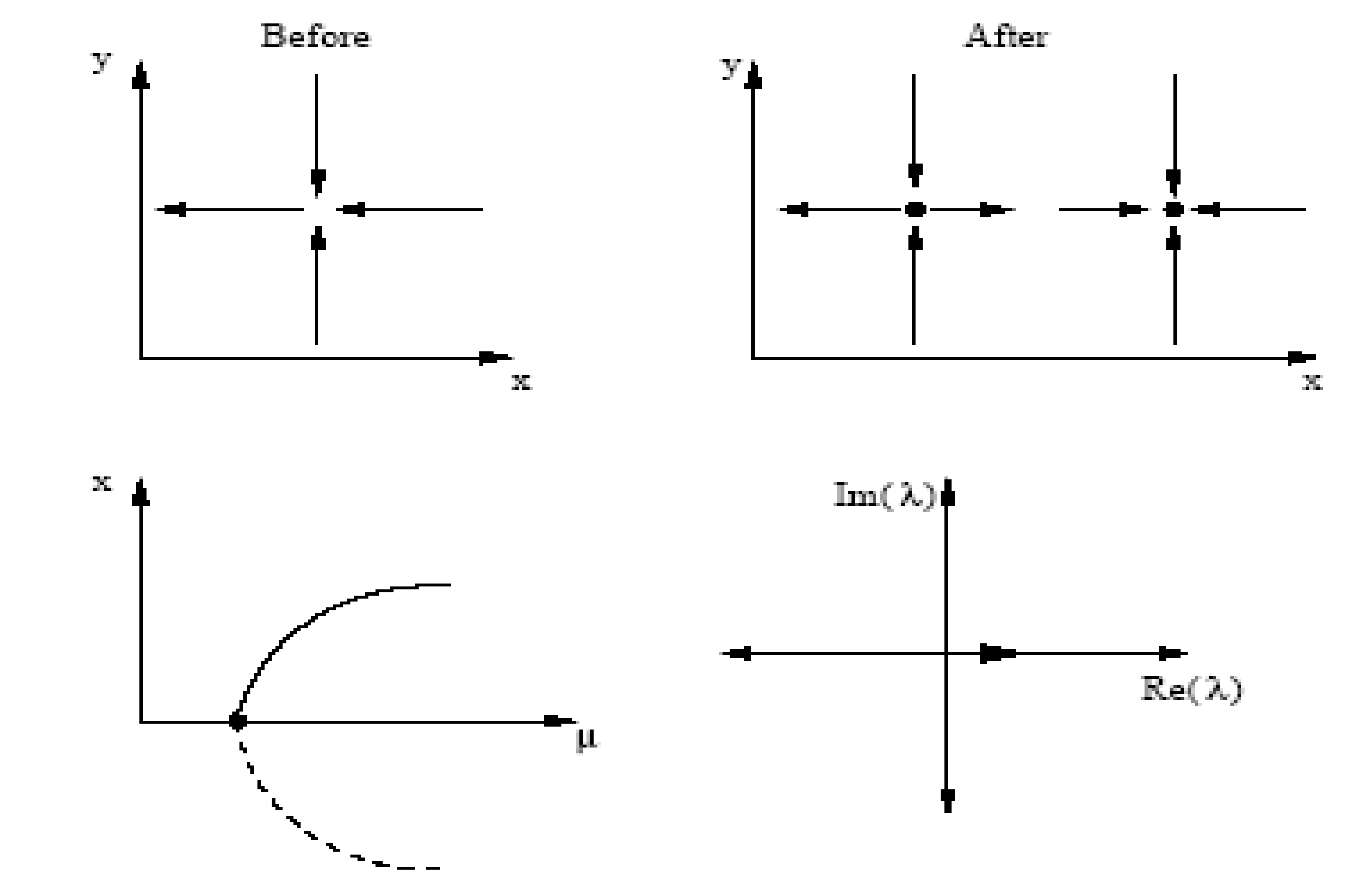
Eigenvalue $\lambda = -2x$

So $x_{-} = \sqrt{\mu}$ is stable and $x_{+} = -\sqrt{\mu}$ is unstable

$(x, \mu) = (0, 0)$ is a saddle node bifurcation point



Saddle-node bifurcation



Pitchfork bifurcation

Consider

$$\dot{x} = \mu x + \alpha x^3$$

Fixed points at

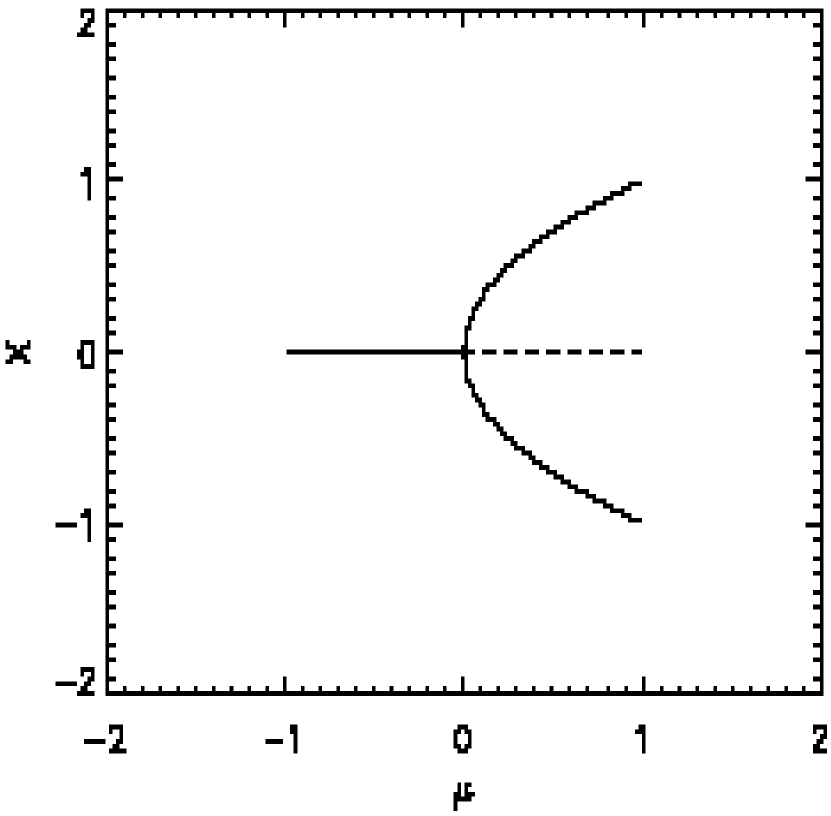
$x = 0, x = \pm\sqrt{-\mu/\alpha}$

Jacobian is $\mu + 3\alpha x^2$

Eigenvalues $\lambda = \mu$ at $x = 0$ and $\lambda = -2\mu$ at $x = \pm\sqrt{-\mu/\alpha}$

$\alpha < 0$, supercritical pitchfork bifurcation

$\alpha > 0$, subcritical pitchfork bifurcation



Transcritical bifurcation

Consider

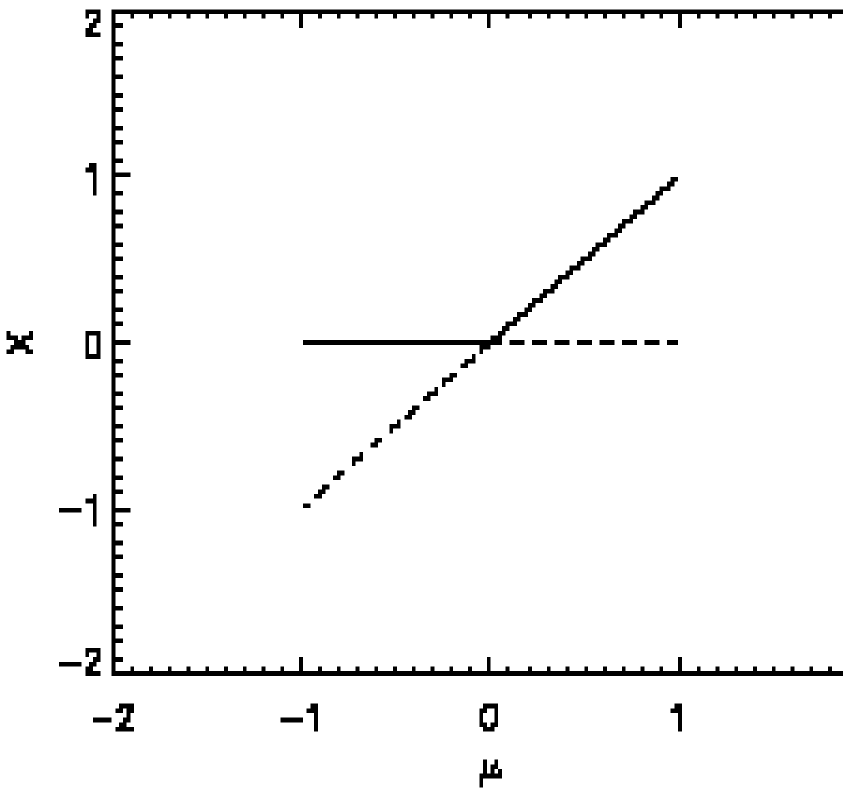
$$\dot{x} = \mu x - x^2$$

Fixed points at $x = 0, x = \mu$

Eigenvalue is $\lambda = \mu$ at $x = 0$
$\lambda = -\mu$ at $x = \mu$

$(x, \mu) = (0, 0)$ is a transcritical bifurcation

Branches swap stability



Hopf bifurcation

A pair of complex conjugate eigenvalues crosses the imaginary axis away from the real line

As a result, the fixed point becomes a limit cycle

Consider system in polar coordinates

$$\dot{r} = -(ar + r^3) \qquad \dot{\theta} = \omega$$

$\dot{r} = 0$ at $r = 0$ and $r^2 = -a$

$a > 0 \Rightarrow r = 0$ is stable

$a < 0 \Rightarrow r = \sqrt{-a}$ is a stable *limit cycle*

In rectangular coordinates

$$\dot{x} = -ax - (x^2 + y^2)x - \omega y$$

$$\dot{y} = -ay - (x^2 + y^2)y - \omega x$$

Hopf bifurcation - 2

Linearization about origin gives

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -a & -\omega \\ \omega & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

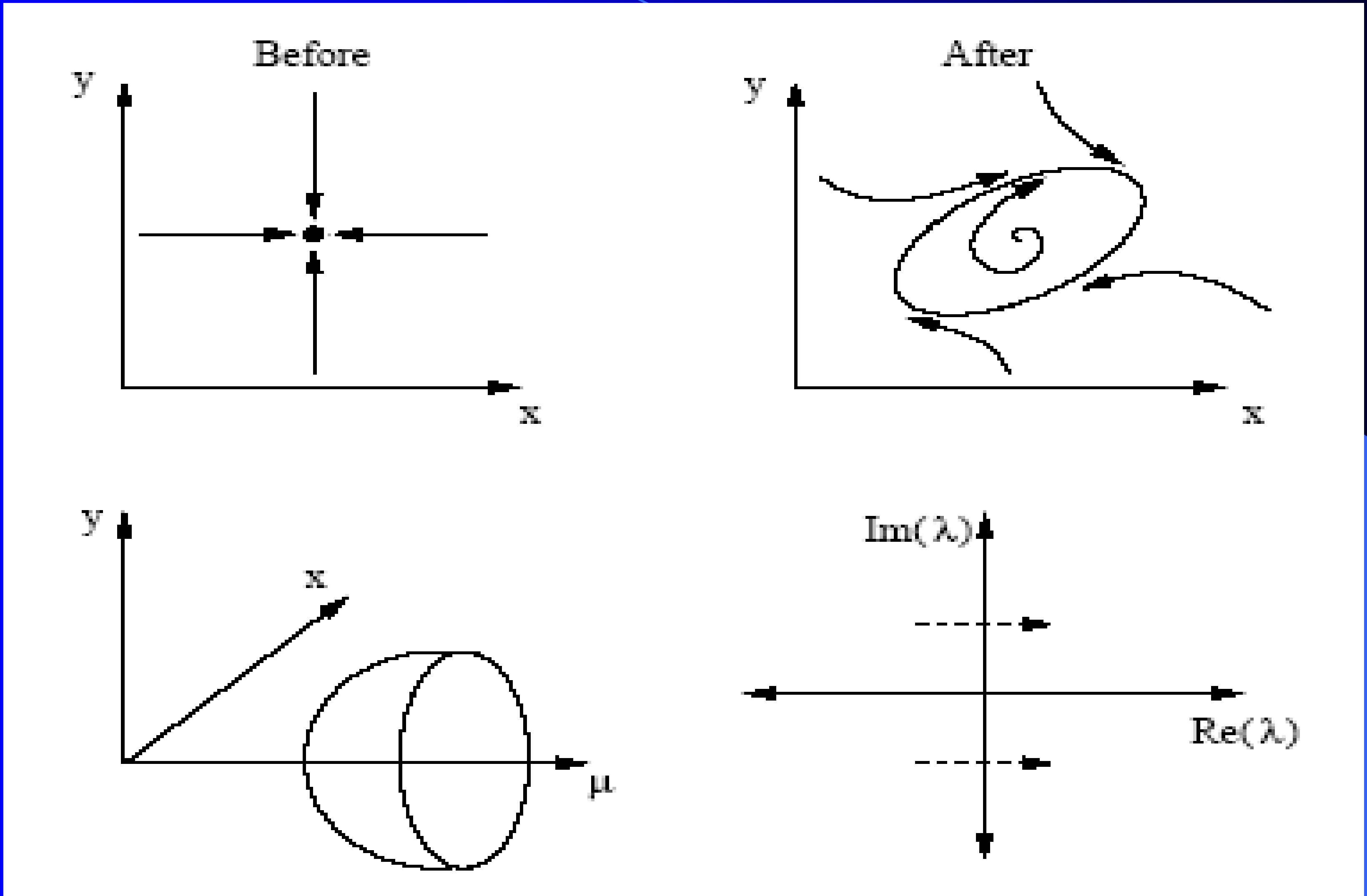
Eigenvalues $\lambda = -a \pm i\omega$

Hopf bifurcation at $a = 0$: conjugate pair of eigenvalues cross imaginary axis

Properties of hopf bifurcation

- Occurs at fixed point

Hopf bifurcation -3



Summary of normal forms

Saddle node bifurcation

$$\dot{x} = \mu + \alpha x^2$$

Transcritical bifurcation

$$\dot{x} = \mu x - \alpha x^2$$

Pitchfork bifurcation

$$\dot{x} = \mu x + \alpha x^3$$

Hopf bifurcation

$$\dot{x} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2)$$

$$\dot{y} = \omega x + \mu y + (\beta x + \alpha y)(x^2 + y^2)$$

Catastrophic bifurcations

A bifurcation is "subtle" if a small change in the control parameter gives rise to a small change in the behaviour

A bifurcation is "catastrophic" if a small change in the control parameter causes a large change in the behaviour

An example of a catastrophic bifurcation is the saddle-node bifurcation where a node (a sink or source) and a saddle (a hyperbolic fixed point) coalesce, leaving no fixed point

When a new attractor appears which is unrelated to the existing attractors, the result can be a catastrophic change in the evolution of the trajectories

For example, if a system has two distinct stable fixed points and one loses its stability, then the system will jump to the remaining fixed point

Properties of maps

If x_0 is a fixed point of f, then it is also a fixed point of f^n:

$$f^n(x_0) = ff\dots f(x_0) = f(x_0) = x_0$$

If x_0, a fixed point of f, becomes unstable, then it is unstable in f^n since

$$\left|\frac{d}{dx}f^n(x_0)\right| = \left|\frac{d}{dx}f(x_0)\right|^n$$

because of the chain rule:

$$\left.\frac{d}{dx}f^n(x)\right|_{x_0} = f'(x_{n-1})f'(x_{n-2})\dots f'(x_1)f'(x_0)$$

where x_k is the kth iterate of x_0 and using the result

$$\frac{d}{dx}f^n(x) = \frac{d}{dx}f(f^{n-1}(x)) = f'(f^{n-1}(x))\frac{d}{dx}f^{n-1}(x)$$

Also known as subharmonic bifurcation since doubling the period halves the frequenc

As control parameter varied you pass through sequence of period-doubling bifurcatioı chaos

First found in noninvertible 1D maps

Consider logistic map

$$x_{n+1} = f(x_n) = ax_n(1 - x_n)$$

For $a < 1$ one fixed point at $x = 0$

When $a > 1$ two fixed points and $|f'(0)| > 1 \Rightarrow$ instability

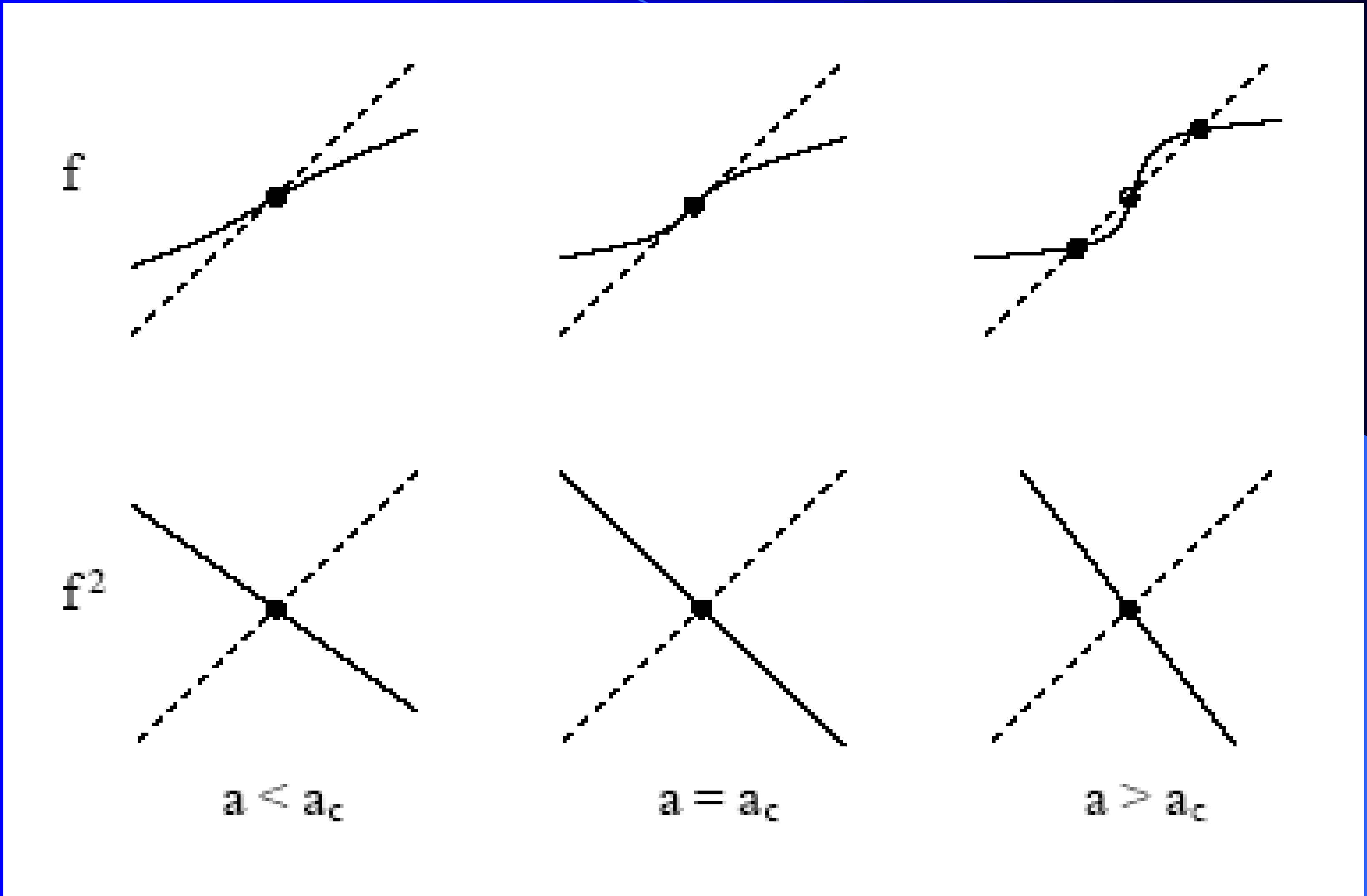
Other fixed point at $x_0 = 1 - 1/a$ and
$|f'(x_0)| = |2 - a|$

When $a > 3$, x_0 becomes unstable and a period-2 orbit forms (pitchfork bifurcation)

Period-2 orbit is fixed point of $f(f(x)) = f^2(x)$

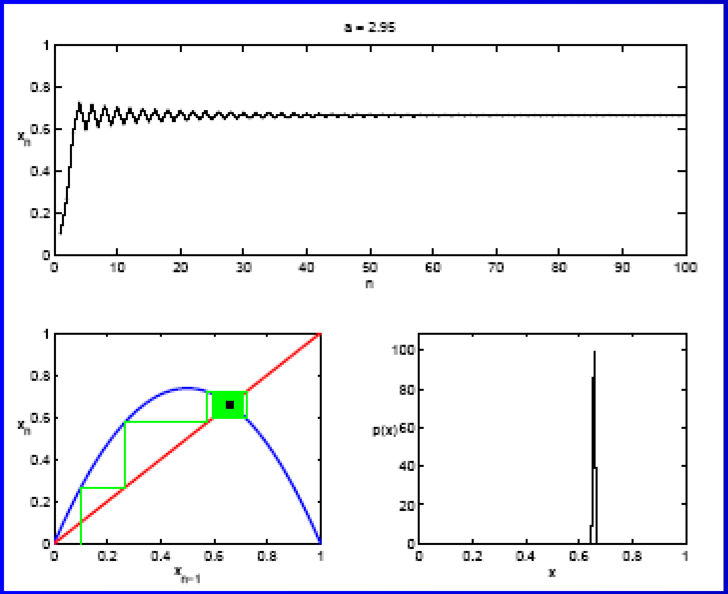
For $a \approx 3.45$ period-4 orbit forms

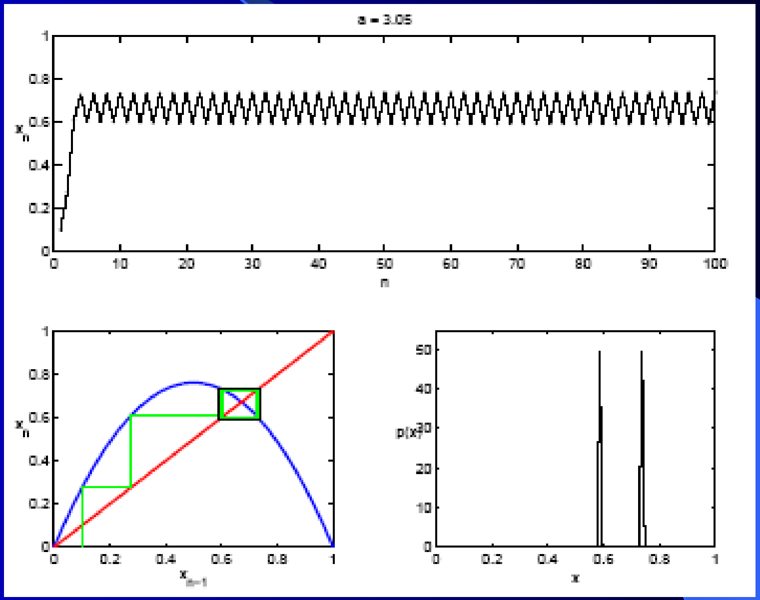
Period doubling

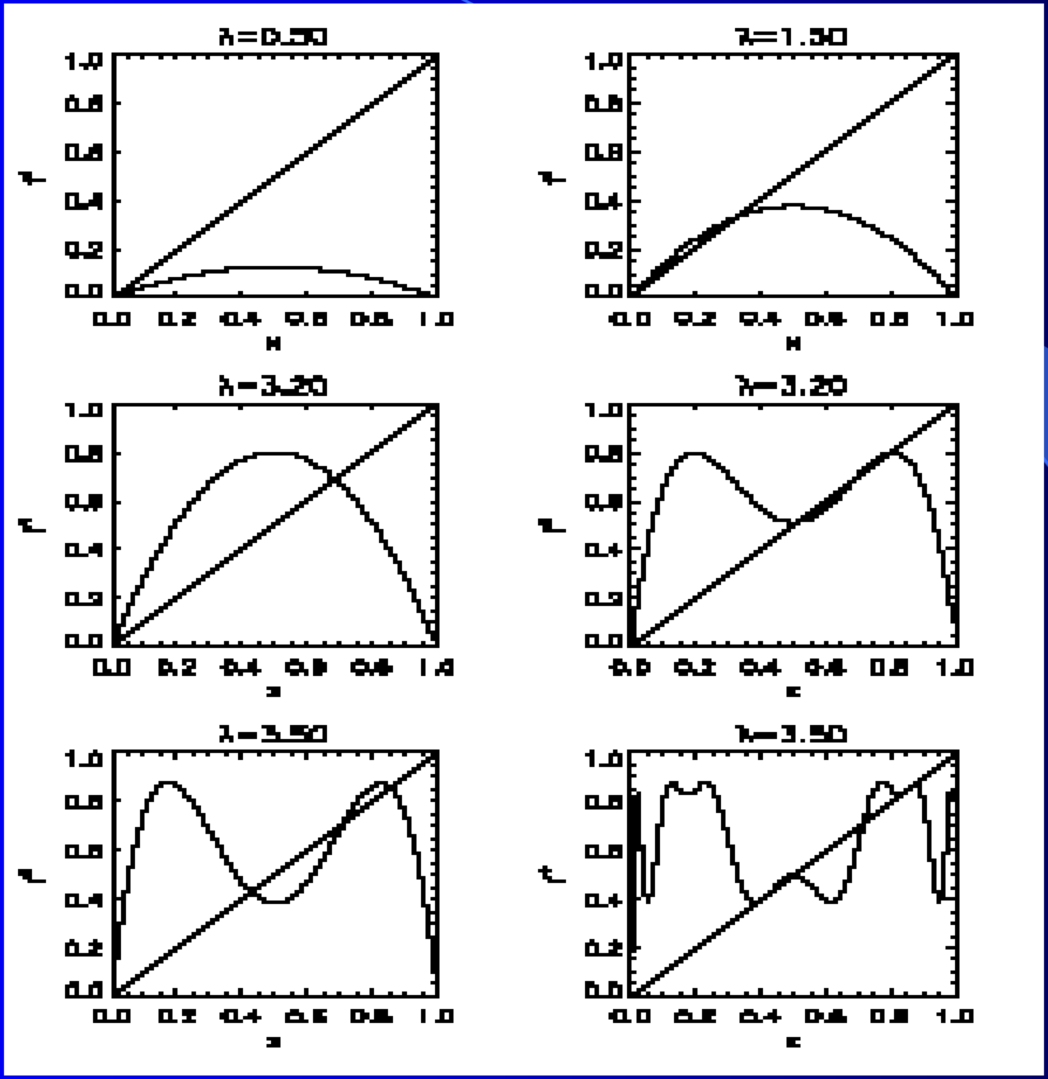


Period doubling – logistic map

- a=2.95
- a=3.95






λ=0.50
λ=1.50
λ=3.20
λ=3.20
λ=3.50
λ=3.50

Circle map

The circle map is a one-dimensional map which maps the circle onto itself,

$$\theta_{n+1} = \left[\theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n)\right] \bmod 1$$

where Ω is an externally applied frequency and K controls the strength of the nonlinearity

Consider the linear case when $K = 0$, giving

$$\theta_{n+1} = \theta_n + \Omega$$

If $\Omega = 2/5 = 0.4$, then the system will return to its original value after five iterations, having made two revolutions

If Ω is rational, then it is known as the map winding number $\Omega = W = \frac{p}{q}$

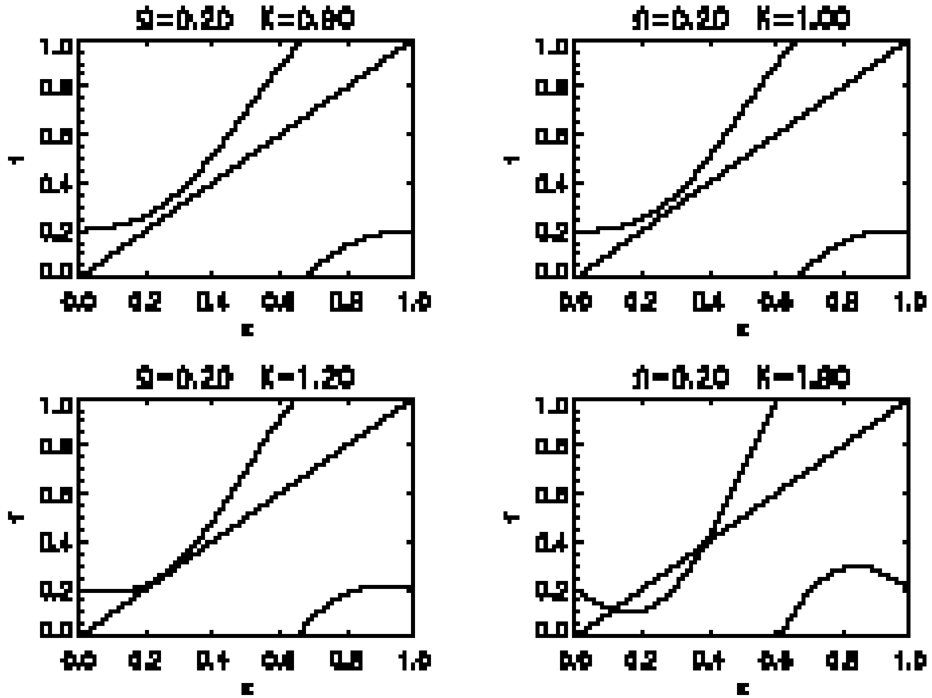
If the winding number is a rational number $W = p/q$ (p and q integers), then the map is cyclic or *periodic*

If the winding number is irrational, then θ does not return exactly to its initial value and the motion is *quasiperiodic*

Quasiperiodic routes in the circle map

Behaviour in (K, Ω) parameter space:

- $K < 1$: Arnold tongues of rational W separated by regions of irrational W
- $K > 1$: map noninvertible, chaotic and nonchaotic regions interwoven
- $K = 1$: regions of irrational W form a cantor set



Mode locking

If the nonlinear term is added ($K \neq 0$), the map can display periodic motion when Ω is irrational

The motion will be periodic is some finite region surrounding each rational Ω

Periodic motion in response to irrational forcing is known as *mode locking*

For $K \neq 0$, the winding number is defined as

$$W = \lim_{n \to \infty} \frac{\theta_n - \theta_0}{n}$$

Consider the (K, Ω) parameter space with regions of periodic mode-locked parameter space plotted around rational Ω (corresponding to map winding numbers W)

These regions widen upwards from 0 at $K = 0$ to a finite width at $K = 1$

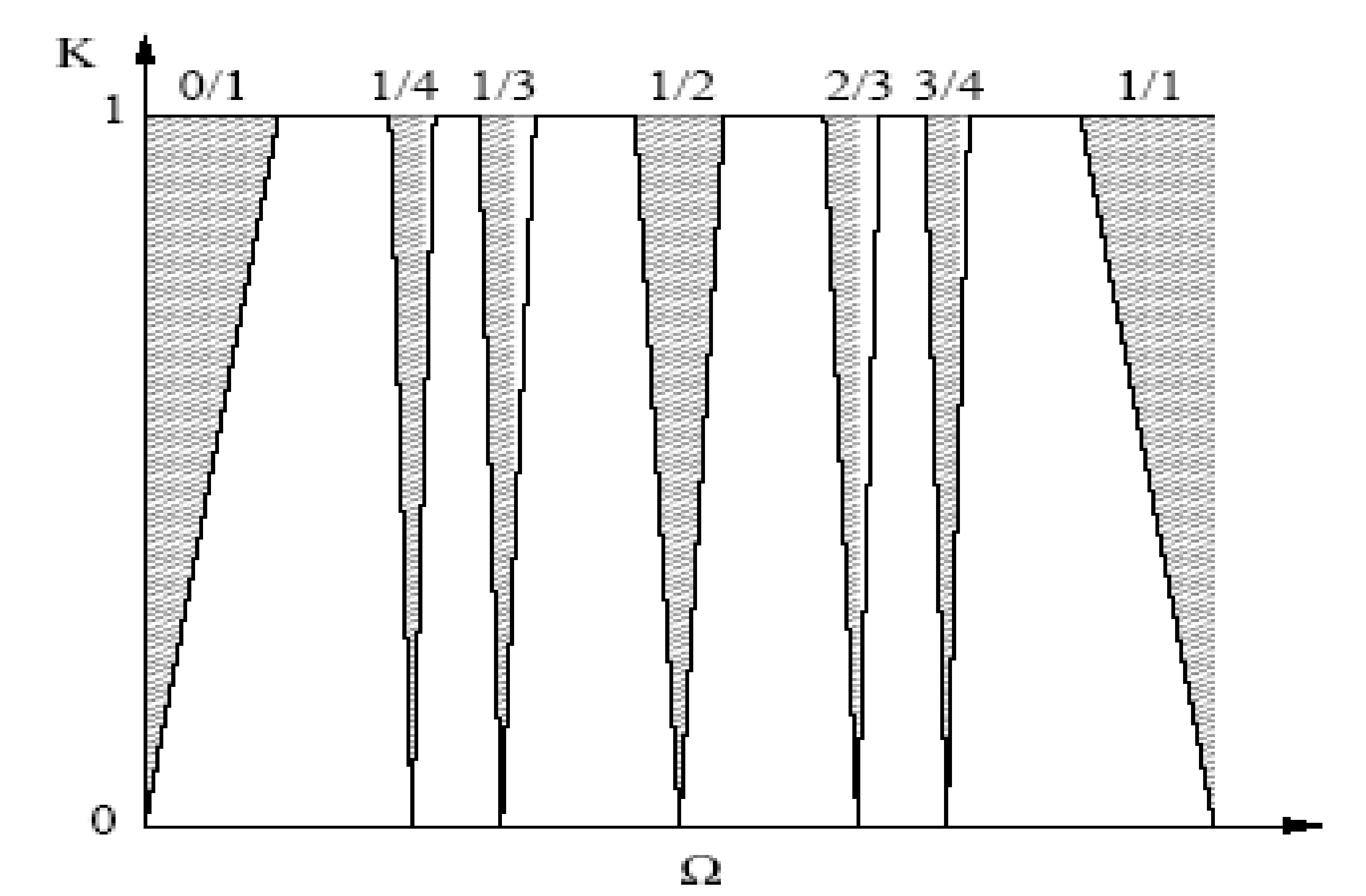
The regions surrounding each rational number is known as an *Arnold tongue*

At $K = 0$ the tongues form an isolated set of measure zero

At $K = 1$, the tongues form a Cantor set of dimension $D \approx 0.087$

For $K > 1$, the tongues overlap and the map becomes noninvertible

Arnold's tongues



Devil's staircase

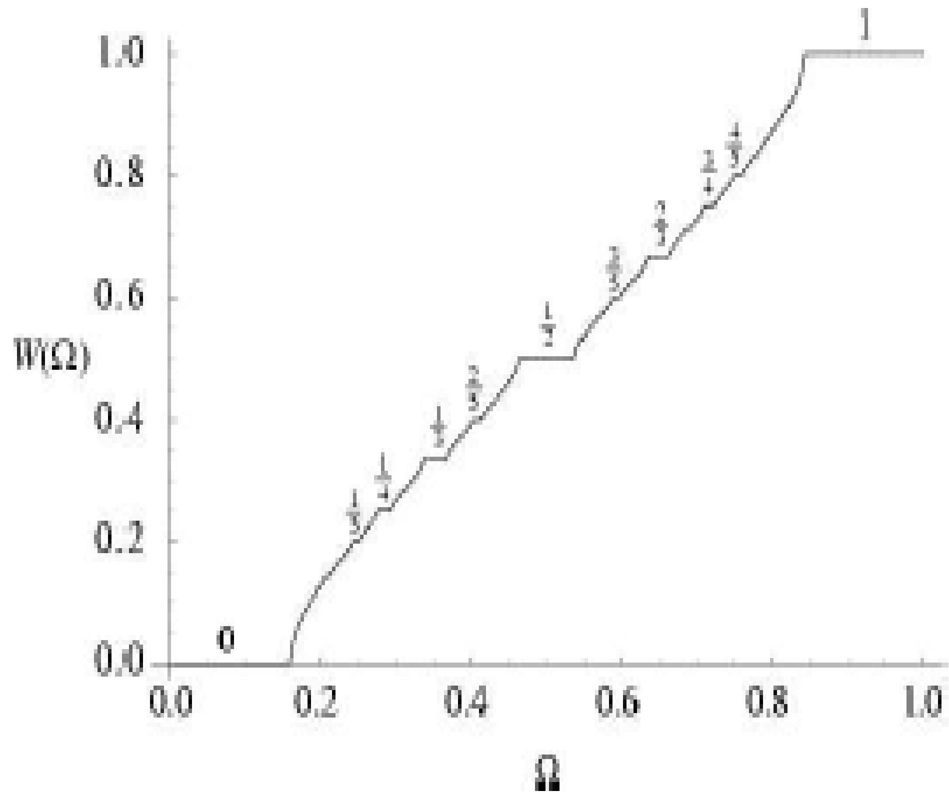
At $K = 1$ only rational winding numbers are available

The mode-locking implies that the winding number is independent of the initial condition

A graph of W versus Ω displays a monotonic increasing *staircase* for which the simplest rational numbers have the largest steps

The *devil's staircase* continuously maps the interval [0,1] onto itself but is constant almost everywhere

It has a dimension $D \approx 0.87$



Universality

Quadratic map

$$x_{n+1} = a x_n (1 - x_n)$$

Let a_n be value of a where number of fixed points changes from 2^{n-1} to 2^n

a_n scale like $a_\infty - a_n \propto \delta^{-n}$ when $n \gg 1$, or

$$\delta = \lim_{n \to \infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n}$$

where $a_\infty = 3.5699456\ldots$

Distances, d_n, of point in 2^n cycle closest to $x = 1/2$

$$\frac{d_n}{d_{n+1}} = \alpha \qquad \text{when } n \gg 1$$

δ and α are *Feigenbaum* constants

$$\delta = 4.6692016091\ldots \qquad \alpha = 2.5029078750\ldots$$

What have we learned so far

- In deterministic systems there exist irregular motions – they do not converge to any equilibrium point, periodic or quasi-periodic orbit – thye have many „random-like” properties (continuous spectra, ergodicity, mixing, „uncertainty due to sensitive dependence on initial conditions or parameters);
- Such motions can be observed in nonlinear, continuous-time, autonomous systems of at least third order, nonlinear contiuous driven systems of at least second order or discrete-time nonlinear systems (maps) of any order (we have seen many examples);
- Fundamental feature of chaos – sensitive dependence on initial conditions;
- Chaos appears in a system via a sequence of bifurcations („route”) – most common are period-doubling (Feigenbaum) sequence, period-adding, torus break-down, intermittencies;
- How can we recognize chaotic motion in practice?

Quantifying dynamics

Lyapunov Exponents

Lyapunov A. M. (1857-1918)

Alexander Lyapunov was born 6 June 1857 in Yaroslavl, Russia in the family of the famous astronomer M.V. Lyapunov, who played a great role in the education of Alexander and Sergey.

Aleksandr Lyapunov was a school friend of Markov and later a student of Chebyshev at Physics & Mathematics department of Petersburg University which he entered in 1876. He attended the chemistry lectures of D.Mendeleev.

In 1885 he brilliantly defends his MSc diploma "On the equilibrium shape of rotating liquids", which attracted the attention of physicists, mathematicians and astronomers of the world.

The same year he starts to work in Kharkov University at the Department of Mechanics. He gives lectures on Theoretical Mechanics, ODE, Probability.

In 1892 defends PhD. In 1902 was elected to Science Academy.

After wife's death 31.10.1918 committed suicide and died 3.11.1918.

What is “chaos”?

Chaos – is an *aperiodic* long-time behavior arising in a *deterministic* dynamical system that exhibits

a *sensitive dependence on initial conditions*.

The nearby trajectories separate exponentially fast

Trajectories which do not settle down to fixed points, periodic orbits or quasiperiodic orbits as $t \to \infty$

Lyapunov Exponent > 0

The system has no random or noisy inputs or parameters – the irregular behavior arises from system’s nonliniarity

Non-wandering set

- a set of points in the phase space having the following property: All orbits starting from any point of this set come arbitrarily close and arbitrarily often to any point of the set.

- **Fixed points:** stationary solutions;
- **Limit cycles:** periodic solutions;
- **Quasiperiodic orbits:** periodic solutions with at least two incommensurable frequencies;
- **Chaotic orbits:** bounded non-periodic solutions.

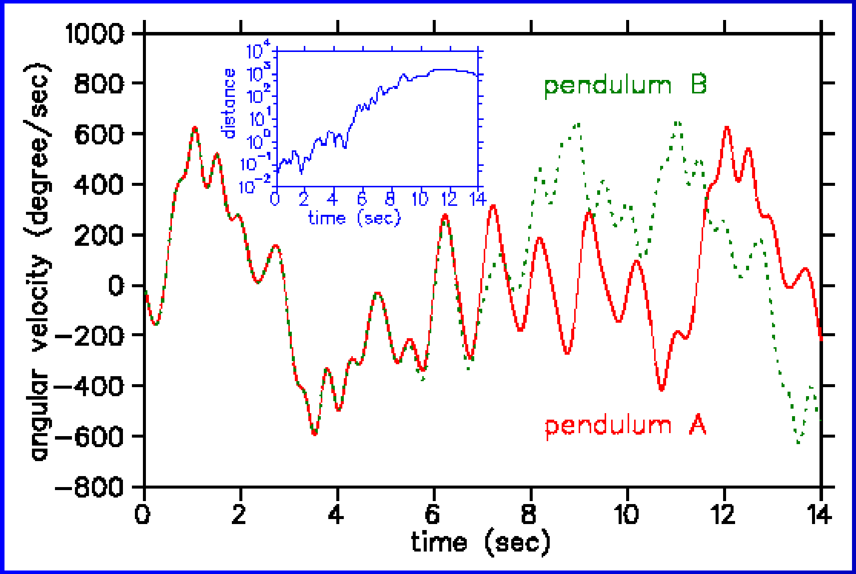
Appears only in nonlinear systems

Attractor

- A non-wandering set may be stable or unstable
- **Lyapunov stability:** Every orbit starting in a neighborhood of the non-wandering set remains in a neighborhood.
- **Asymptotic stability:** In addition to the Lyapunov stability, every orbit in a neighborhood approaches the non-wandering set asymptotically.
- **Attractor:** Asymptotically stable minimal non-wandering sets.
- **Basin of attraction:** is the set of all initial states approaching the attractor in the long time limit.
- **Strange attractor:** attractor which exhibits a sensitive dependence on the initial conditions.

Sensitive dependence on the initial conditions

- Definition: A set S exhibits *sensitive dependence* if ∃r>0 s.t. ∀ε>0 and ∀x∈S ∃y s.t |x-y|<ε and $|x_n - y_n| > r$ for some n.



pendulum A: φ = -140° , *d*φ/*dt* = 0
pendulum B: φ = -140° 1', *d*φ/*dt* = 0

Demonstration

The sensitive dependence of the trajectory on the initial conditions is a key element of deterministic chaos!

However...

Sensitivity on the initial conditions also happens in linear systems

Why?

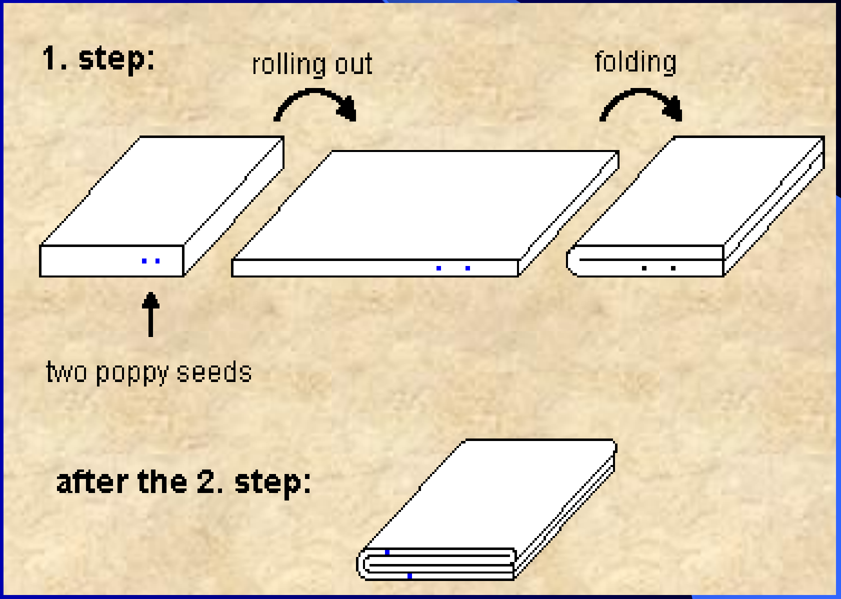
There is no boundness.

(Lagrange stability)

But this is "explosion process", not the deterministic chaos!

$$x_{n+1} = 2x_n$$

- SIC leads to chaos only if the trajectories are bounded (the system cannot blow up to infinity).
- With linear dynamics either SIC or bounded trajectories. With nonlinearities could be both.



There is no folding without nonlinearities!

The Lyapunov Exponent

- A quantitative measure of the sensitive dependence on the initial conditions is **the Lyapunov exponent λ.** It is the averaged rate of divergence (or convergence) of two neighboring trajectories in the phase space.
- Actually there is a whole spectrum of Lyapunov exponents. Their number is equal to the dimension of the phase space. If one speaks about *the* Lyapunov exponent, the largest one is meant.

Definition of Lyapunov Exponents

- Given a continuous dynamical system in an *n*-dimensional phase space, we monitor the long-term evolution of an *infinitesimal n-sphere* of initial conditions.
- The sphere will become an n-ellipsoid due to the locally deforming nature of the flow.
- The *i*-th one-dimensional Lyapunov exponent is then defined as following:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}$$

On more formal level

Consider a map M in an n-dimensional phase space, x_0 an initial condition, x_n the corresponding orbit.

Consider an infinitesimal displacement from x_0 in the direction of tangent vector y_0 then the evolution of the tangent vector is:

$$y_n = DM^n(x_0)y_0,$$

where $DM^n(x_0) = DM(x_{n-1}) \cdot \ldots \cdot DM(x_0)$

Define a Lyapunov exponent for initial condition x_0, and initial orientation of the infinitesimal displacement given by $u_0 = y_0/|y_0|$:

$$L(x_0, u_0) = \lim_{n\to\infty} \frac{1}{n} \ln(|y_n|/|y_0|) = \lim_{n\to\infty} \frac{1}{n} \ln |DM^n(x_0) \cdot u_0|.$$

- The Multiplicative Ergodic Theorem of Oseledec states that this limit exists for almost all points x_0 and almost all directions of infinitesimal displacement in the same basin of attraction.

- Order: $\lambda_1 > \lambda_2 > \ldots > \lambda_n$
- The linear extent of the ellipsoid grows as $2^{\lambda_1 t}$
- The area defined by the first 2 principle axes grows as $2^{(\lambda_1+\lambda_2)t}$
- The volume defined by the first 3 principle axes grows as $2^{(\lambda_1+\lambda_2+\lambda_3)t}$ and so on…
- The sum of the first *j* exponents is defined by the long-term exponential growth rate of a *j*-volume element.

Signs of the Lyapunov exponents

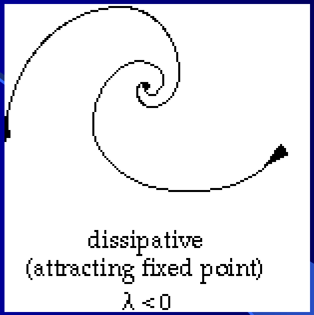
- Any continuous time-dependent DS without a fixed point will have ≥1 zero exponents.
- The sum of the Lyapunov exponents must be negative in dissipative DS $\Rightarrow \exists$ at least one negative Lyapunov exponent.
- A positive Lyapunov exponent reflects a "direction" of *stretching* and *folding* and therefore determines chaos in the system.

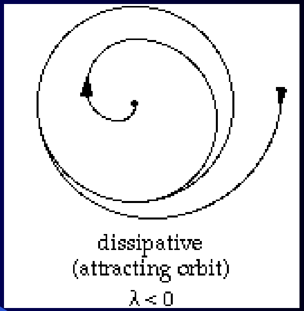
The signs of the Lyapunov exponents provide a qualitative picture of a system's dynamics

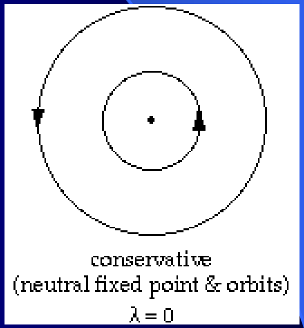
- 1D maps: ∃! $\lambda_1=\lambda$:
  - $\lambda=0$ – a marginally stable orbit;
  - $\lambda<0$ – a periodic orbit or a fixed point;
  - $\lambda>0$ – chaos.
- 3D continuous dissipative DS: $(\lambda_1,\lambda_2,\lambda_3)$
  - (+,0,-) – a strange attractor;
  - (0,0,-) – a two-torus;
  - (0,-,-) – a limit cycle;
  - (-,-,-) – a fixed point.

The sign of the Lyapunov Exponent

- **$\lambda<0$** - the system attracts to a fixed point or stable periodic orbit. These systems are non conservative (dissipative) and exhibit *asymptotic stability*.
- **$\lambda=0$** - the system is neutrally stable. Such systems are conservative and in a steady state mode. They exhibit *Lyapunov stability*.
- **$\lambda>0$** - the system is chaotic and unstable. Nearby points will diverge irrespective of how close they are.







Computation of Lyapunov Exponents

- Obtaining the Lyapunov exponents from a system with known differential equations is no real problem and was dealt with by Wolf.
- In most real world situations we do not know the differential equations and so we must calculate the exponents from a time series of experimental data. Extracting exponents from a time series is a complex problem and requires care in its application and the interpretation of its results.

Calculation of Lyapunov spectra from ODE (Wolf et al.)

- A "fiducial" trajectory (the center of the sphere) is defined by the action of nonlinear equations of motions on some initial condition.
- Trajectories of points on the surface of the sphere are defined by the action of linearized equations on points infinitesimally separated from the fiducial trajectory.
- Thus the principle axis are defined by the evolution via linearized equations of an initially orthonormal vector frame $\{e_1, e_2, \ldots, e_n\}$ attached to the fiducial trajectory

Problems in implementing:

- Principal axis diverge in magnitude.
- In a chaotic system each vector tends to fall along the local direction of most rapid growth. (Due to the finite precision of computer calculations, the collapse toward a common direction causes the tangent space orientation of all axis vectors to become indistinguishable.)

Solution

Gram-Schmidt reorthonormalization (GSR) procedure!

GSR:

- The linearized equations act on $\{e_1, e_2, \ldots, e_n\}$ to give a set $\{v_1, v_2, \ldots, v_n\}$.
- Obtain $\{v'_1, v'_2, \ldots, v'_n\}$.
- Reorthonormalization is required when the magnitude or the orientation divergences exceeds computer limitation.

$$v'_1 = \frac{v_1}{\|v_1\|}$$

$$v'_2 = \frac{v_2 - < v_2, v'_1 > v'_1}{\|v_2 - < v_2, v'_1 > v'_1\|}$$

$$\vdots$$

$$v'_n = \frac{v_n - \sum_{k=1}^{n-1} < v_n, v'_k > v'_k}{\|v_n - \sum_{k=1}^{n-1} < v_n, v'_k > v'_k\|}$$

- GSR never affects the direction of the first vector, so v_1 tends to seek out the direction in tangent space which is most rapidly growing, $|v1| \sim 2^{\lambda_1 t}$;
- v_2 has its component along v_1 removed and then is normalized, so v_2 is not free to seek for direction, however…
- $\{v'_1, v'_2\}$ span the same 2D subspace as $\{v_1, v_2\}$, thus this space continually seeks out the 2D subspace that is most rapidly growing $|S(v_1,v_2)| \sim 2^{(\lambda_1+\lambda_2)t}$ …

  $|S(v_1,v_2,\ldots,v_k)| \sim 2^{(\lambda_1+\lambda_2+\ldots+\lambda_k)t}$ k-volume
- So monitoring k-volume growth we can find first *k* Lyapunov exponents.

Example for Henon map

$$X_{n+1} = 1 - 1.4X_n^2 + Y_n$$

$$Y_{n+1} = 0.3X_n$$

The linearization of this map is:

$$\begin{pmatrix} \delta X_{n+1} \\ \delta Y_{n+1} \end{pmatrix} = J_n \begin{pmatrix} \delta X_n \\ \delta Y_n \end{pmatrix}$$

where $J_n = \begin{bmatrix} -2.8X_n & 1 \\ 0.3 & 0 \end{bmatrix}$

and X_n is the $(n-1)$st iterate of an arbitrary initial condition X_1.

An orthonormal frame of principal axis vectors {(0,1),(1,0)} is evolved by applying the product Jacobian to each vector:

$$\begin{pmatrix} \delta X_n \\ \delta Y_n \end{pmatrix} = J_{n-1}(J_{n-2} \ldots J_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

or by regrouping the terms:

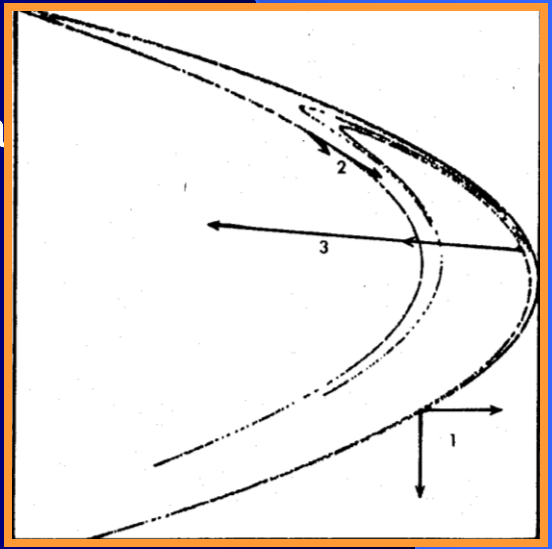
$$\begin{pmatrix} \delta X_n \\ \delta Y_n \end{pmatrix} = [J_{n-1}J_{n-2} \ldots J_1] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Result of the procedure:

- Consider:

$$\begin{pmatrix} \delta X_n \\ \delta Y_n \end{pmatrix} = J_{n-1}(J_{n-2} \ldots J_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

- J multiplies each *current* axis vector, which is the initial vector multiplied by all previous Js.
- The magnitude of each current axis vector diverges and the angle between the two vectors goes to zero.
- GSR corresponds to the replacement of each current axis vector.
- Lyapunov exponents are computed from of length of the 1-st vector and the a



$\lambda_1 = 0.603; \lambda_2 = -2.34$

Lyapunov spectrum for experimental data (Wolf et al.)

- Experimental data usually consist of discrete measurements of a single observable.
- Need to reconstruct phase space with delay coordinates and to obtain from such a time series an attractor whose Lyapunov spectrum is identical to that of the original one.

Procedure for λ_1

Given the time series $x(t)$. Choose m a dimension of the phase space and τ - a delay time: a point on the attractor is given by: $\{x(t), x(t+\tau), \ldots, x(t+[m-1]\tau)\}$.

1. Take the initial point $\{x(t_0), \ldots, x(t_0+[m-1]\tau)\}$. Find the nearest neighbor (in the Euclidean sense) and denote the distance between them $L(t_0)$.
2. At time t_1: $L(t_0) \to L'(t_1)$. We want $(t_1 - t_0)$ to be small enough so that only small scale attractor structure to be examined.
3. Look for a new data point s.t.: a) its separation $L(t_1)$ form the evolved fiducial point is small; b) the angle between evolved and replacement elements is small. If cannot find such point - retain used points.
4. Repeat the procedure until the fiducial trajectory has traversed the entire data and estimate:

$$\lambda_1 = \frac{1}{t_M - t_0} \sum_{k=1}^{M} \log_2 \frac{L'(t_k)}{L(t_{k-1})}$$

where M is the total number of replacement steps.

Procedure for $\lambda_1+\lambda_2$

The procedure is similar:

1. Take the initial point $\{x(t_0), \ldots, x(t_0 + [m-1]\tau)\}$. Find the 2 nearest neighbors and denote the defined by these 3 points area as $A(t_0)$.

2. At time t_1: $A(t_0) \to A'(t_1)$.

3. Look for a 2 new points to obtain a smaller $A(t_1)$ that best preserves the $A'(t_1)$ orientation.

4. Propagation and replacement steps repeated until the fiducial trajectory has traversed the entire data and estimate:

$$\lambda_1 + \lambda_2 = \frac{1}{t_M - t_0} \sum_{k=1}^{M} \log_2 \frac{A'(t_k)}{A(t_{k-1})}$$

Implementation Details

- Selection of embedding dimension and delay time: emb.dim>2*dim. of the underlying attractor, delay must be checked in each case.
- Evolution time between replacements: maximizing the propagation time, minimizing the length of the replacement vector, minimizing orientation error.
- Henon attractor: the positive exponent is obtained within 5% with 128 points defining the attractor.
- Lorenz attractor: the positive exponent is obtained within 3% with 8192 points defining the attractor.

Finite-Time Lyapunov Exponent

The evolution of an infinitesimal uncertainty over a finite time Δt is determined by the linear propagator $M(x_0, \Delta t)$ along the trajectory $x(t)$, that is:

$$\epsilon(t_0 + \Delta t) = M(x_0, \Delta t)\epsilon(t_0)$$

where $x_0 = x(t_0)$ and, for a flow,

$$M(x_0, \Delta t) = \exp\left(\int_{t_0}^{t_0+\Delta t} J(x(t))dt\right)$$

.

For discrete time maps, the linear propagator is simply the product of the Jacobians along the trajectory:

$$M(x_0, k) = J(x_{k-1})J(x_{k-2})\dots J(x_0)$$

SVD

The linear dynamics are most easily examined through the singular value decomposition (SVD) of M:

$$M = U\Sigma V^T$$

where U and V are orthogonal and $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$ s.t. $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$

With this ordering, the first singular vectors correspond to the direction which will have grown the most between t_0 and $t_0 + \Delta t$. The right singular vectors v_i are "initial time" singular vectors defined at t_0 evolve into the left singular vectors u_i at time $t_0 + \Delta t$. Under the action of M:

$$Mv_i = \sigma_i u_i$$

For a given x and Δt, the σ_i define **The finite-time Lyapunov exponents**:

$$\lambda_i(x, \Delta t) = \frac{1}{\Delta t} \log_2 \sigma_i$$

The Global Lyapunov Exponent $\Lambda_1 = \lim_{\Delta t \to \infty} \lambda_i(x, \Delta t)$ for almost all x and almost all initial orientations ϵ_0.

Lyapunov exponents for maps

System	$\langle\lambda_i^{(\tau)}\rangle$	Conf. Int.	D_{Lyap}	Conf. Int.
Hénon	0.6049	-0.0035, +0.0036	1.2583	-0.0019, +0.0019
	-2.3418	-0.0036, +0.0035		
Ikeda	0.7323	-0.0044, +0.0045	1.7066	-0.0073, +0.0074
	-1.0363	-0.0045, +0.0044		

Finite time Lyapunov exponent spectrum (in bytes per unit time) for $\tau = 2^{18}$. The last columns contain D_{Lyap} and associated confidence intervals derived from the Lyapun exponents

Lyapunov exponents for flows

System	$\langle\lambda_i^{(\tau)}\rangle$	Conf. Int.	D_{Lyap}	Conf. Int.
Lorenz†	1.3067	-0.0051, +0.0047	2.0622	-0.0003, +0.0002
	-0.0001	-0.0002, +0.0002		
	-21.0233	-0.0048, +0.0052		
Moore-Spiegel	0.2461	-0.0568, +0.0539	2.1453	-0.0434, +0.0445
	-0.0009	-0.0106, +0.0097		
	-1.6879	-0.0575, +0.0608		
Rössler	0.1285	-0.0216, +0.0202	2.0091	-0.0019, +0.0017
	0.0001	-0.0044, +0.0034		
	-14.1398	-0.0415, +0.0538		

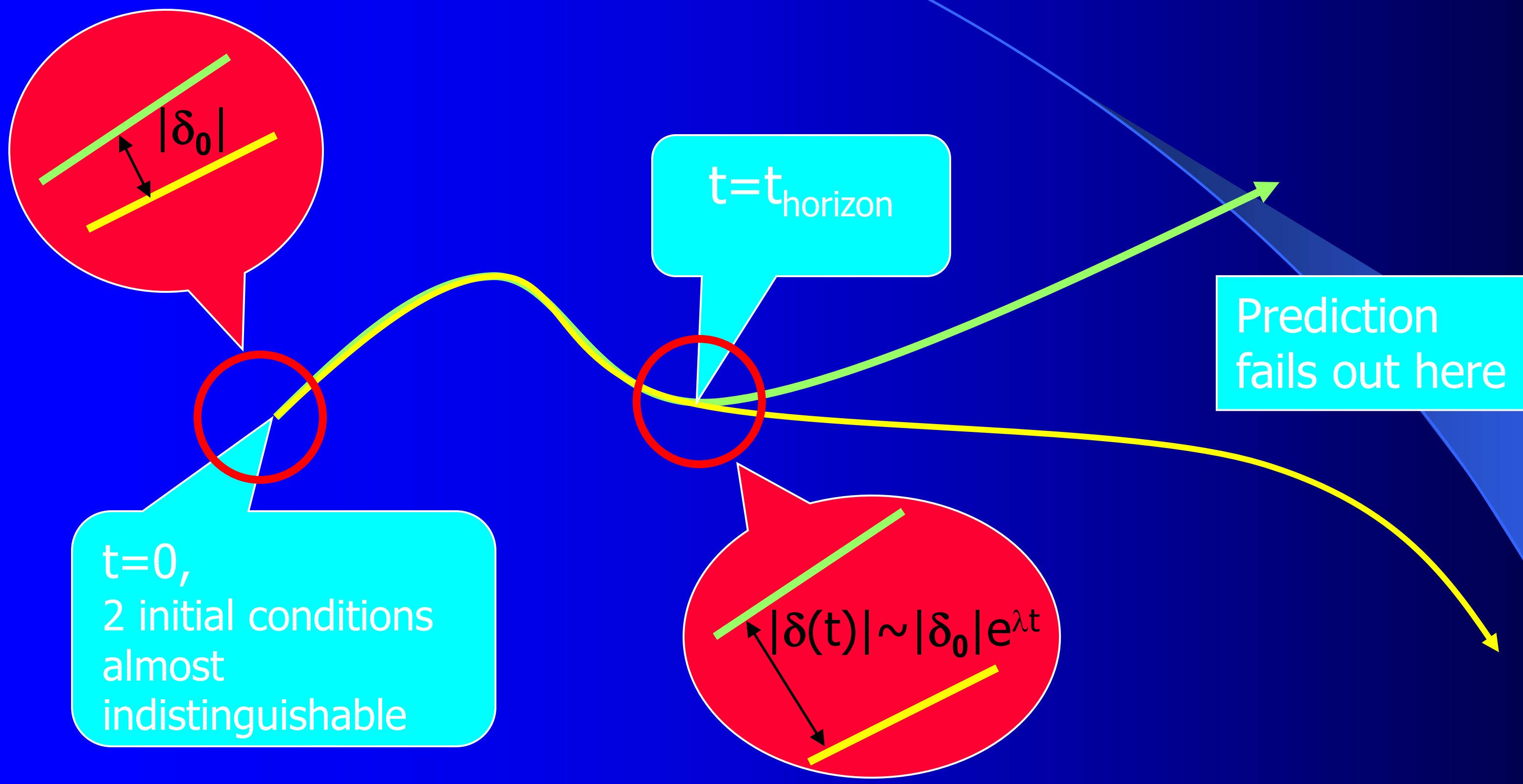
Finite time Lyapunov exponent spectrum (in bytes per unit time) for $\tau = 2^{10}$, except for the Lorenz system superscripted by †, for which $\tau = 2^{15}$. The last two columns contain D_{Lyap} and associated confidence intervals derived from the Lyapunov exponents

Time Horizon

- When the system has a positive λ, there is a time horizon beyond which prediction breaks down (even qualitative).
- Suppose we measure the initial condition of an experimental system very accurately, but no measurement is perfect – let $|\delta_0|$ be the error.
- After time *t*: $|\delta(t)| \sim |\delta_0| e^{\lambda t}$, if *a* is our tolerance, then our prediction becomes intolerable when $|\delta(t)| \geq a$ and this occurs after a time

$$t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{||\delta_0||}\right)$$

Demonstration



- No matter how hard we work to reduce measurement error, we cannot predict longer than a few multiples of $1/\lambda$.

Example

Let $a = 10^{-3}$, initial error $||\delta_0|| = 10^{-7}$ then

$$t_{horizon} \approx \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-7}} = \frac{4 \ln 10}{\lambda}$$

If we improve initial error to $||\delta_0|| = 10^{-13}$ then

$$t_{horizon} \approx \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-13}} = \frac{10 \ln 10}{\lambda}$$

So, after a million fold improvement in the initial uncertainty, we can predict only 2.5 times longer!

Some philosophy

- The notion of chaos seems to conflict with that attributed to Laplace: given precise knowledge of the initial conditions, it should be possible to predict the future of the universe. However, Laplace's dictum is certainly true for any <u>deterministic</u> system.
- The main consequence of chaotic motion is that given imperfect knowledge, the predictability horizon in a deterministic system is much shorter than one might expect, due to the exponential growth of errors.

State-space reconstruction

To improve understanding of a particular dynamical system

Underlying processes are unknown

Collect observations (univariate, multivariate, spatio-temporal)

Need to construct a model state space where each vector uniquely defines a state

Trajectories of a deterministic system should not have self-intersections

Self-intersections are caused by using an insufficient dimension

Define extra state vector components using delays (or derivatives)

Henon reconstruction

Two-dimensional Hénon map in the (x, y) space:

$$x_{n+1} = 1 + y_n - ax_n^2$$

$$y_{n+1} = bx_n$$

Parameter values:

$$a = 1.4,\ b = 0.3$$

Consider observations of the x variable, $\{x_i\}_{i=0}^{N}$

One can substitute delayed values of x for the y variable

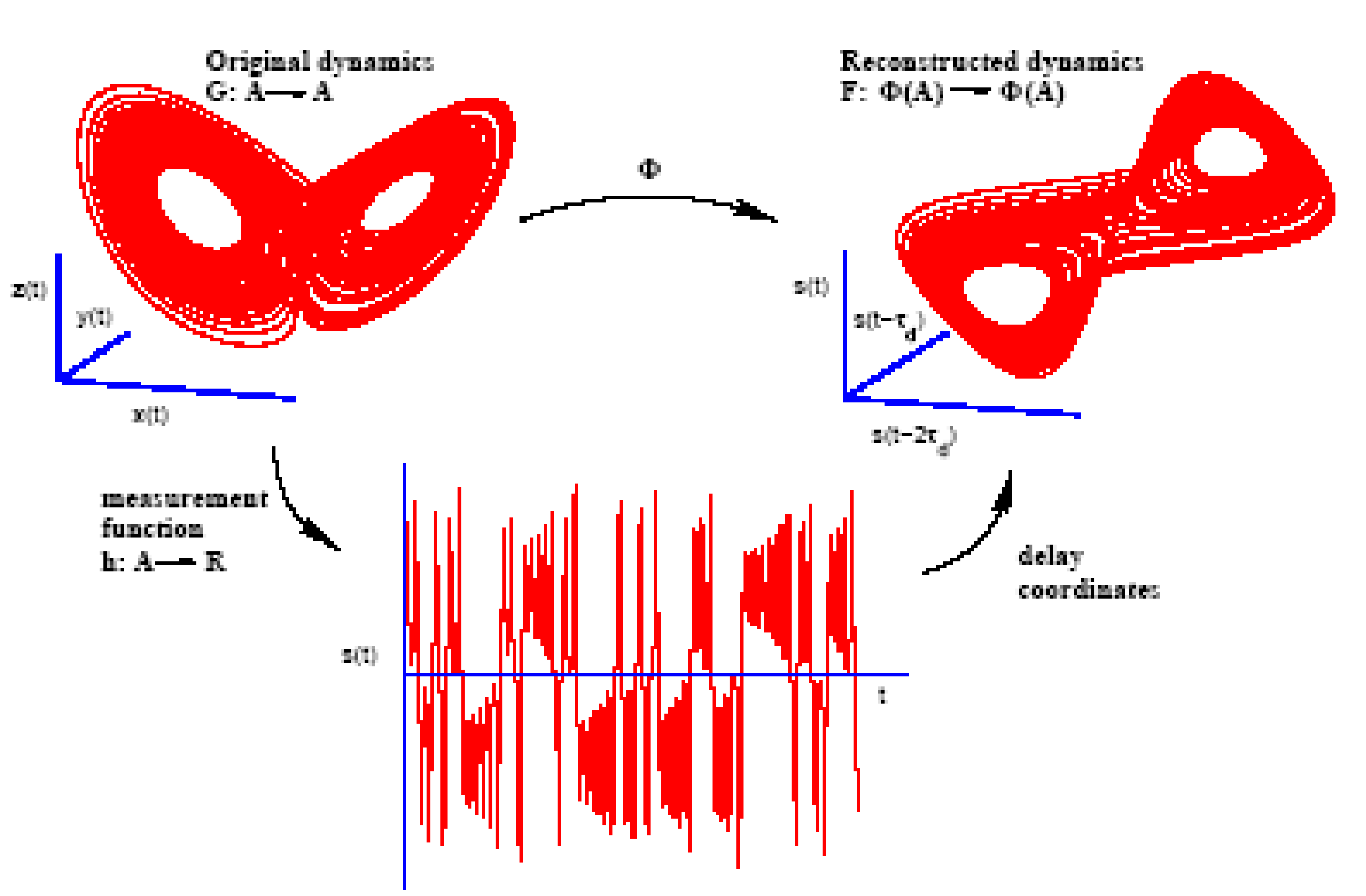
Investigate dynamics in (x_{n-1}, x_n) model state space

Works perfectly in the case of the Hénon map since the system equations can also be expressed as

$$x_{n+1} = 1 + bx_{n-1} - ax_n^2$$

Knowledge of x_{n-1} and x_n allows one to determine x_{n+1}

Lorenz reconstruction



Embedding

Lorenz 63 model

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz$$

This can also be written as

$$x\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2}\frac{dx}{dt} + (2+b)\frac{d^2x}{dt^2}x - 2\left(\frac{dx}{dt}\right)^2 - b\sigma r x^2$$

$$+2b\frac{dx}{dt}x + bx^2 - \frac{x^3}{\sigma}\frac{dx}{dt} - \frac{x^4}{\sigma} = 0$$

Derivatives may be approximated using:

$$\frac{dx}{dt} \approx \frac{x(t+\tau) - x(t-\tau)}{2\tau}$$

$$\frac{d^2x}{dt^2} \approx \frac{x(t+\tau) - 2X(t) + x(t-\tau)}{\tau^2}$$

...and so on...

Delay reconstruction

Derivatives are sensitive to noise (especially higher orders)

One can also obtain derivatives by fitting polynomials locally in time

Similar information is obtained using *delay coordinate vectors*

Consider a discrete time series s_t

Define M-dimensional vectors

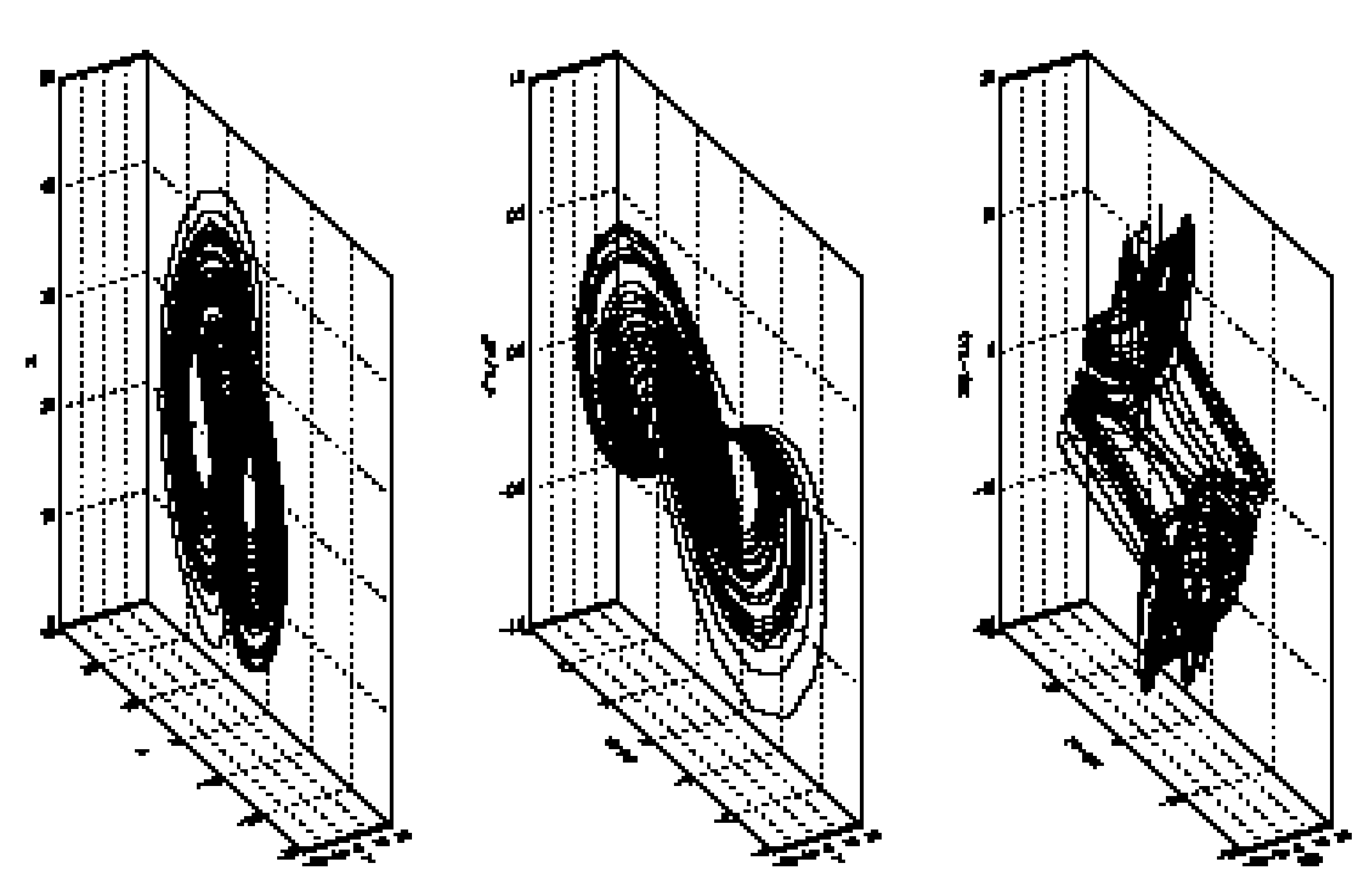
$$\mathbf{x}_t = \left[s_{t-(M-1)\tau}, \ldots, s_{t-\tau}, s_t\right]$$

These vectors define a *model state space*

For sufficiently large dimension M, trajectories in model state space are topologically equivalent to trajectories in system state space

One can also reduce noise by applying filters to coordinates (e.g. SVD)

Delay reconstruction for Lorenz systems



Euclid dimension

- In Euclid geometry, dimensions of objects are defined by integer numbers.
- 0 - A point
- 1 - A curve or line
- 2 - Triangles, circles or surfaces
- 3 - Spheres, cubes and other solids

- For a square we have N^2 self-similar pieces for the magnification factor of N

dimension=log(number of self-similar pieces) /log(magnification factor)

=log(N^2)/logN=2

For a cube we have N^3 self-similar pieces

dimension=log(number of self-similar pieces) /log(magnification factor)

=log(N^3)/logN=3

Sierpinski triangle consists of three self-similar pieces with magnification factor 2 each

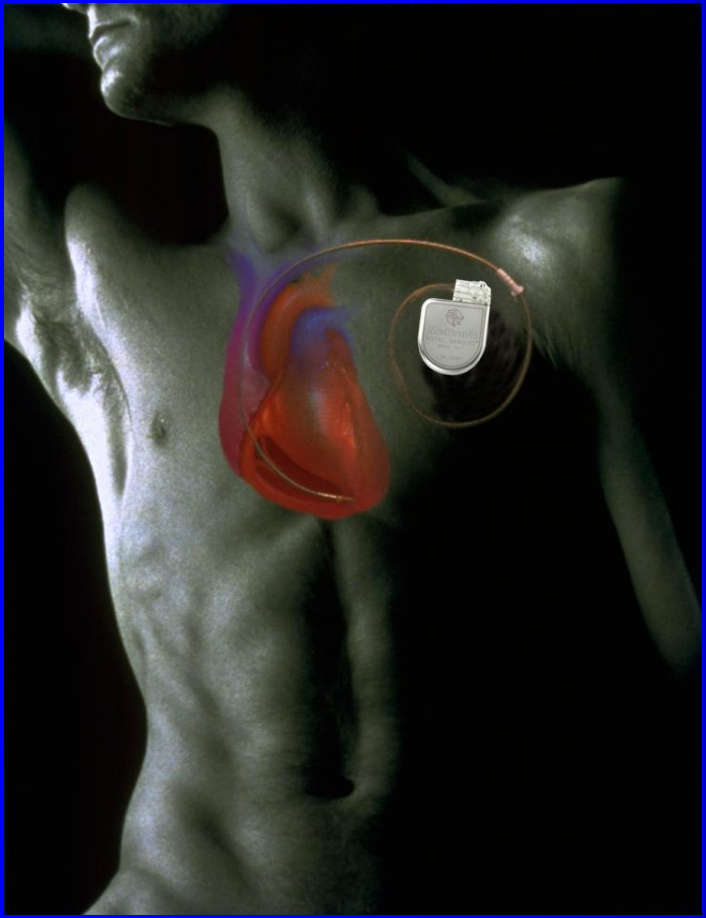
dimension=log3/log2=1.58

APPLIED CHAOS

Removing chaos

- Heart defibrillation – „hardware reset”

What is ICD (Implantable Cerdioverter Defibrillator) Therapy?



- ICD Therapy consists of pacing, cardioversion, and defibrillation therapies to treat brady and tachy arrhythmias.
- An external programmer is used to monitor and access the device parameters and therapies for each patient.

Goals of ICD Therapy

TODAY

- Termination of ventricular tachycardia or fibrillation
- Treatment of co-morbidities of AT/AF and heart failure

- Reduction of sudden cardiac death
- Improvement in quality of life
- Prolongation of life

FUTURE

– Prevention of life-threatening episodes of VT/VF

– Expanding the understanding and management of sudden cardiac death (SCD)

Medtronic Implantable Defibrillators (1989-2003)

209 cc | 120 cc | 80 cc | 80 cc | 72 cc | 54 cc

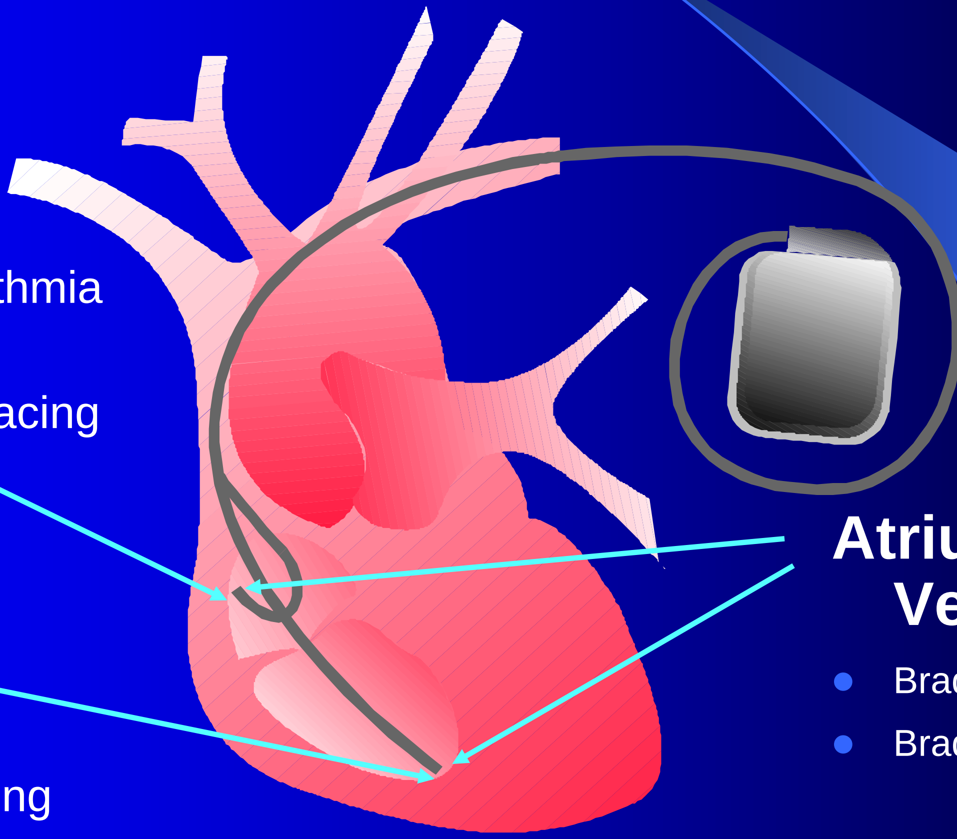
62 cc | 49 cc | 39.5 cc | 39.5 cc | 39.5 cc | 38 cc | 36 cc

83% size reduction since 1989!

Therapies Provided by Today's Dual-Chamber ICDs

Atrium

- AT/AF tachyarrhythmia detection
- Antitachycardia pacing
- Cardioversion



Ventricle

- VT/ VF detection
- Antitachycardia pacing
- Cardioversion
- Defibrillation

Atrium & Ventricle

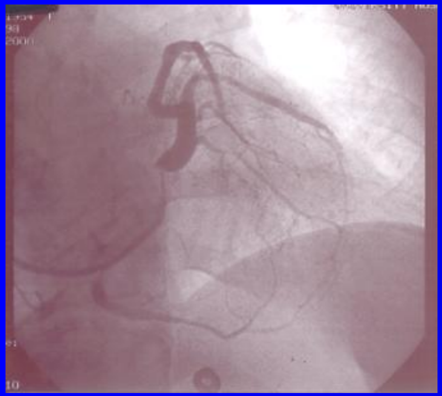
- Bradycardia sensing
- Bradycardia pacing

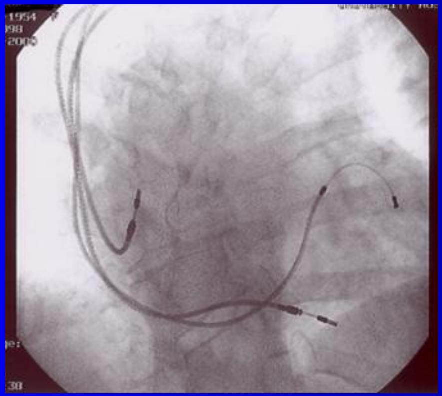
Achieving Cardiac Resynchronization

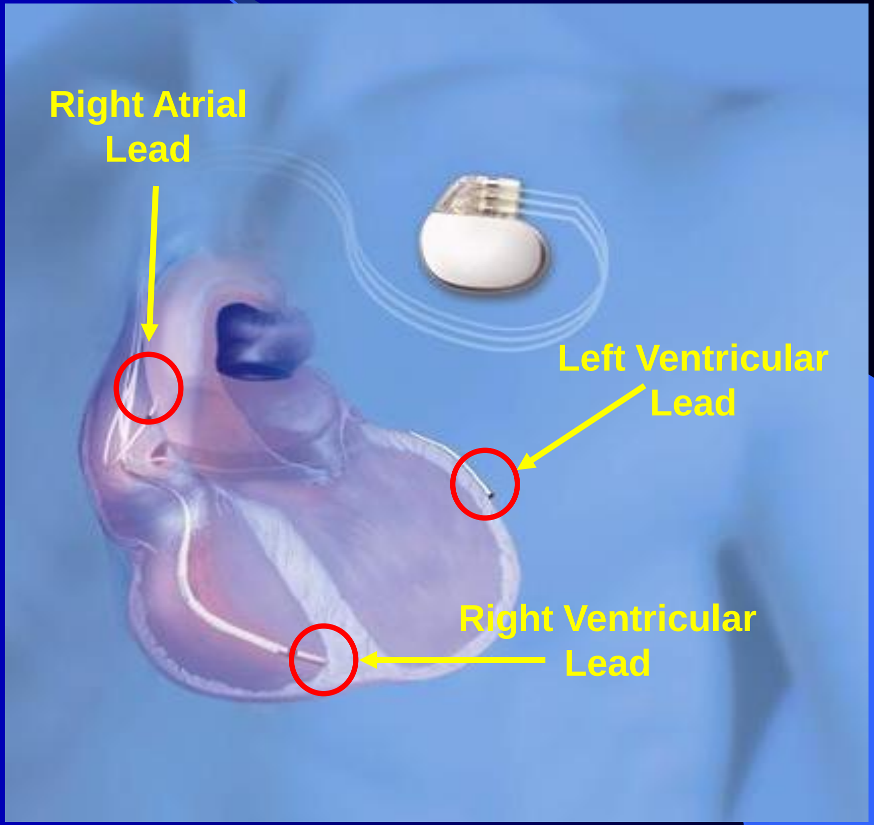
Goal: Atrial synchronous biventricular pacing

Transvenous approach for left ventricular lead via coronary sinus

Back-up epicardial approach





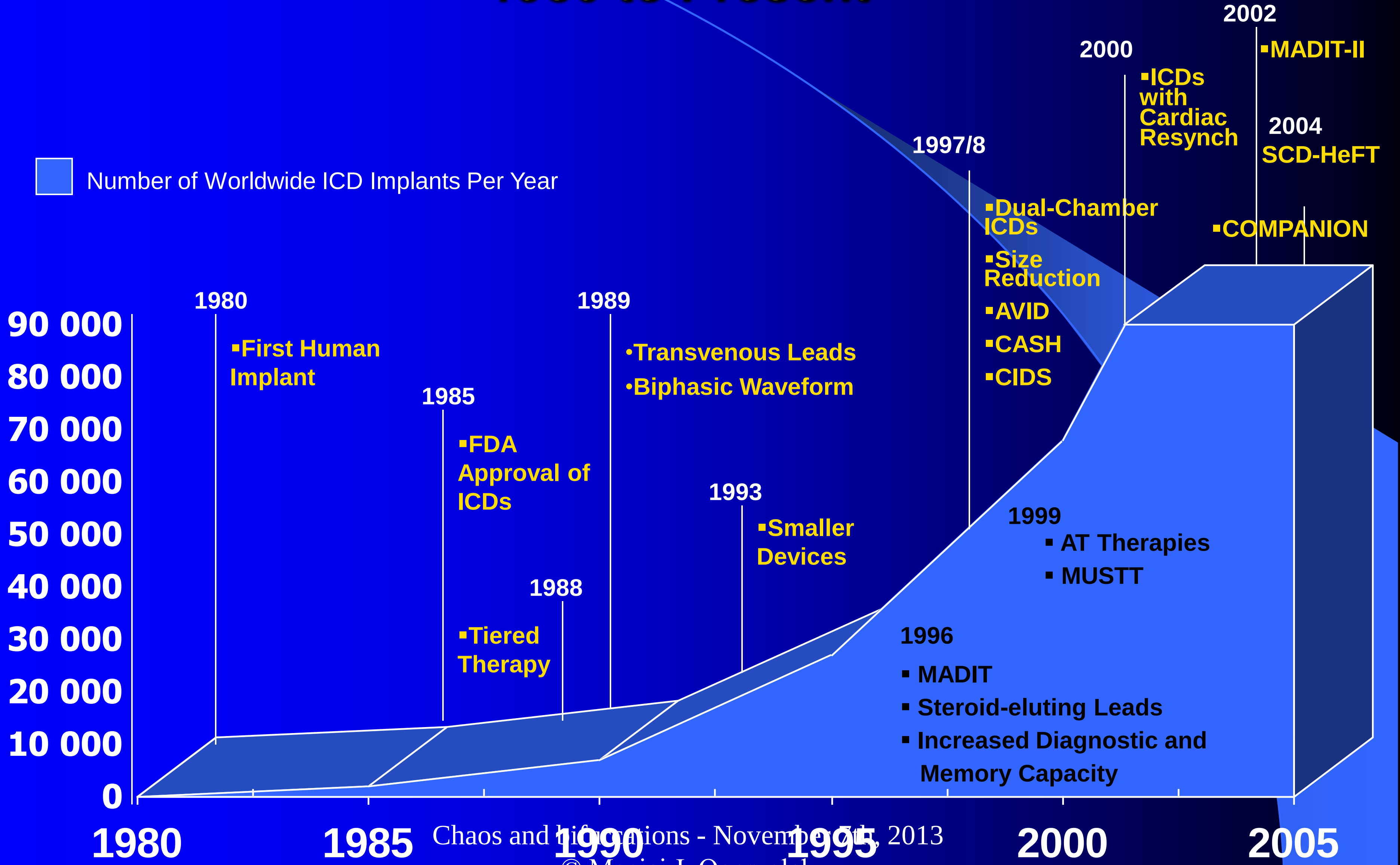


InSync Marquis ™ ICD & Cardiac Resynchronization System

Charge Times: 5.9 sec BOL / 7.5 ERI
Output: 30 Joules
Size: 38 cc, 77 g, 14 mm

- **Powerful ICD & Resynchronization Therapy**
  - Powerful 30 J therapy
  - Fast charge times
  - Proven cardiac resynchronization therapy for patients with ventricular dysynchrony
- **Better, Faster & Easier**
  Heart Failure Patient Management
  - 14 months of patient specific data provided by Cardiac Compass ™ trends
  - Follow up efficiency with RapidRead™ telemetry, Leadless ™ ECG, Painless High Voltage lead impedance
- **Implant Confidence & Efficiency**
  - Most complete family of left-heart leads & delivery systems
  - Lead placement flexibility, enhanced telemetry distance, one-stop defibrillation testing

Evolution of ICD Therapy: 1980 to Present



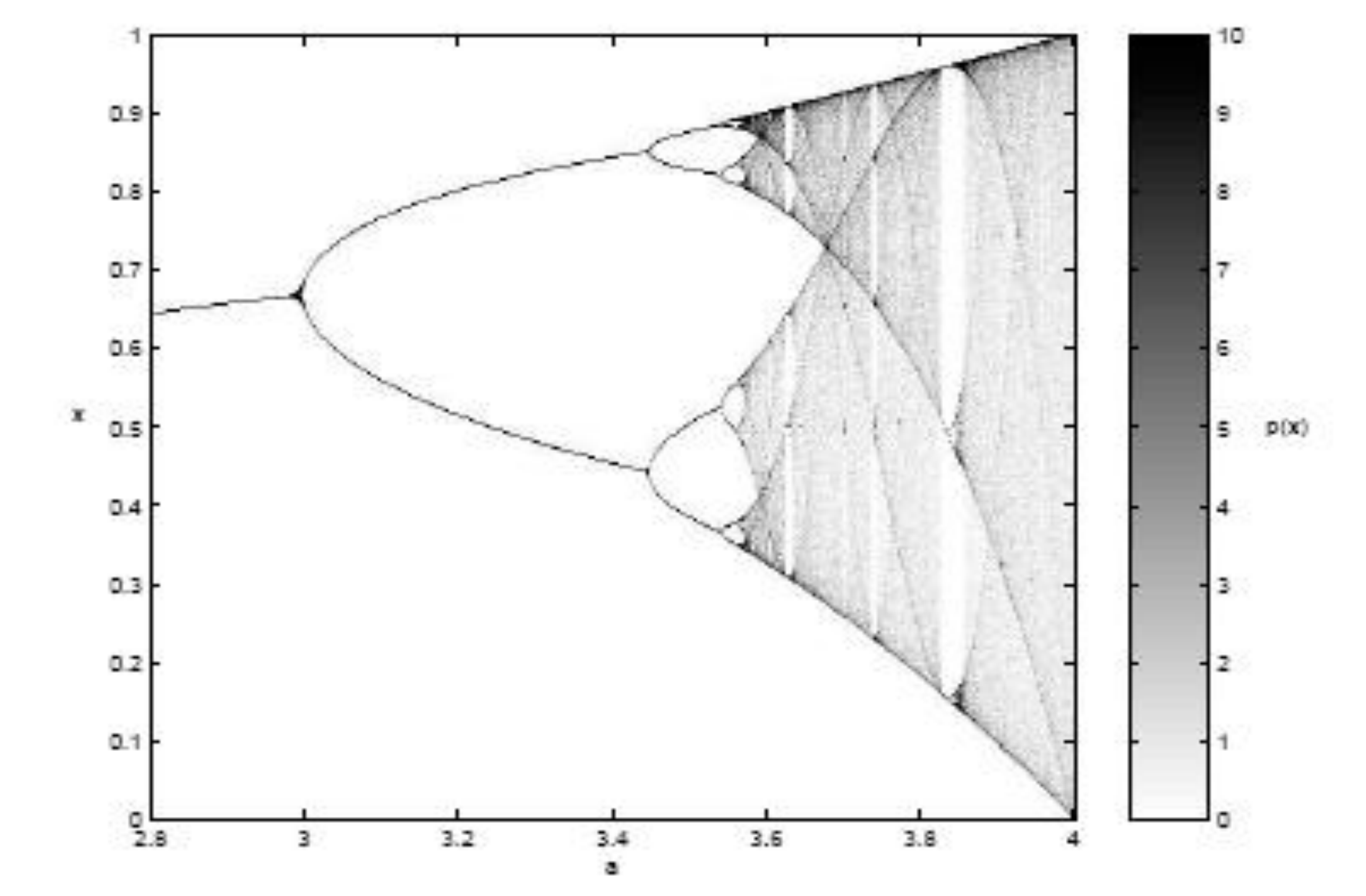
Future ICD Technology

- Enhanced automaticity:
  - Device software that suggests programming options to the clinician based on the patient' s history and demographics
- Continued reductions in device size:
  - Will require advancements in battery, capacitor and circuitry technology and/or decreasing the delivered energy output.

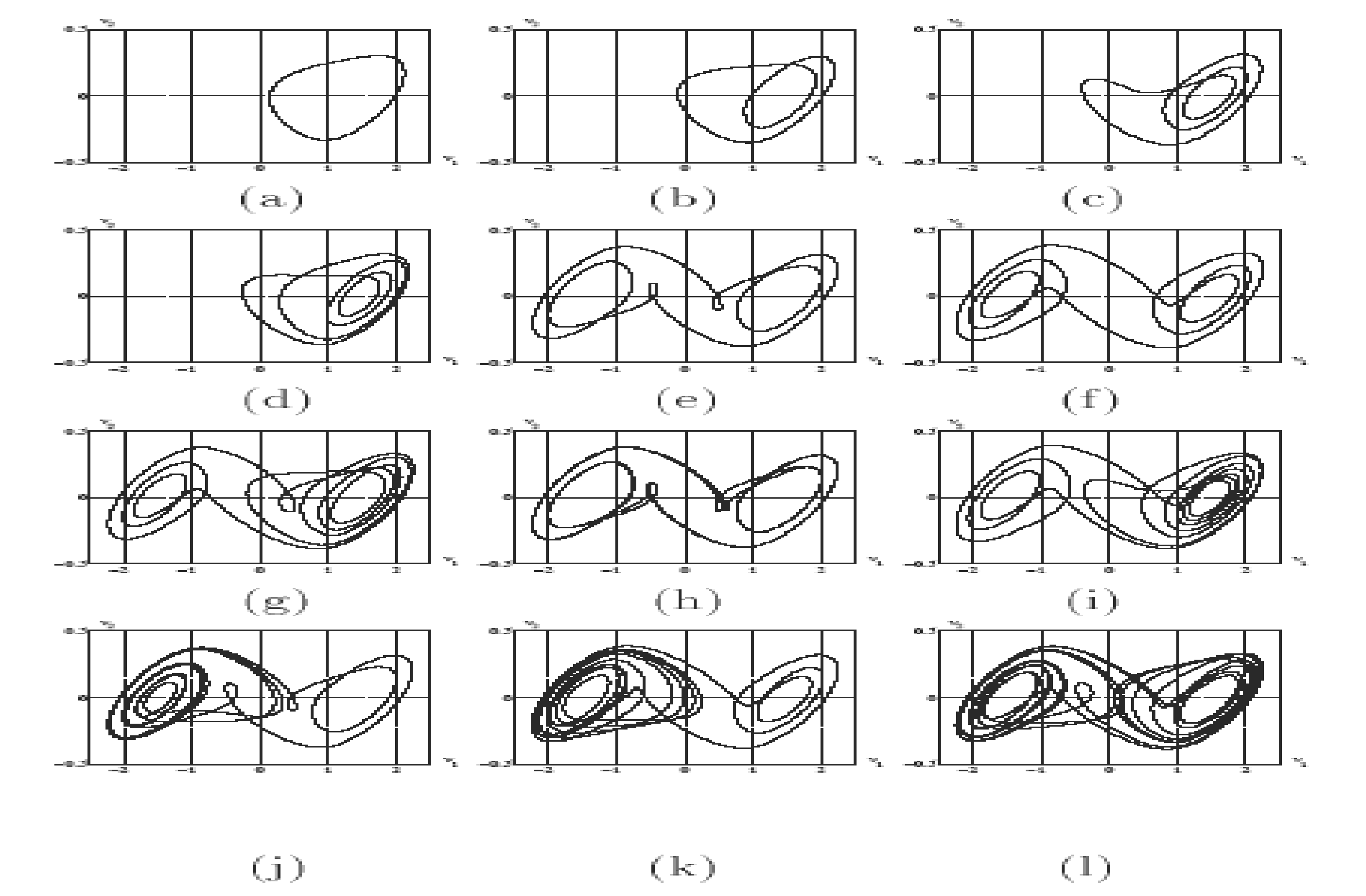
Control concept – eliminate chaos!

- Switch onto one of the unstable periodic orbits existing within the attractor
- Feasible because every trajectory is dense on the attractor ie. It will eventually pass arbitrarily close to any of these unstable orbits
- Possible stabilization by very small energy perturbation (sensitive dependence on initial condition)
- Best to use the Poincare section concept and apply control on the section plane

Bifurcation diagram for the logistic map



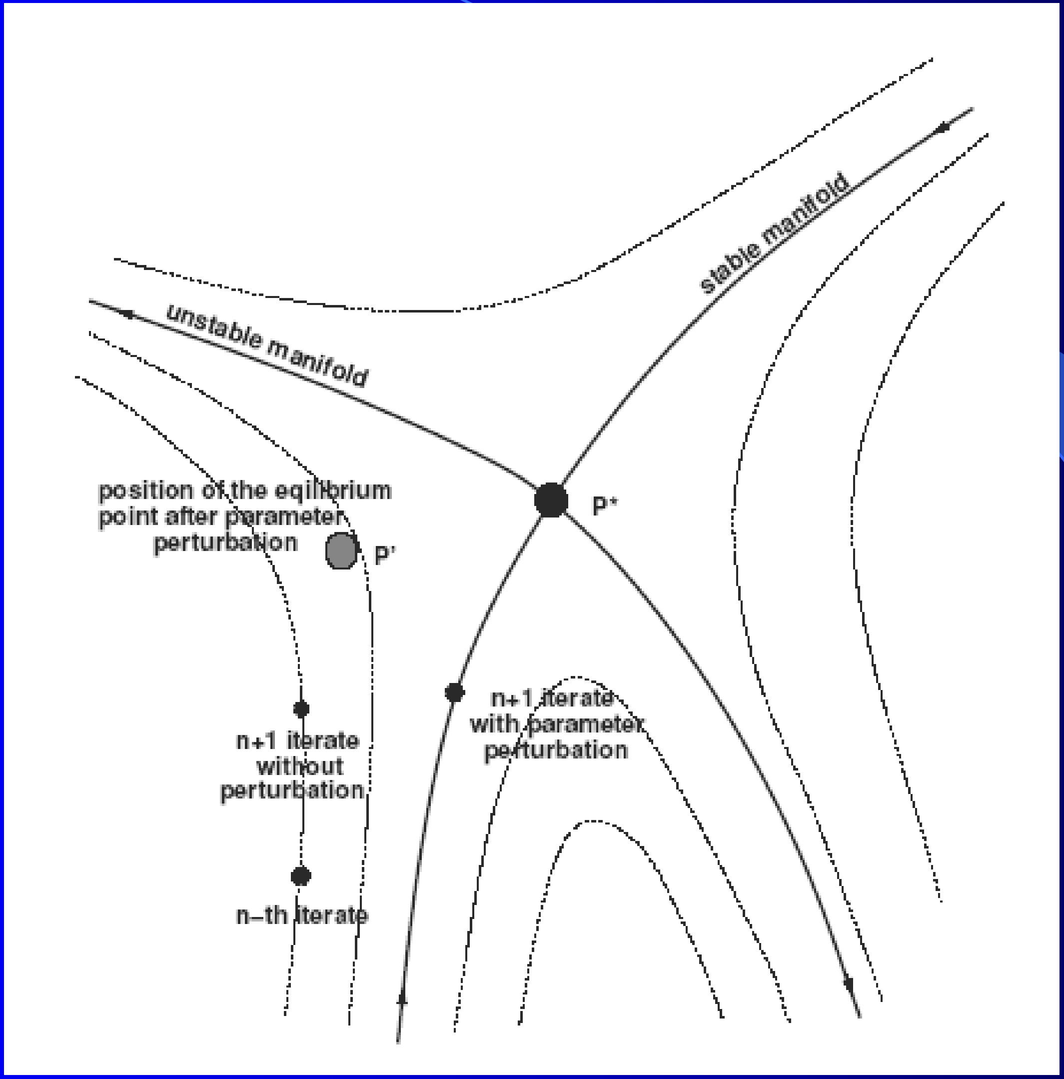
Existence of unstable periodic orbits



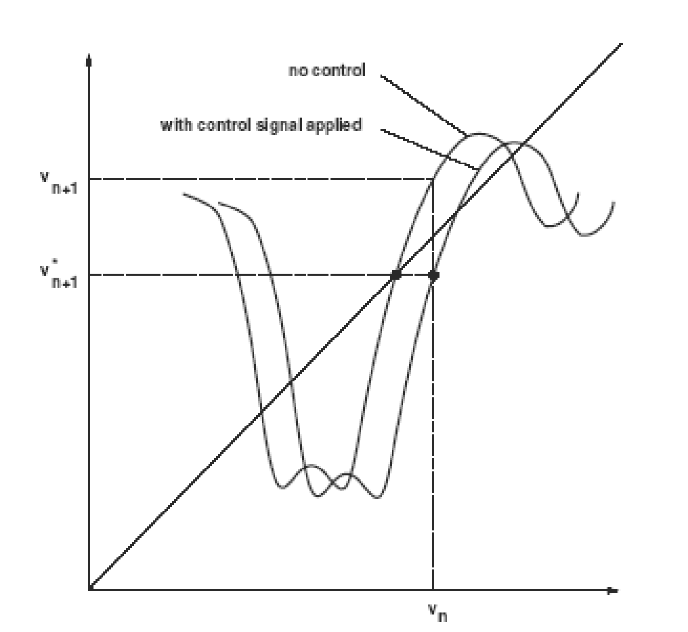
New approach

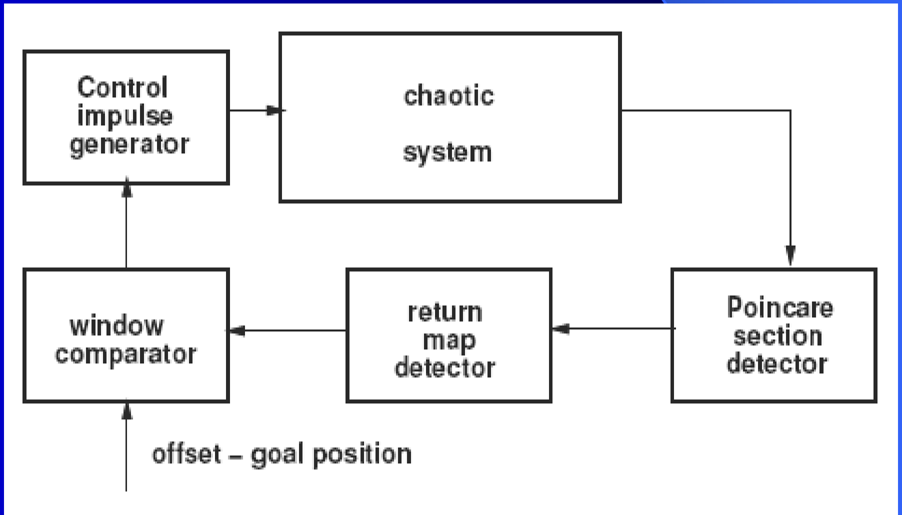
- Continuous analysis of heart waveforms and extraction of features of arrythmia
- Identification of unstable periodic orbit(s)
- Calculation of control signal
- Delivery of control signal to the heart

Ott-Grebogi-Yorke scheme

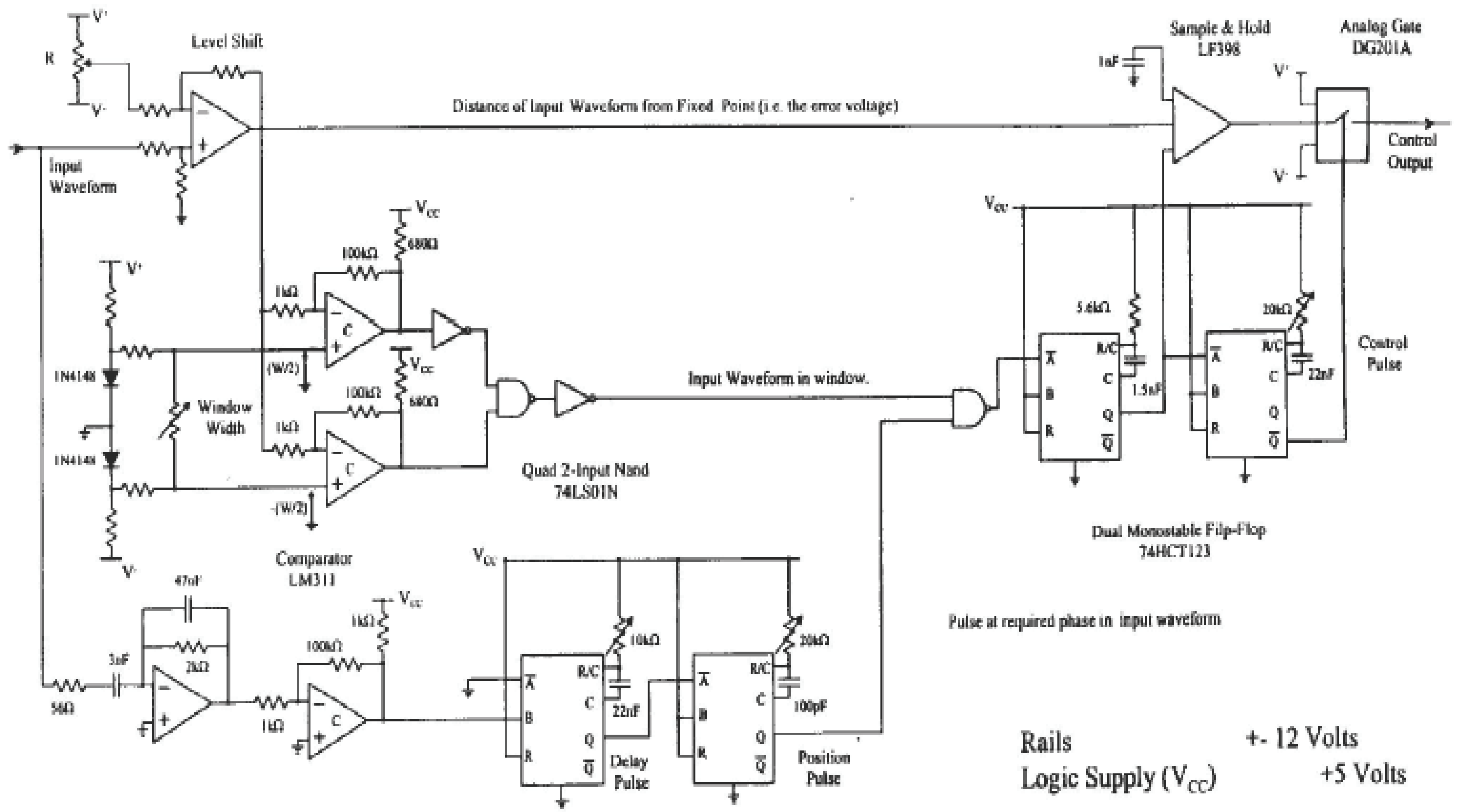


OPF chaos controller

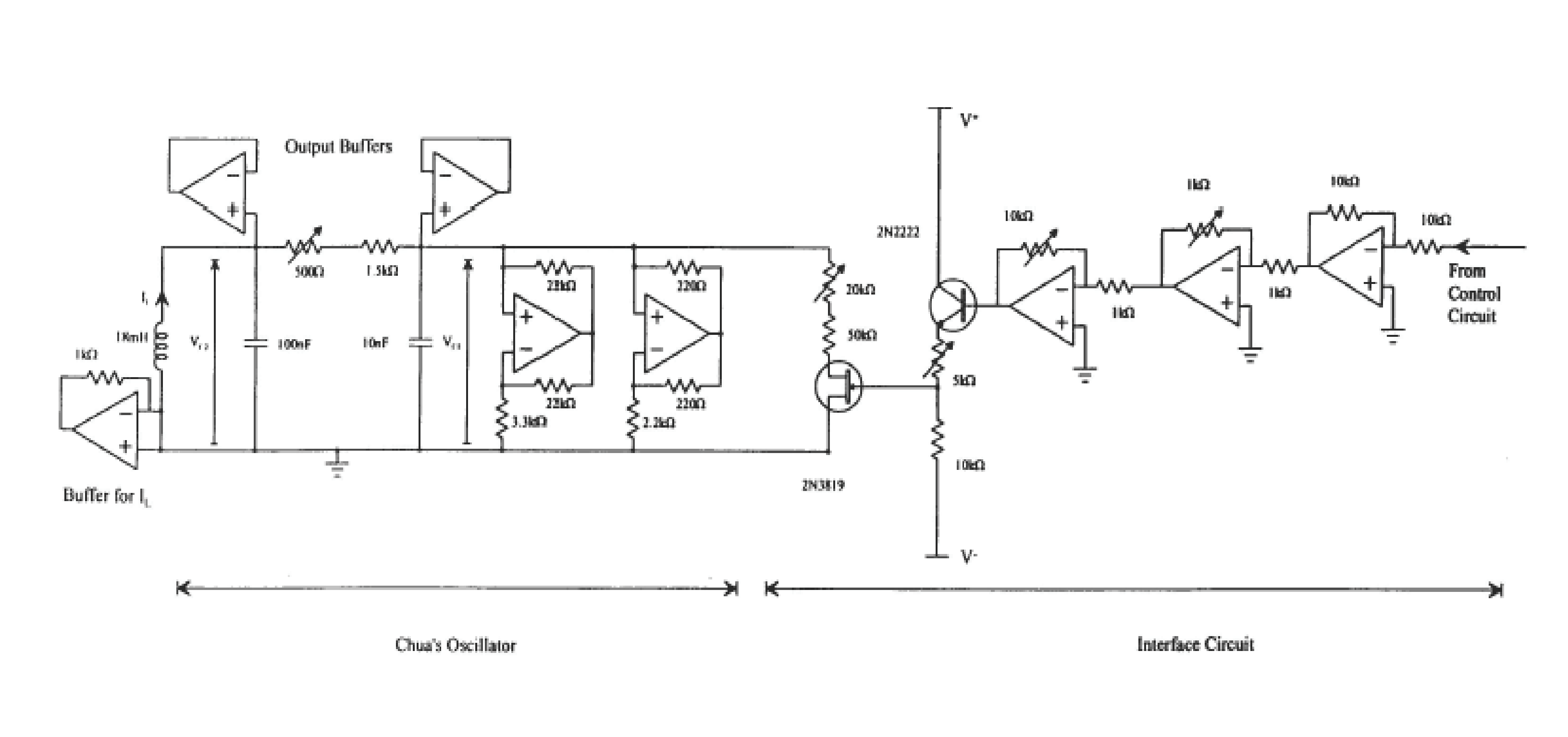




Electronic chaos controller



Control of chaos in Chua's circuit



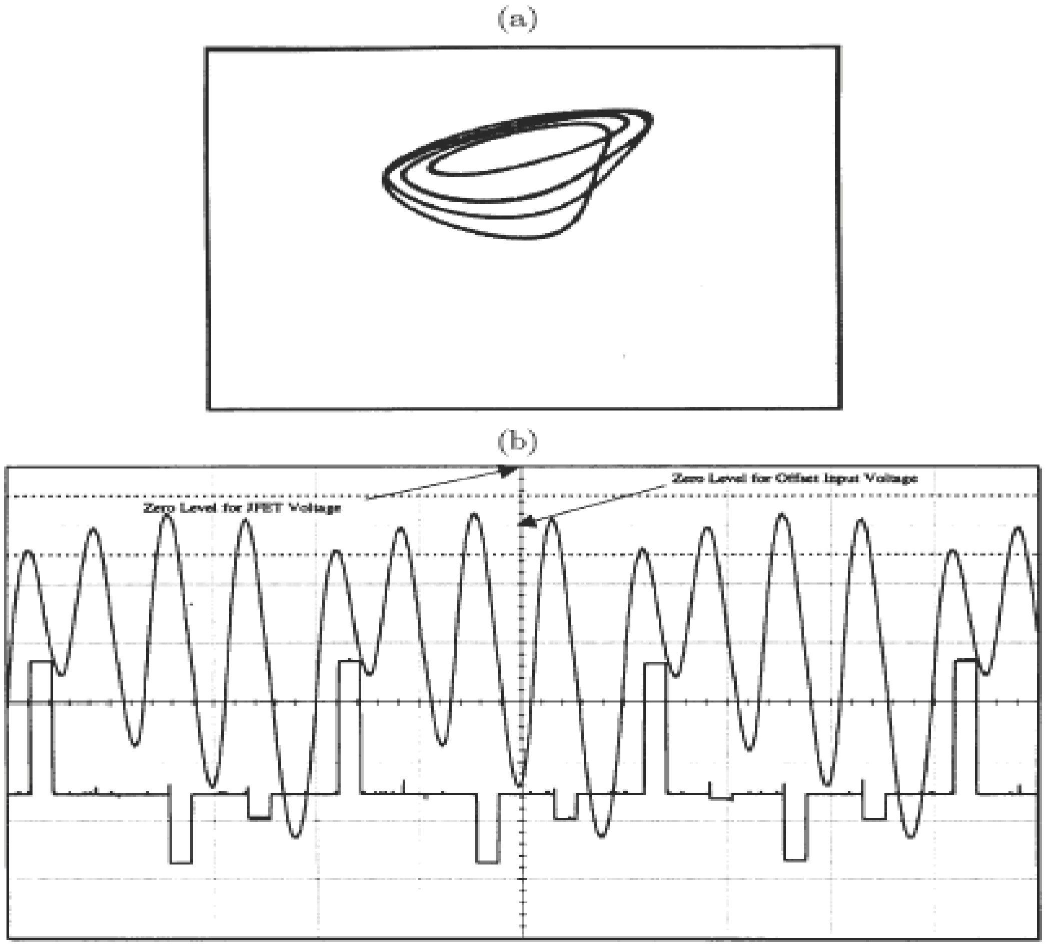
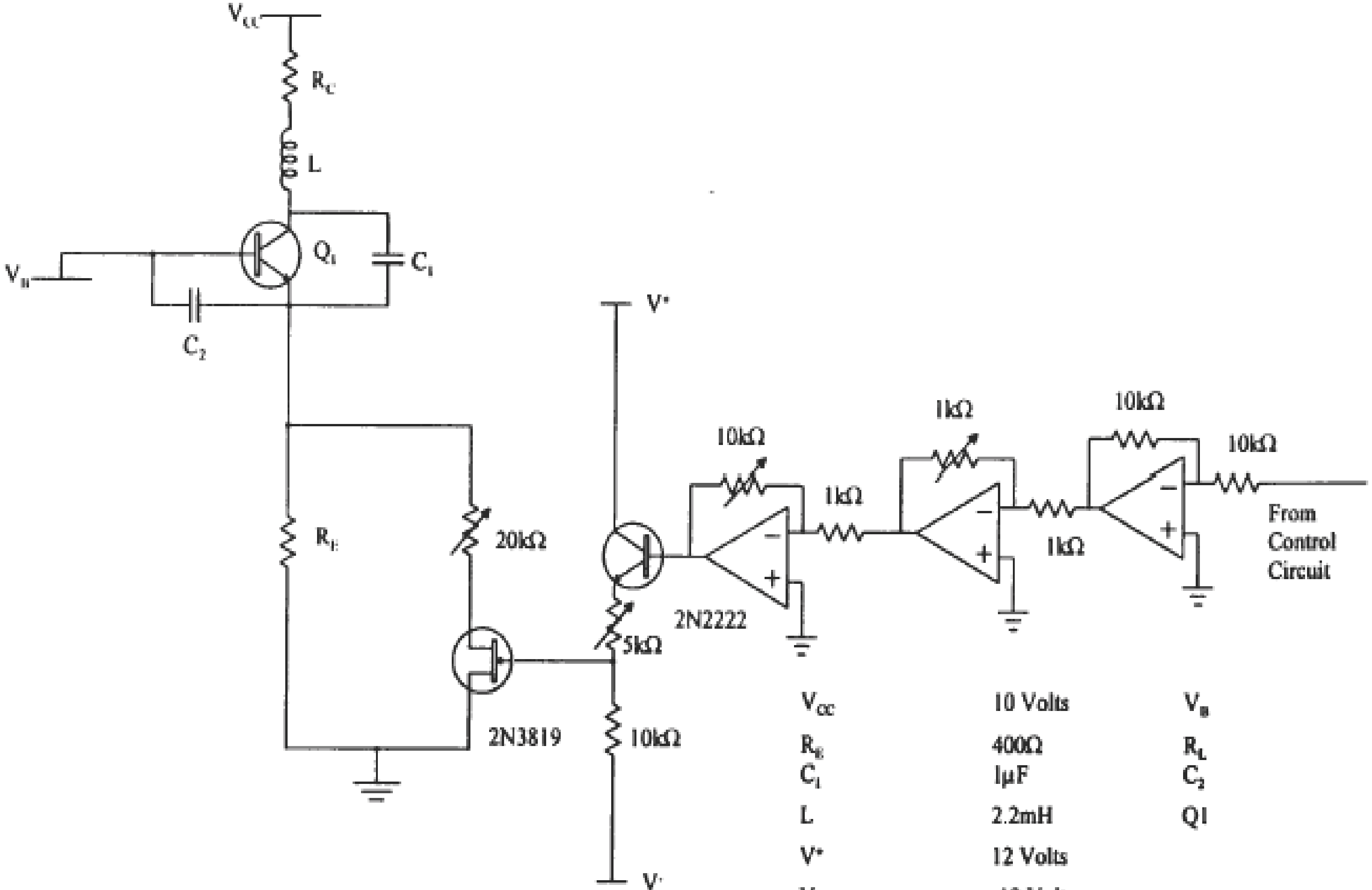
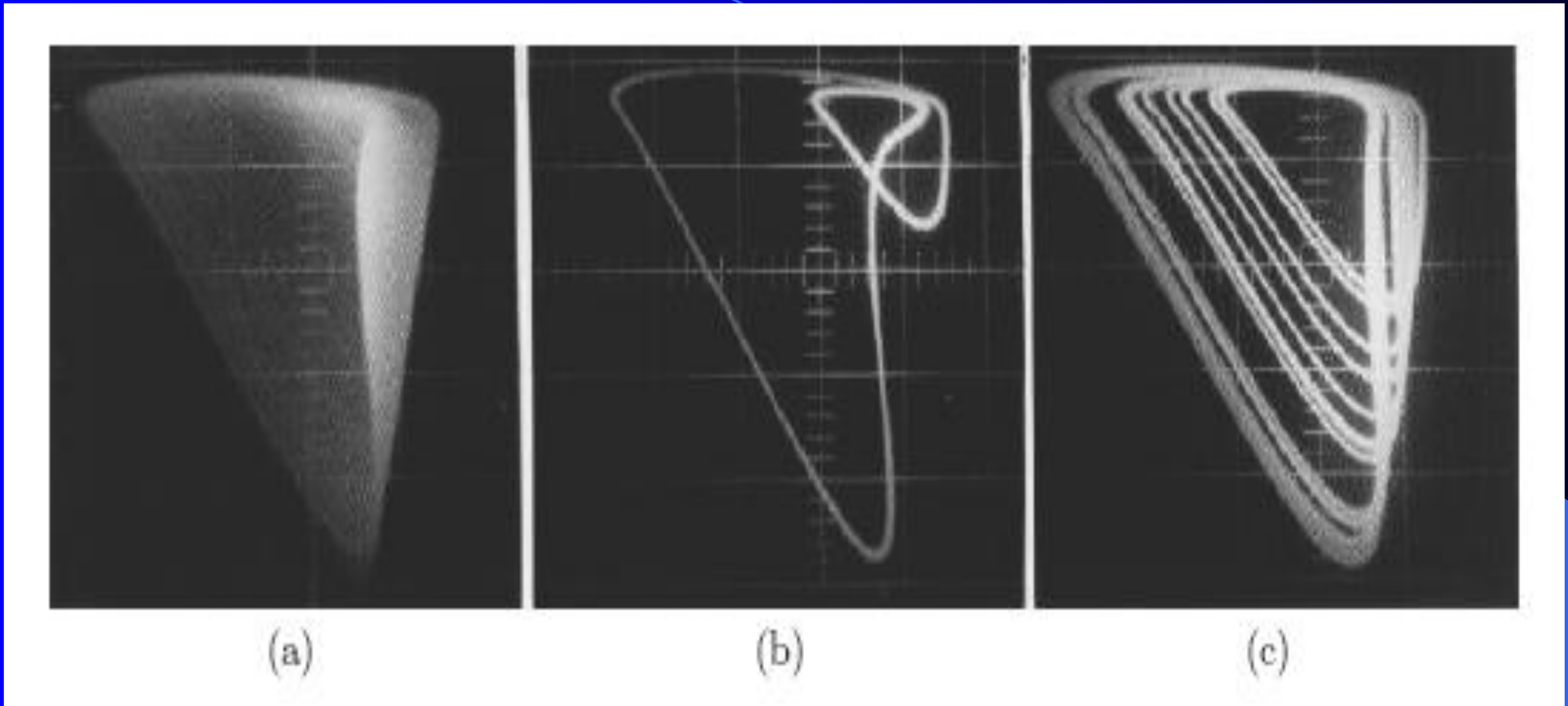


Figure 23: Period-4 orbit stabilized in Chua's circuit using improved chaos controller

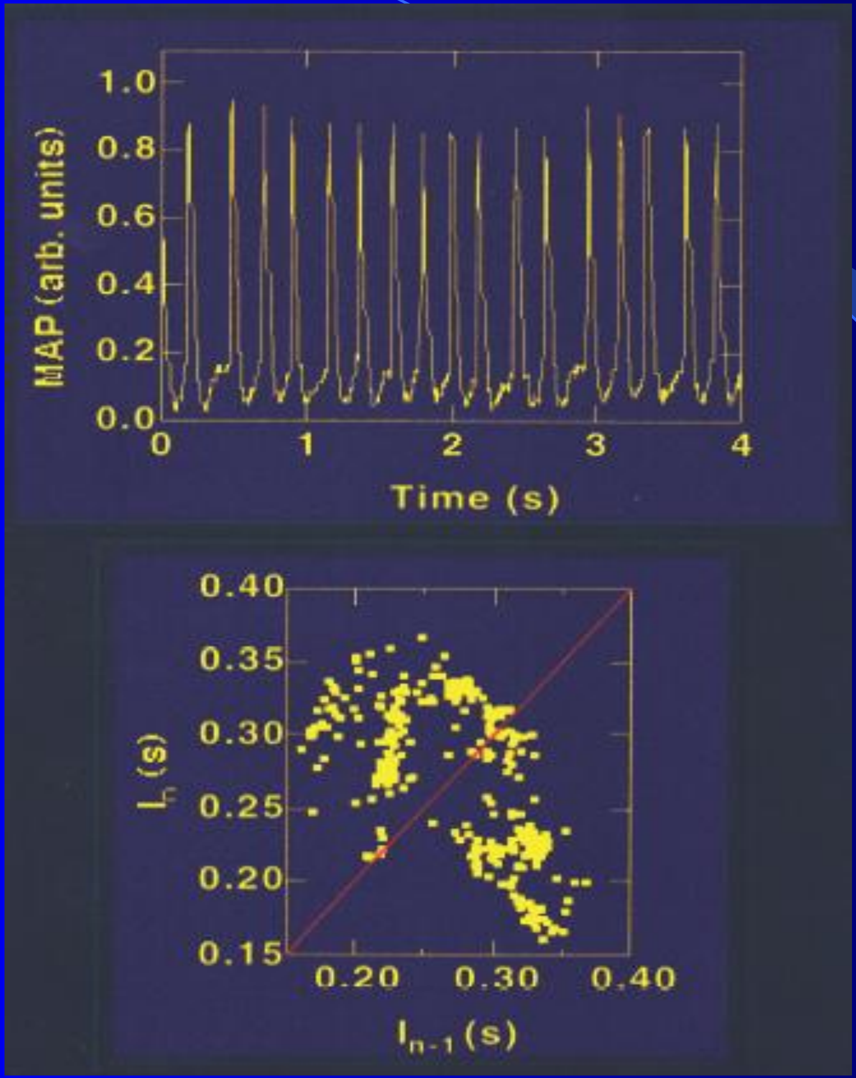


V_{cc}	10 Volts	V_B	3.5 Volts
R_E	400Ω	R_L	35Ω
C_1	1μF	C_2	1μF
L	2.2mH	Q1	2N2222A
V+	12 Volts		
V-	-12 Volts		

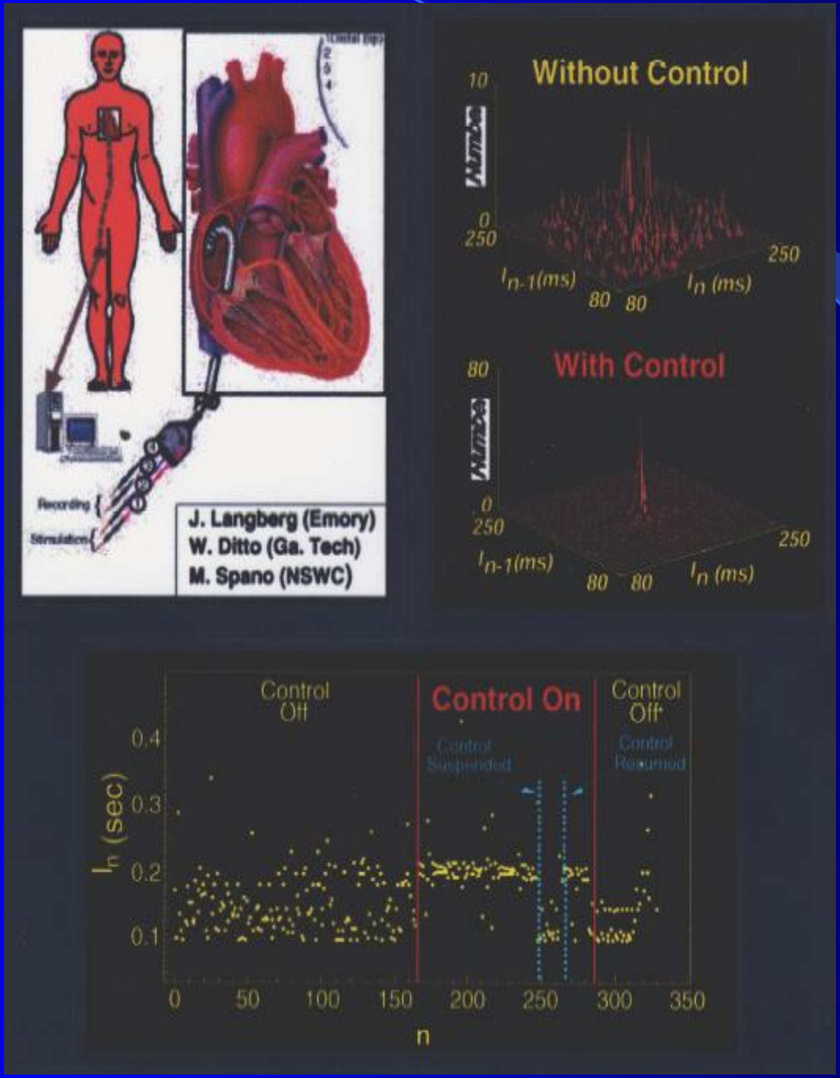
Control of chaos in the Colpitts oscillator



Cardiac chaos

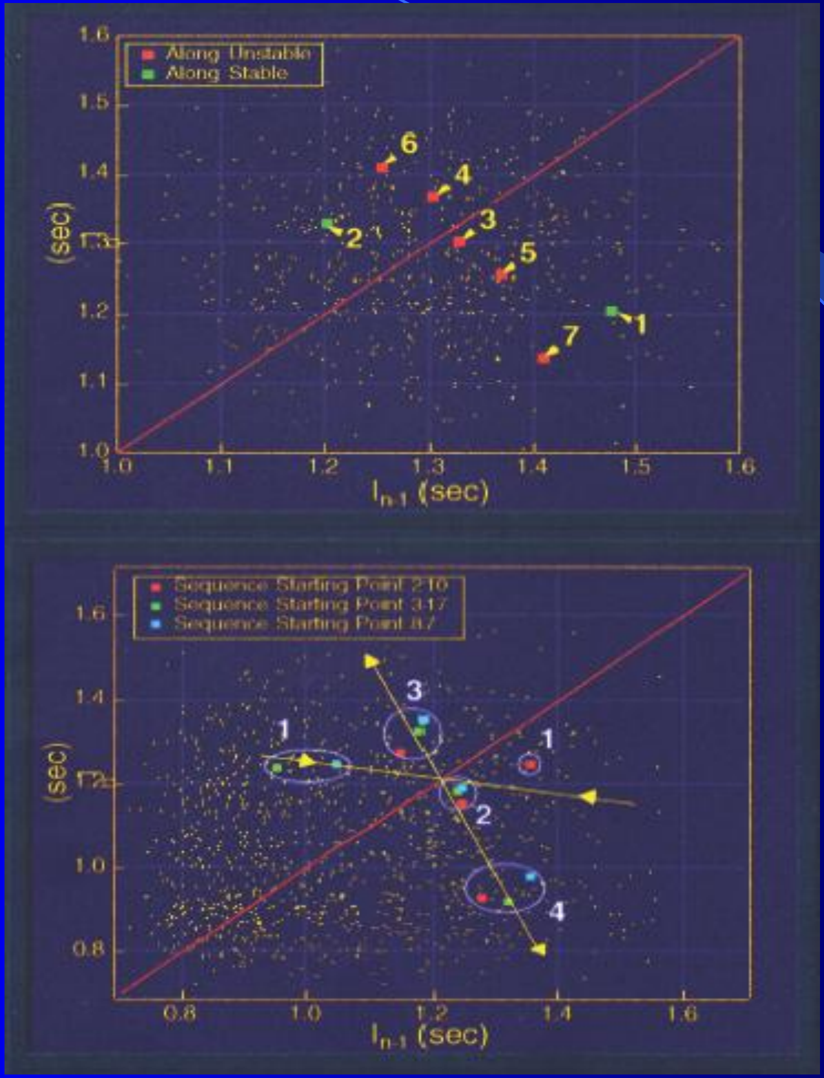


Controlling atrial fibrillation in humans

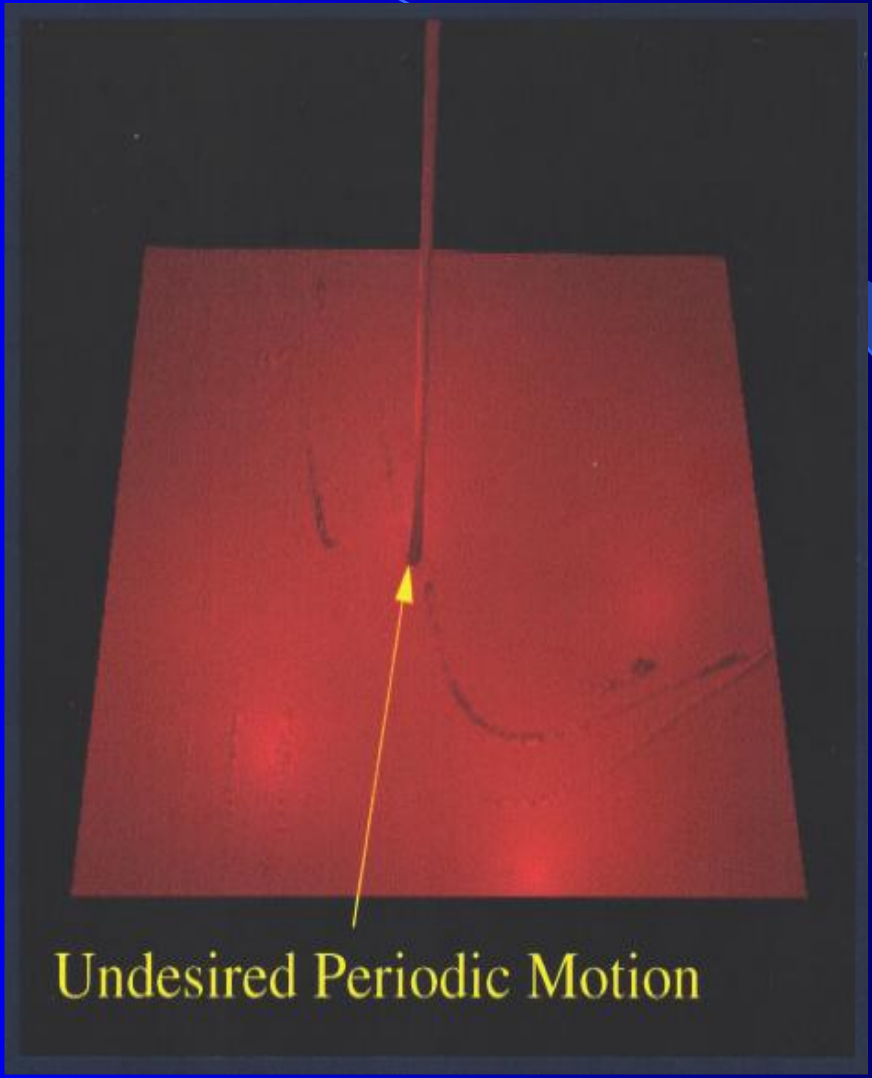


Chaos-based epilepsy treatment

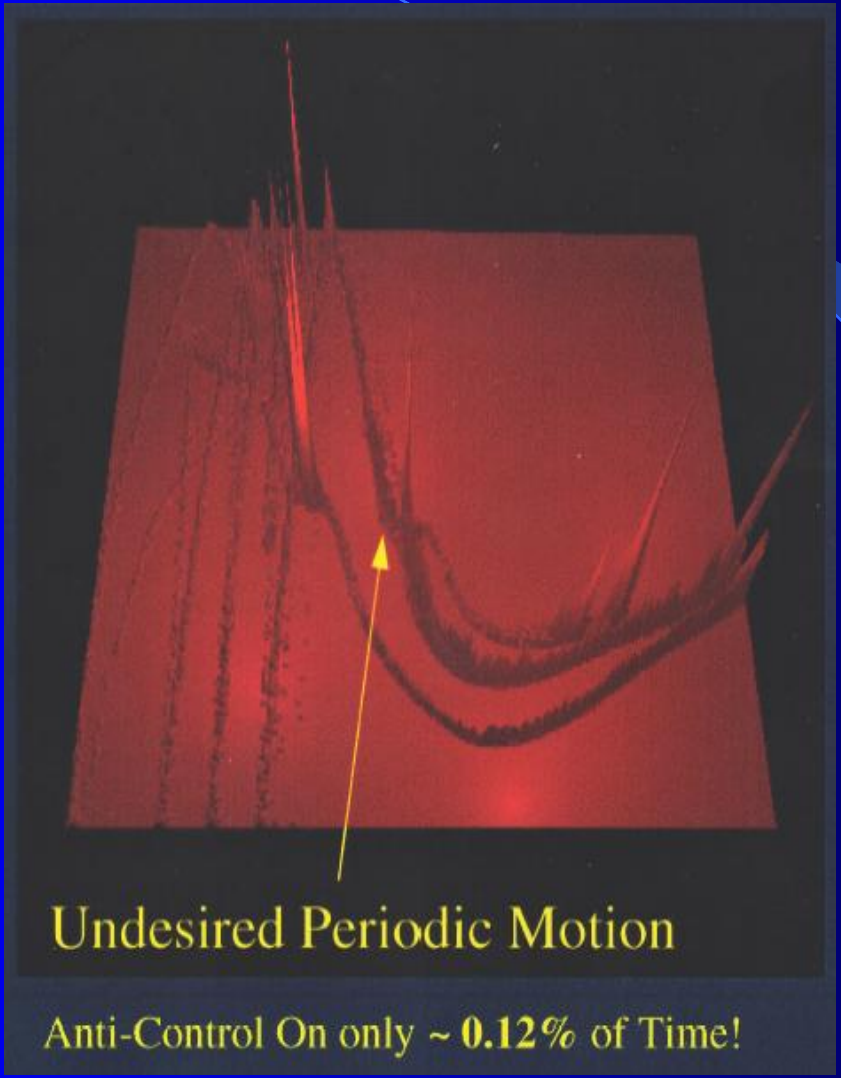
Unstable fixed points



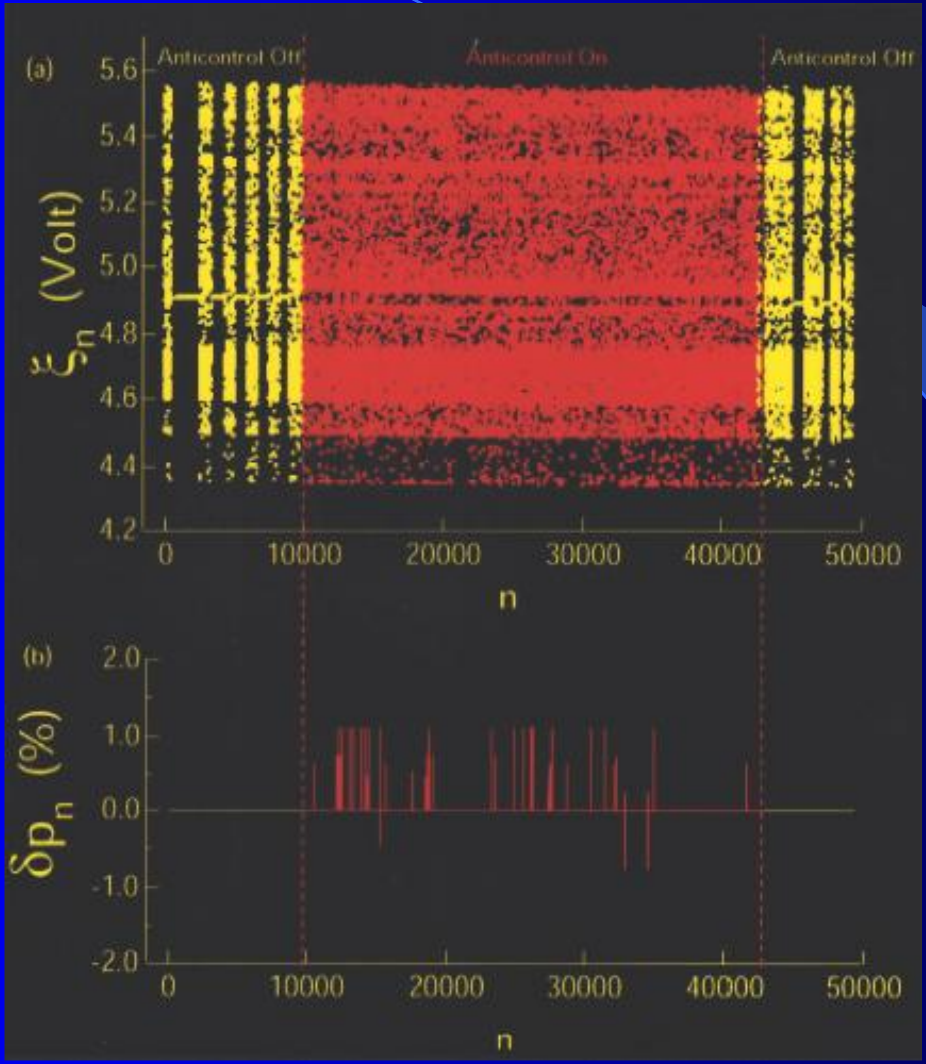
Anti-control „off"



Anti-control „on”

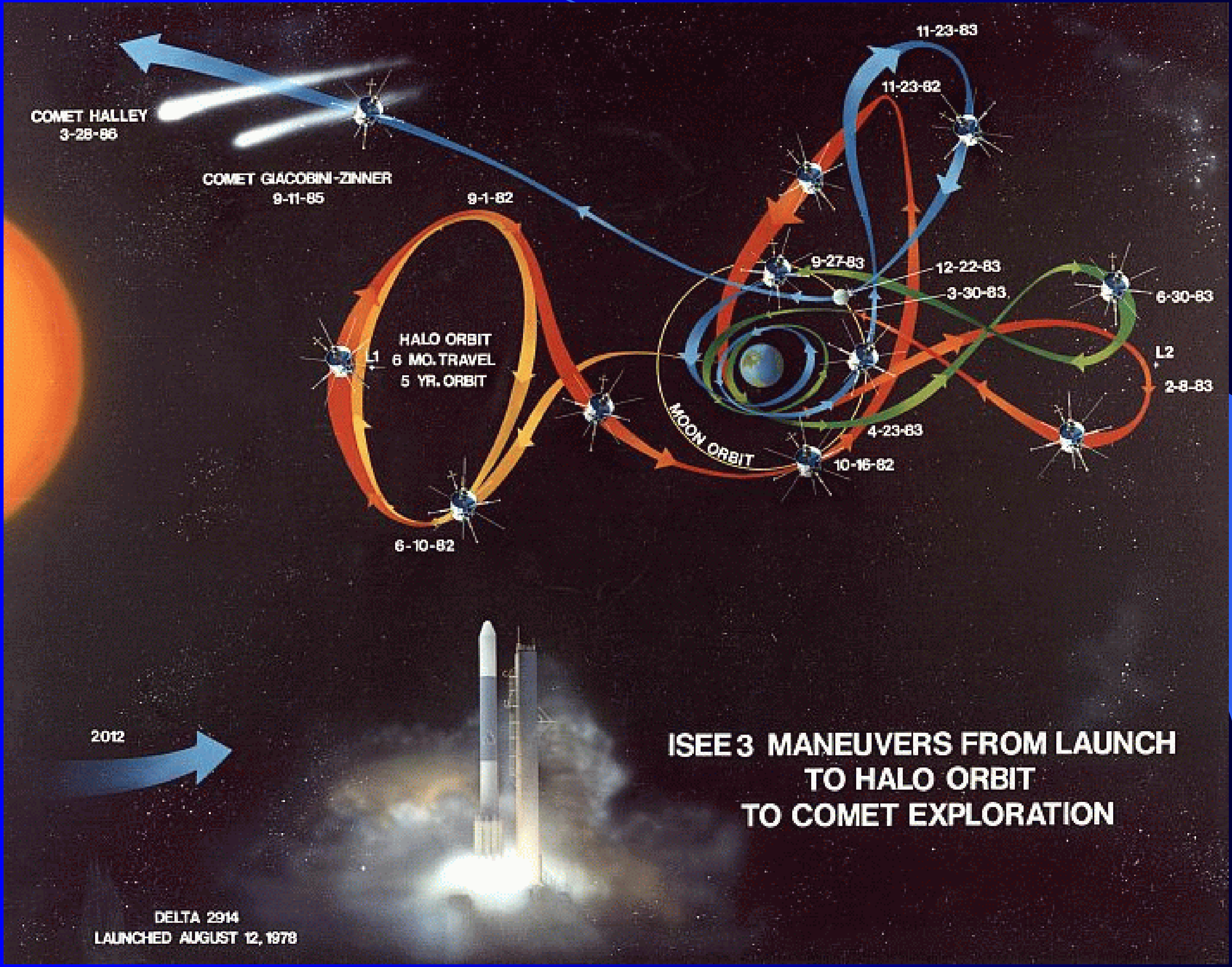


Chaos anti-control



Laboratory experiments

Targetting


COMET HALLEY
3-28-86
COMET GIACOBINI-ZINNER
9-11-85
9-1-82
11-23-83
11-23-82
9-27-83
12-22-83
3-30-83
6-30-83
HALO ORBIT
6 MO. TRAVEL
5 YR. ORBIT
L1
L2
2-8-83
MOON ORBIT
4-23-83
10-16-82
6-10-82
2012
ISEE 3 MANEUVERS FROM LAUNCH
TO HALO ORBIT
TO COMET EXPLORATION
DELTA 2914
LAUNCHED AUGUST 12, 1978

Infinitesimal dynamics

The evolution of a perturbation $\epsilon(t)$ about an initial condition $\mathbf{x}_0$ is described by an m-dimensional Taylor expansion about $\mathbf{x}_0$:

$$\dot{\epsilon}^{(i)}(t) = \epsilon_0^{(j)} \frac{\partial f^{(i)}}{\partial x^{(j)}}(\mathbf{x}_0) + \epsilon_0^{(j)} \epsilon_0^{(k)} \frac{1}{2!} \frac{\partial^2 f^{(i)}}{\partial x^{(j)} \partial x^{(k)}}(\mathbf{x}_0) + \cdots,$$

where $\epsilon_0 = \epsilon(t_0)$ and repeated indices denote summation

First order approximation describing the linear evolution of the perturbed trajectory i

$$\dot{\epsilon}(t) = \mathcal{J}(\mathbf{x}(t))\epsilon(t),$$

where elements of the Jacobian matrix $\mathcal{J}$ are given by $\mathcal{J}_{ij} = \partial f^{(i)} / \partial x^{(j)}$

The accuracy of this approximation depends on

- (i) the particular initial condition $\mathbf{x}_0$,
- (ii) the magnitude of ϵ_0 and
- (iii) the orientation of ϵ_0

The uncertainty after an arbitrary time $\tau = k\Delta t$ may be linearly approximated by

$$\epsilon(t_0 + \tau) = \mathcal{M}(\mathrm{x}_0, \tau)\epsilon(t_0),$$

where $\mathcal{M}(\mathrm{x}_0, \tau)$ is the *linear propagator* defined as

$$\mathcal{M}(\mathrm{x}_0, \tau) = \exp\left[\int_{t_0}^{t_0+\tau} \mathcal{J}(\mathrm{x}(t))dt\right].$$

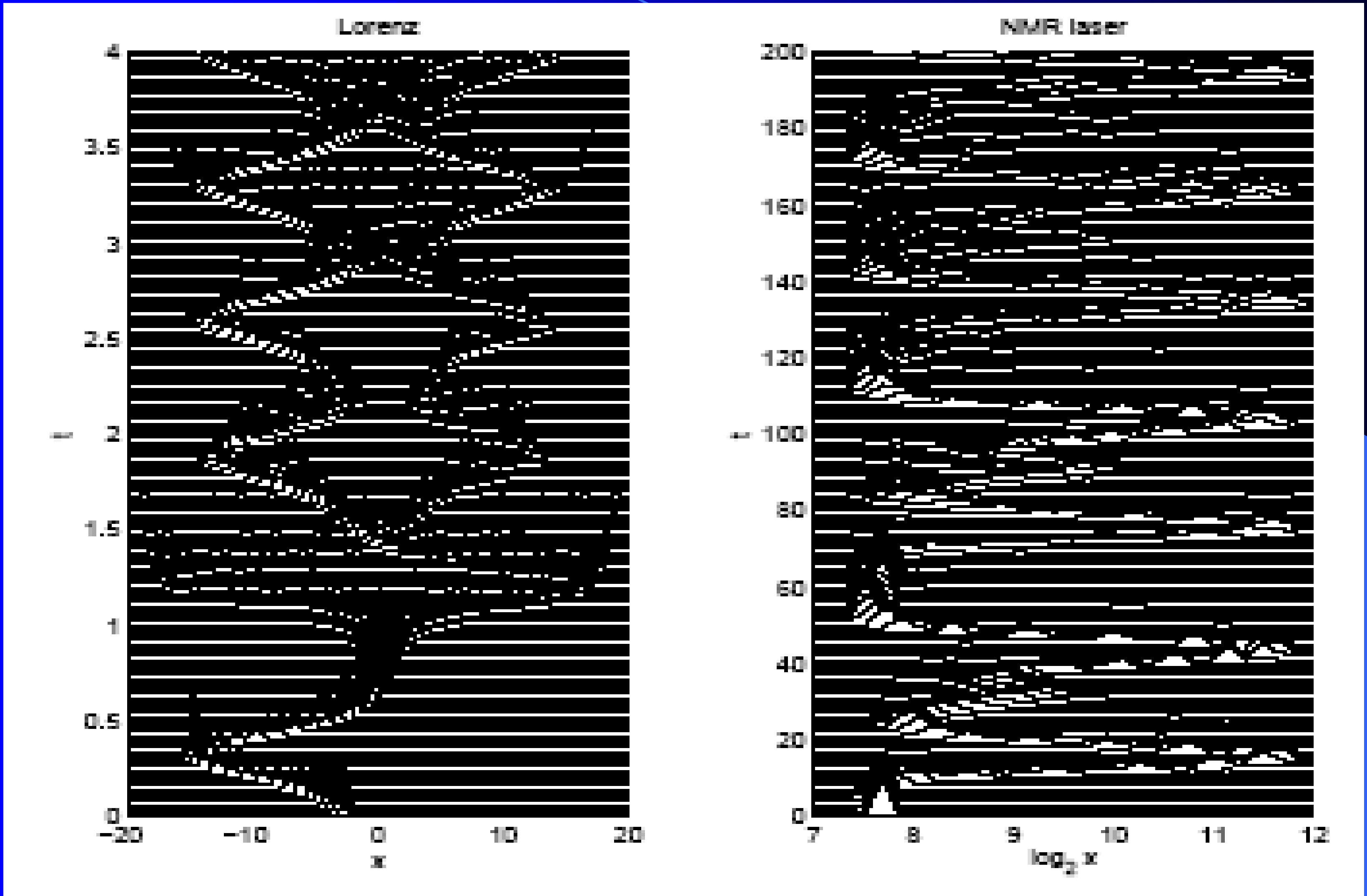
The magnitude of the uncertainty after a time τ has passed is

$$\epsilon(t_0 + \tau) = ||\epsilon(t_0 + \tau)|| = \sqrt{\epsilon(t_0 + \tau)^T \epsilon(t_0 + \tau)},$$

or

$$\epsilon(t_0 + \tau) = \sqrt{\epsilon(t_0)^T \mathcal{M}(\mathrm{x}_0, \tau)^T \mathcal{M}(\mathrm{x}_0, \tau)\epsilon(t_0)}.$$

PDF evolution



Estimation of Lyapunov exponents

The finite time Lyapunov exponents $\lambda_i^{(k)}$ depend on the singular values of

$$\begin{aligned} A(x_0, k\Delta t) &= \mathcal{M}(x_0, k\Delta t)^T \mathcal{M}(x_0, k\Delta t) \\ &= \mathcal{M}(x_0, \Delta t)^T \cdots \mathcal{M}(x_{k-1}, \Delta t)^T \mathcal{M}(x_{k-1}, \Delta t) \cdots \mathcal{M}(x_0, \Delta \end{aligned}$$

For short times, singular value decomposition (SVD) may be used to determine this s value spectrum

For longer times, computation of the matrix product $A(x_0, k\Delta t)$ may be complicated

- (i) becoming singular, and
- (ii) having individual elements which become extremely large, thereby causing numerical overflow problems

Recursive QR decomposition

The QR decomposition of an arbitrary matrix X yields

$$\mathbf{X} = \mathbf{Q}\mathbf{R},$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper-triangular matrix

A factorised product of n matrices

$$\mathbf{A} = \mathbf{A}_n \mathbf{A}_{n-1} \cdots \mathbf{A}_2 \mathbf{A}_1,$$

is recursively replaced by n QR decompositions

$$\mathbf{A}_i \mathbf{Q}_{i-1} = \mathbf{Q}_i \mathbf{R}_i,$$

where $\mathbf{Q}_0 = \mathbf{I}$ and $i = 1, 2, \ldots, n$, yielding

$$\mathbf{A} = \mathbf{Q}_n \mathbf{R}_n \mathbf{R}_{n-1} \cdots \mathbf{R}_2 \mathbf{R}_1.$$

Applying this recursive QR decomposition yields

$$\mathbf{A}^{(1)} = \mathbf{Q}_{2k}^{(1)} \mathbf{R}_{2k}^{(1)} \mathbf{R}_{2k-1}^{(1)} \cdots \mathbf{R}_2^{(1)} \mathbf{R}_1^{(1)},$$

where the superscript denotes the recursive iteration step

Recursive QR decomposition -2

A new matrix $\mathbf{A}^{(2)}$ satisfying

$$\mathbf{A}^{(2)} = \mathbf{Q}_{2k}^{(1)T} \mathbf{A}^{(1)} \mathbf{Q}_{2k}^{(1)},$$

is computed which has the same $\mathbf{R}$ matrices as $\mathbf{A}^{(1)}$, but has the matrix $\mathbf{Q}_{2k}^{(1)}$ on the

$$\mathbf{A}^{(2)} = \mathbf{R}_{2k}^{(1)} \mathbf{R}_{2k-1}^{(1)} \cdots \mathbf{R}_{2}^{(1)} \mathbf{R}_{1}^{(1)} \mathbf{Q}_{2k}^{(1)}.$$

The matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ have the same eigenvalues because of their relationshi
shown above

This process of QR decomposition is repeated p times yielding the sequence of matri
$\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(p)}$.

Recursive QR decomposition -3

As p increases, $Q_{2k}^{(p)}$ converges to the identity and $\mathbf{A}^{(p)}$ is then upper-triangular

The Lyapunov exponents are related to the diagonal elements of the $R_j^{(p)}$s:

$$\lambda_i^{(k)}(x_0) = \frac{1}{2k}\sum_{j=1}^{2k}\log_2\left[R_j^{(p)}\right]_{ii}.$$

This results from the fact that the eigenvalues of a product of upper-triangular matrices equals the product of the eigenvalues of the individual matrices

In practice, numerical calculations with $p < 10$ are usually sufficient to obtain elements of $Q_{2k}^{(p)}$ which differ by less than 10^{-6}

Conjecture of Kaplan-Yorke

Kaplan & Yorke (1979) conjectured that there is a relationship between the Lyapunov exponents and the fractal dimension of a typical chaotic attractor

The Kaplan-Yorke dimension (or Lyapunov dimension) is defined as

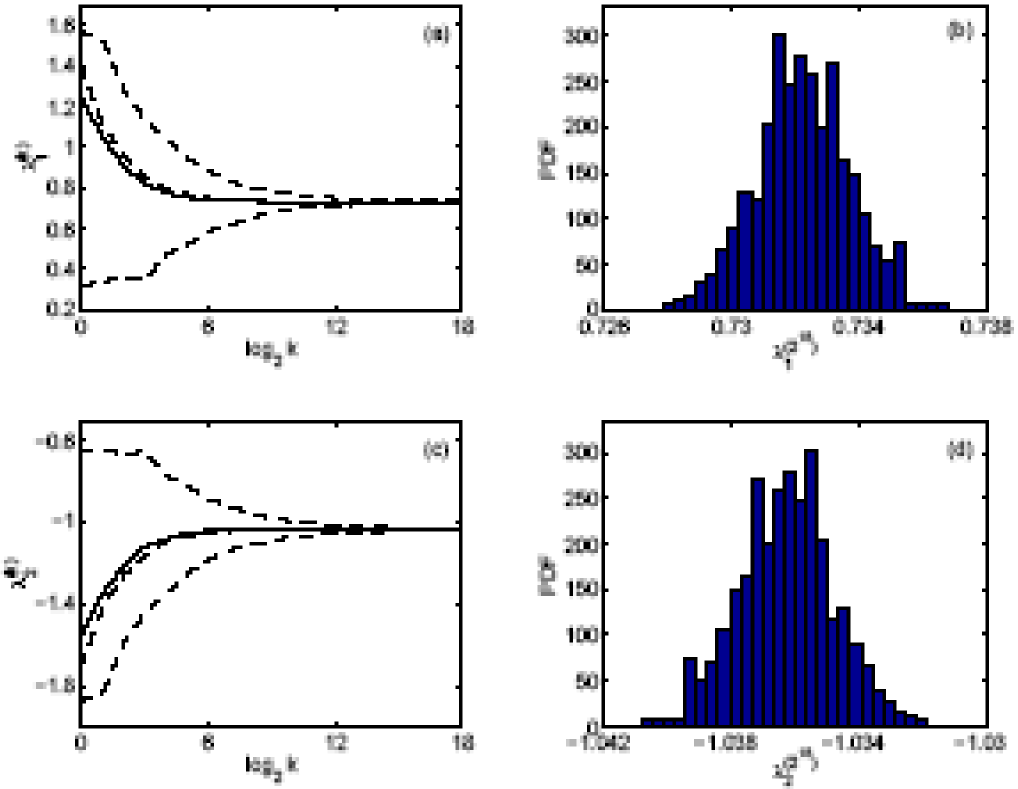
$$D_{\mathrm{Lyap}} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|},$$

where k is defined such that $\sum_{i=1}^{k} \lambda_i \geq 0$ and $\sum_{i=1}^{k+1} \lambda_i < 0$

The conjecture states that $D_{Lyap} = D_1$, the information dimension

While the conjecture may be proved for some $2D$ maps, counter-examples exist for hi dimensional systems

Convergence



Lyapunov spectra for the Ikeda map: (a) convergence of $\lambda_1^{(k)}$ with iteration time k, (b) PDF of $\lambda_1^{(k)}$ for $k = 2^{18}$, (c) convergence of $\lambda_2^{(k)}$ with iteration time k, and (d) PDF of $\lambda_2(k)$ for $k = 2^{15}$. In both (a) and (c), the dashed lines reflect the 5%, 50%, and 95% percentiles, and the solid line is the mean $\langle \lambda_i^{(k)} \rangle$

Consistency test 1 (Henon)

Hénon map: $D_2 = 1.2$, $D_Lyap = D_{KY} = D_1 = 1.2583$ and $D_0 = 1.26$

Satisfies Renyi dimensions: $D_2 \leq D_1 \leq D_0$

In dissipative systems, the state space volume contracts at the rate given by the sum Lyapunov exponents

This contraction can be equated with that from the stability analysis formula for maps:

$$V(t+1) = |\det \mathcal{J}| V(t)$$

Equating these gives:

$$\sum_{i=1}^{M} \Lambda_i = \log_2 |\det \mathcal{J}|$$

The Hénon map has contant contraction everywhere given by $|\det \mathbf{J}| = b = 0.3$ and $\log_2 |\det \mathbf{J}| = -1.7370$

Estimated Lyapunov exponents: $\sum_{i=1}^{M} \lambda_i = -1.7369$

Consistency test 2 (Lorenz)

Recall that the Lorenz equations are

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= -xz + rx - y,\\ \dot{z} &= xy - bz,\end{aligned}$$

where $\sigma = 10$, $b = 8/3$, and $r = 28$

The state space volume contracts at the rate given by the sum of the Lyapunov exponents

This contraction can be equated with that from the stability analysis formula for flows:

$$V(t+\tau) = \exp(\nabla \mathcal{J} \tau) V(t)$$

Equating these gives:

$$\sum_{i=1}^{M} \Lambda_i = \log_2 \exp(\nabla \mathcal{J})$$

The Lorenz flow has constant contraction everywhere given by
$\nabla \mathcal{J} = -(\sigma + 1 + b) = -(10 + 1 + \frac{8}{3}) = -\frac{41}{3}$ and $\log_2 \exp(\nabla \mathcal{J}) = -19.7168$

Estimated Lyapunov exponents: $\sum_{i=1}^{M} \lambda_i = -19.7167$

Box-counting dimension

The *box-counting dimension* of a self-similar point set is calculated by counting the number of hyper-cubes $N(\epsilon)$ of side ϵ required to cover the set.

For a self-similar set $N(\epsilon) \propto \epsilon^{-D_0}$ as $\epsilon \to 0$ and

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}.$$

Recall the invariant measure $d\mu(\mathbf{x})$ generated by chaotic dynamics on an attractor

Let $p_\epsilon(\mathbf{x}) = \int_{B_\epsilon(\mathbf{x})} d\mu(\mathbf{x})$ denote the probability of finding a point along a typical orbit in a ball $B_\epsilon(\mathbf{x})$ of radius ϵ around $\mathbf{x}$

Renyi dimension

Renyi (1971) defined a family of generalised dimensions (Renyi dimensions) which di the way regions with different densities are weighted

These dimensions give more weight to regions which are visited more frequently and contain larger fractions of the measure

A generalised correlation integral is defined by

$$C_q(\epsilon) = \int p_\epsilon(x)^{q-1} d\mu(x) = \langle p_\epsilon^{q-1} \rangle,$$

so that for a self-similar set, $C_q(\epsilon) \propto \epsilon^{(q-1)D_q}$ as $\epsilon \to 0$

The *Renyi dimensions* are then given by

$$D_q = \frac{1}{q-1} \lim_{\epsilon \to 0} \frac{\ln C_q(\epsilon)}{\ln \epsilon},$$

where D_q is a decreasing function of q: $D_{q1} \leq D_{q2}$ if $q_1 > q_2$

A measure for which D_q varies with q is called a *multi-fractal* measure

For $q = 0$, the box-counting dimension D_0 is recovered

Information dimension

Applying l'Hopital's rule yields the *information dimension* D_1:

$$D_1 = \lim_{\epsilon \to 0} \frac{\langle \ln p_i \rangle}{\ln \epsilon},$$

where $\langle \ln p_i \rangle$ is the average information needed to specify a point x with accuracy ϵ

Note that the local scaling of the natural measure in a small ball $B_\epsilon(x)$ is described by D_1

Let the *pointwise dimension* of the measure μ at x be

$$\alpha(x) = \lim_{\epsilon \to 0} \frac{\ln \mu\,[B_\epsilon(x)]}{\ln \epsilon}.$$

It can be shown that if x is some point on a typical orbit, then $\alpha(x) = D_1$ and therefore $\mu\,[B_\epsilon(x)] = \epsilon^{D_1}$

Correlation dimension

The *correlation dimension* D_2 may be estimated from experimental data. This may b[e]
by first writing

$$C_2(\epsilon) = \int d\mu(\mathbf{x})\left[\int d\mu(\mathbf{y})\Theta(\epsilon - \|\mathbf{x}-\mathbf{y}\|)\right],$$

where Θ is the Heaviside step function: $\Theta(x) = 0$ if $x \leq 0$ and $\Theta(x) = 1$ if $x > 0$

For a finite sequence of points $\{\mathbf{x}_i\}_{i=1}^{N}$, $C_2(\epsilon)$ may be approximated by

$$\hat{C}_2(\epsilon) = \frac{2}{N(N-1)}\sum_{i=1}^{N}\left[\sum_{j=i+1}^{N}\Theta(\epsilon - \|\mathbf{x}_i - \mathbf{x}_j\|)\right].$$

Expect $\hat{C}_2(\varepsilon)$ to scale like a power law: $\hat{C}_2(\epsilon) \propto \epsilon^{D_2}$

Let $d_2(\varepsilon, N) = \frac{\partial \ln \hat{C}_2(N,\varepsilon)}{\partial \ln \varepsilon}$ and define

$$D_2 = \lim_{\varepsilon \to 0}\lim_{N \to \infty} d_2(\varepsilon, N)$$

Implementation of the algorithm

Consider the plot of $\log \hat{C}_2(\varepsilon, N)$ versus $\log \varepsilon$

Need to find a scaling region over which the slope is constant

Corresponds to a plateau in $d_2(\varepsilon, N)$ versus ε

Estimation is limited at small ε by statistics (not enough data points)

For experimental observations, the estimation is also limited by *noise* at small ε

Estimation is limited at large ε by the finite size of the attractor ($D_2 \to 0$ as ε becomes greater than the size of the attractor)

Criteria suggest that number of data points required N, grows exponentially with the dimension D_2:

- Tsonis criterion: $N \sim 10^{2+\frac{2}{5}D_2}$
- Smith criterion: $N \sim D_2^{42}$

Minimum data

Suppose the data are uniformly distributed on the unit interval

$$Prob(|x-y|<\varepsilon)=\varepsilon(2-\varepsilon)$$

Probability of x and y both being in an M-dimensional hyper-cube of size ε is

$$Prob(|\mathbf{x}-\mathbf{y}|<\varepsilon)=[Prob(|x-y|<\varepsilon)]^M=[\varepsilon(2-\varepsilon)]^M$$

And $C_2(\varepsilon)=Prob(|\mathbf{x}-\mathbf{y}|<\varepsilon)=[\varepsilon(2-\varepsilon)]^M$ gives

$$\hat{D}_2=\frac{d\log C(\varepsilon)}{d\log\varepsilon}\approx M\left(1-\frac{\varepsilon}{2}\right)$$

We want $\hat{D}_2\geq QM$ where $0\leq Q\leq 1$

So $\varepsilon_{max}=2(1-Q)$

If $R=\varepsilon_{max}/\varepsilon_{min}$ then $\varepsilon_{min}=2(1-Q)/R$ so

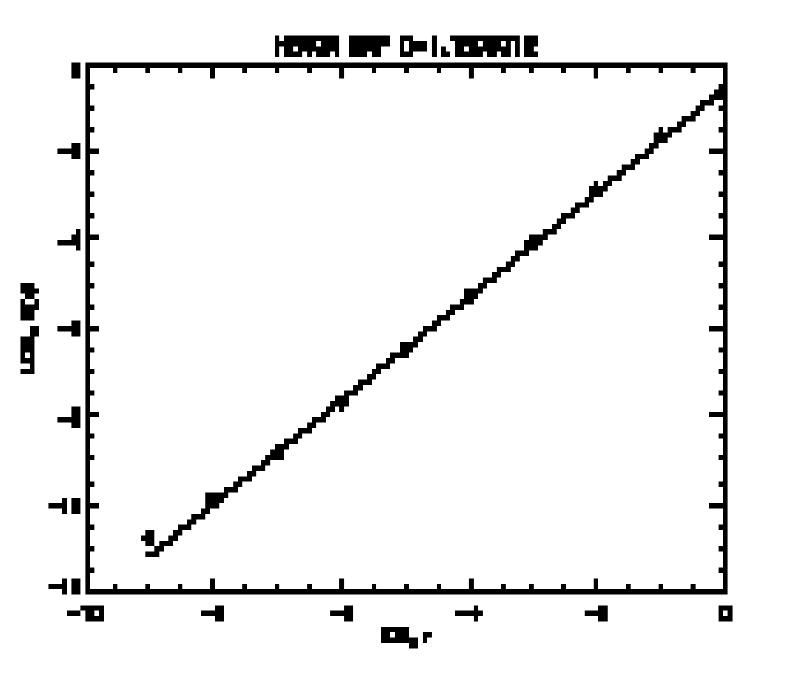
$$N_{min}\geq\left(\frac{1}{\varepsilon_{min}}\right)^M=\left(\frac{R}{2(1-Q)}\right)^M\approx O(10^M)$$

Hénon two-dimensional map:

$$x_{n+1} = 1 + y_n - a x_n^2$$
$$y_{n+1} = b x_n$$

Parameter values: $a = 1.4$ and $b = 0.3$

$D_2 \approx 1.2 \pm 0.1$



Oseledec theorem

For a time $t = k\Delta t$, define

$$\mathcal{O}(\mathbf{x}, k\Delta t) = \left[\mathcal{M}(\mathbf{x}, k\Delta t)^T \mathcal{M}(\mathbf{x}, k\Delta t)\right]^{\frac{1}{2k\Delta t}},$$

and

$$\mathcal{O}(\mathbf{x}) = \lim_{k \to \infty} \mathcal{O}(\mathbf{x}, k\Delta t)$$

where X^T denotes the transpose of X

Oseledec (1968) proved that if the limit $k \to \infty$ exists, then under a wide range of con
the eigenvalues of $\mathcal{O}(\mathbf{x})$ are independent of $\mathbf{x}$ for almost all $\mathbf{x}$ in the same basin of at

An m-dimensional dynamical system thus has m Lyapunov exponents, Λ_i, $i = 1, 2, .$
defined via σ_i, the eigenvalues of $\mathcal{O}(\mathbf{x})$, as $\Lambda_i = \log_2(\sigma_i)$, $i = 1, 2, .., m$; by conventi
$\Lambda_i \geq \Lambda_j$ for $i < j$

The eigenvalues of the matrix $\mathcal{M}^T\mathcal{M}$ are simply the squares of the singular values

Global Lyapunov exponents

In the limit $k \to \infty$, the logarithm of the first singular value of $\mathcal{M}(\mathbf{x}, \Delta t)$, σ_1 approaches the first global Lyapunov exponent, Λ_1:

$$\Lambda_1 = \lim_{k \to \infty} \frac{1}{k \Delta t} \log_2(\sigma_1)$$

All Lyapunov exponents are expressed using the logarithm to the base two, thus the exponents have units of bits per second

Lyapunov exponents are often used for the characterisation of a dynamical system's behaviour. The classifications for a three-dimensional flow are as follows:

Λ_1	Λ_2	Λ_3	Classification	Topological dimension
$-$	$-$	$-$	Fixed point attractor	zero
0	$-$	$-$	Limit cycle attractor	one
0	0	$-$	Two-torus attractor	two
$+$	0	$-$	Chaotic attractor	three

A dynamical system is said to be *chaotic* if there exists average expansion for at least one direction and almost every $\mathbf{x}_0$, that is, if $\Lambda_1 > 0$

Finite-time Lyapunov exponents

The *finite time* Lyapunov exponents are defined as

$$\lambda_i^{(k)}(\mathbf{x}) = \frac{1}{k\Delta t}\log_2(\sigma_i^{(k)})$$

Note that the finite time Lyapunov exponent $\lambda_1^{(k)}$ describes the maximum possible lin growth over the time $\tau = k\Delta t$ for which the linear propagator was defined

The growth for times shorter (longer) than $k\Delta t$ need not be smaller (larger) than thos implied by $\lambda_1^{(k)}$

Finite time Lyapunov exponents are statistical estimates and as such should be accompanied by an evaluation of their uncertainty

This is particularly important when determining the sign of the leading global Lyapunc exponent, Λ_1

Sensitive dependence on initial conditions

An exploration of sensitivity to initial conditions involves investigating perturbations ab fiducial trajectory $\mathbf{x}(t)$, which originates at the initial condition $\mathbf{x}_0$

While each initial condition uniquely determines its future trajectory in a deterministic system, imperfect measurements of initial conditions give rise to a PDF of consistent conditions

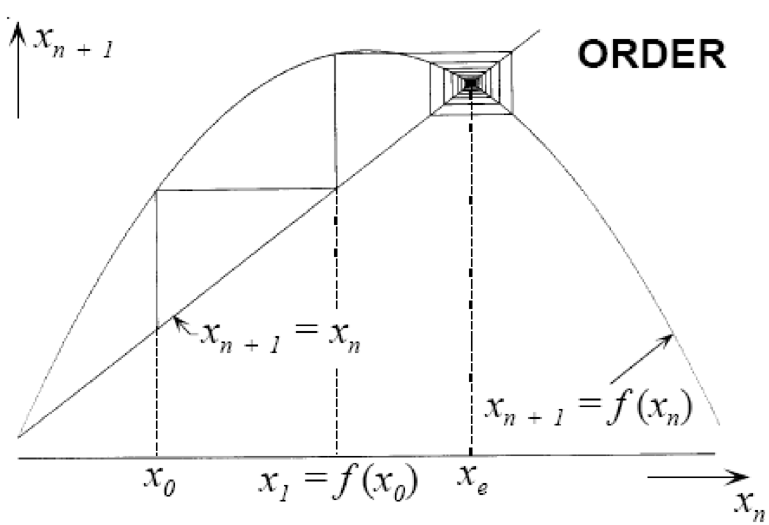
The evolution of this PDF depends on both the initial shape of the PDF and the initial location in state space

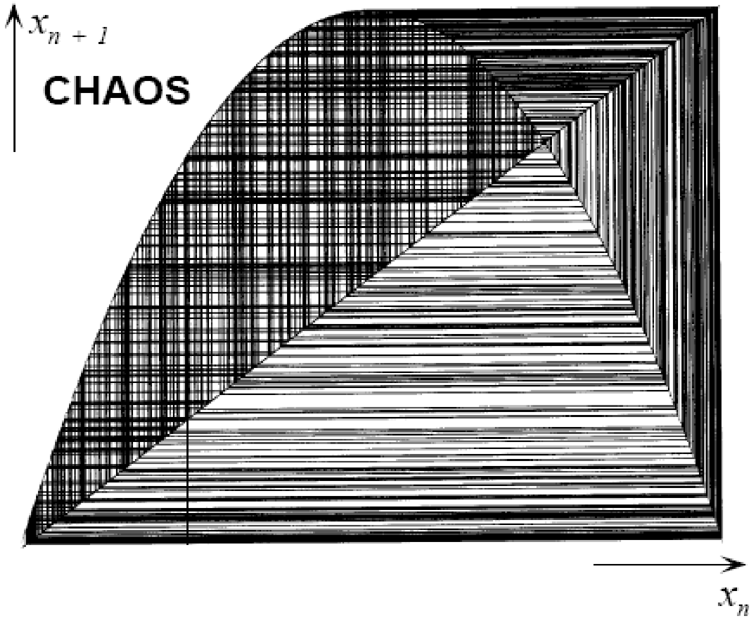
The disparity between trajectories arising from different (but equally likely) initial cond corresponds with the PDF spreading out over the system's attractor

Even in the case of a perfect model and a perfect initial PDF (all members on the syst attractor), predictability is eventually lost when the PDF is indistinguishable from the system's invariant measure

Logistic map

- Originated as a population dynamics model (Verhulst, 1844 & 1847).
- Dynamical system (1-D map): $x_{n+1} = \mu x_n(1 - x_n) =: f(x_n)$, $0 \leq \mu \leq 4$
- Sample orbits from this map's rich set of dynamics:



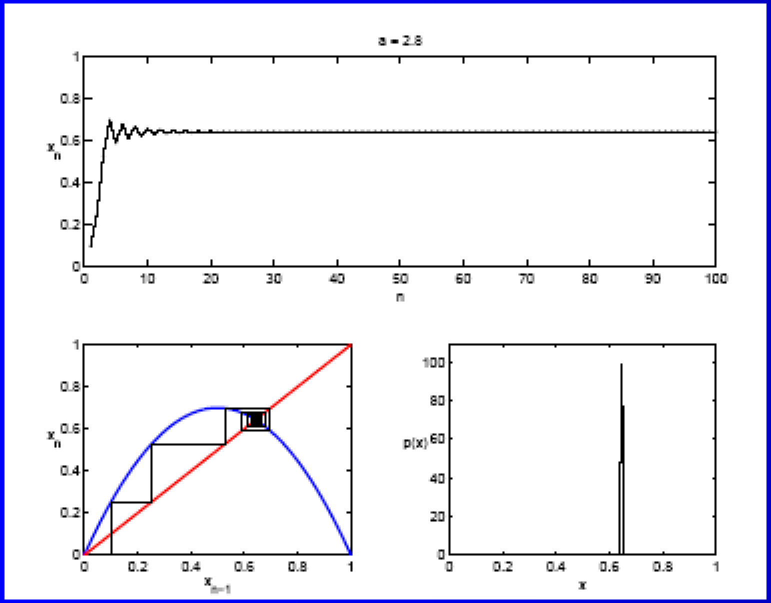


Stable fixed point x_e
$|f'(x_e)| < 1$
$\mu < 4$

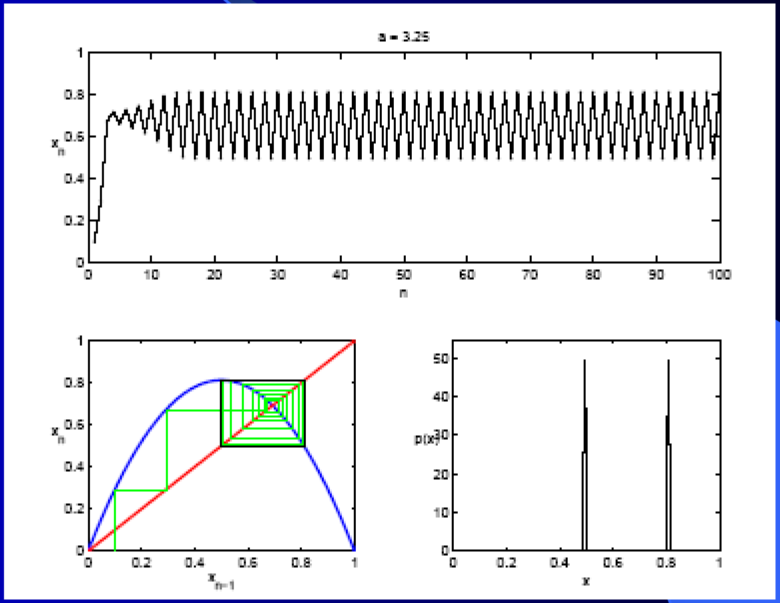
Unstable fixed point x_e
$|f'(x_e)| > 1$
$\mu = 4$

Logistic map

- a=2.8

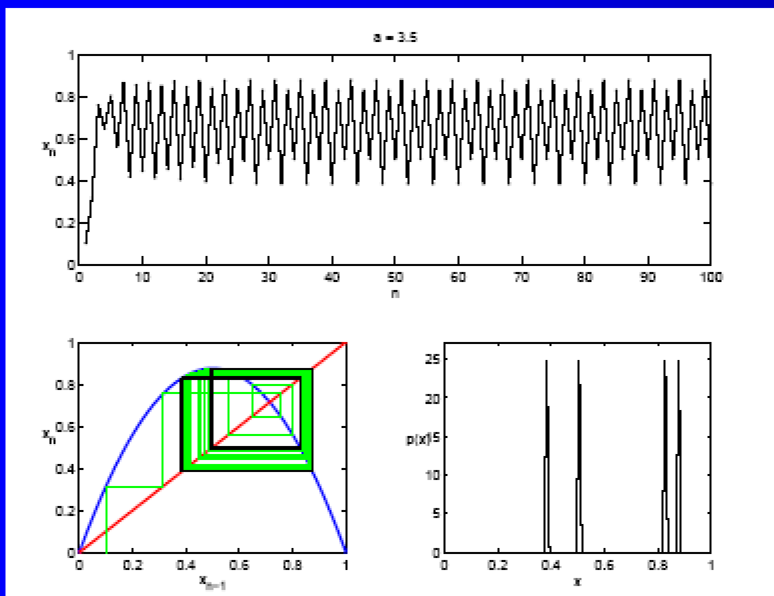


- a=3.25



Logistic map (continued)

- a=3.5



- a=3.8