Synchronization in electrical and biological oscillatory networks

Prof. Fernando Corinto
(Politecnico di Torino - Italy)

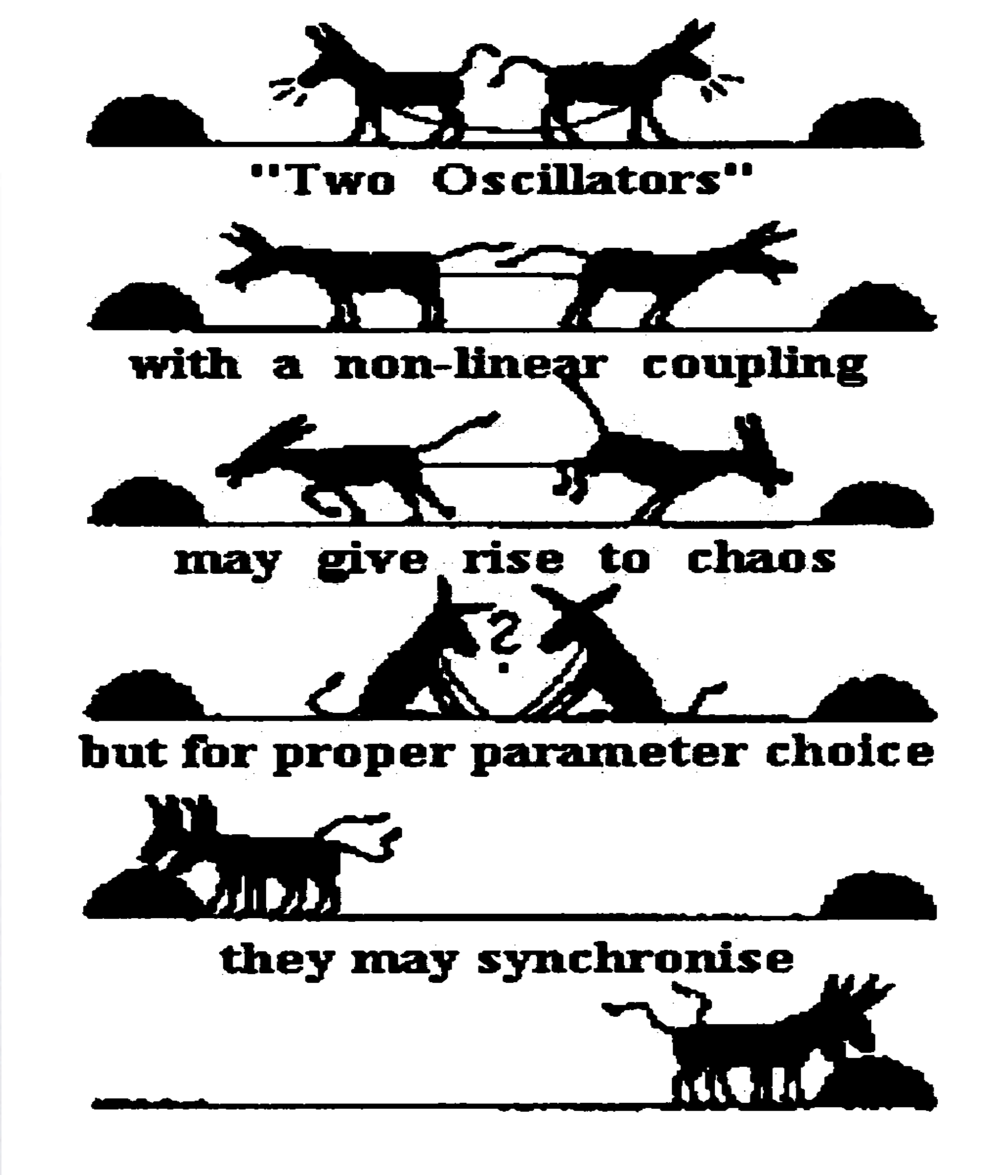
Outline

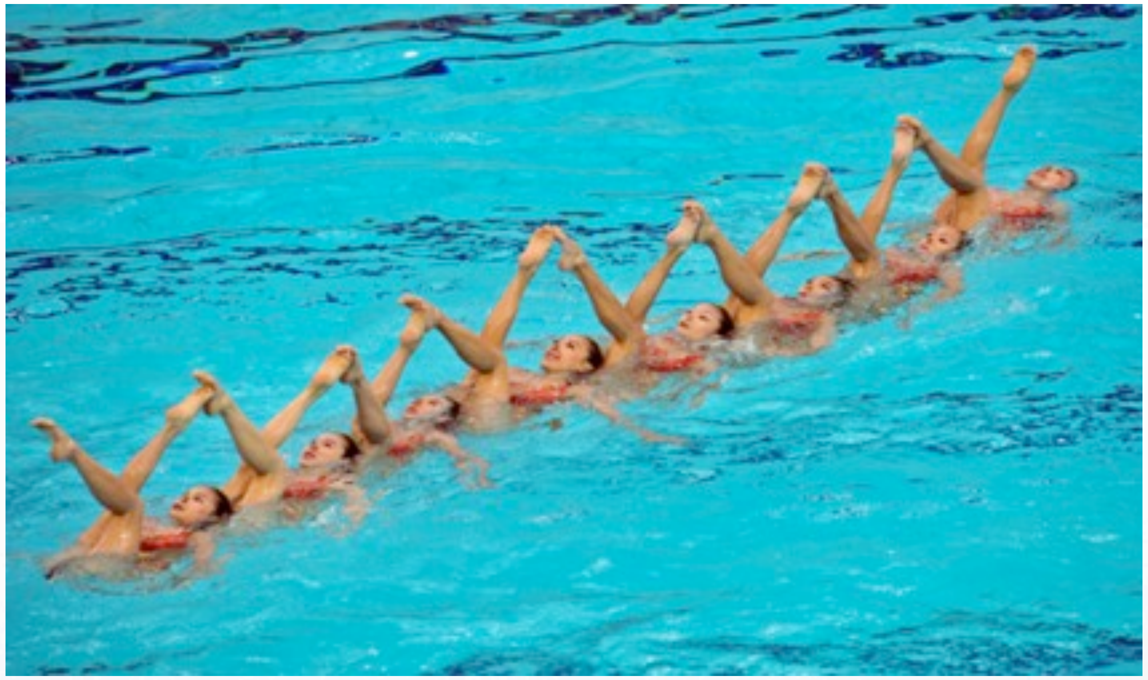
- Introduction
  - A historical perspective and motivations
- Problem formulation
  - Syncronization manifold
  - Perturbative techniques
- Applications
  - associative memories and pattern recognition systems
- Conclusions

What is synchronization?

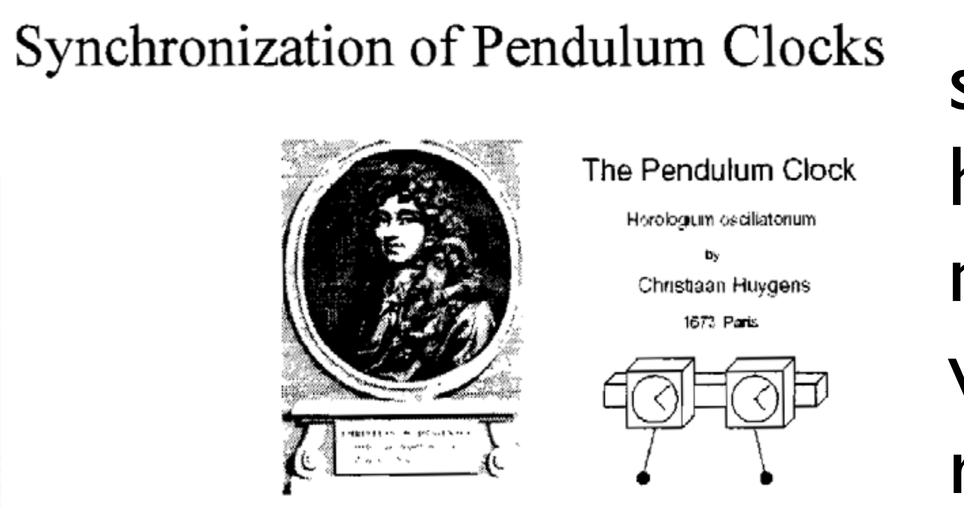
- Synchronize: to agree in time, to happen at the same time, to represent or arrange (events) to indicate coincidence or coexistence
- It is an important concept in: Physics, Biology, Telecommunication, Computer science, Cryptography, Multimedia, Photography, Music (rhythm)
- Synchronicity is a word coined by the Swiss psychologist Carl Jung to describe the "temporally coincident occurrences of acausal events."

What is synchronization?





A historical perspective
Christiaan Huygens (1658)



"It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."

A historical perspective
Engelbert Kaempfer (1680)

Synchronization in a large population of oscillating systems

Engelbert Kaempfer (1680)

The glowworms represent another shew, which settle on some Trees, like a fiery cloud, with this surprising circumstance, that a whole swarm of these insects, having taken possession of one Tree, and spread themselves over its branches, sometimes hide their Light all at once, and a moment after make it appear again with the utmost regularity and exactness

This very early observation reports on synchronization in a large population of oscillating systems. The same physical mechanism that makes the insects to keep in sync is responsible for the emergence of synchronous clapping in a large audience or onset of rhythms in neuronal populations.

A historical perspective

- Sleep-Wake rhythms: biological systems can adjust their rhythms to external signals. Under natural conditions, biological clocks tune their rhythms (i.e. synchronize) in accordance with the 24-hour period of the Earth's daily cycle (First observed by J.J. Dortous de Mairan, 1729)
- Synchronization of triode oscillators (Appleton, van der Pol, van der Mark, 1922-1928)

A historical perspective

Mutual synchronization of cardiac pacemaker cells

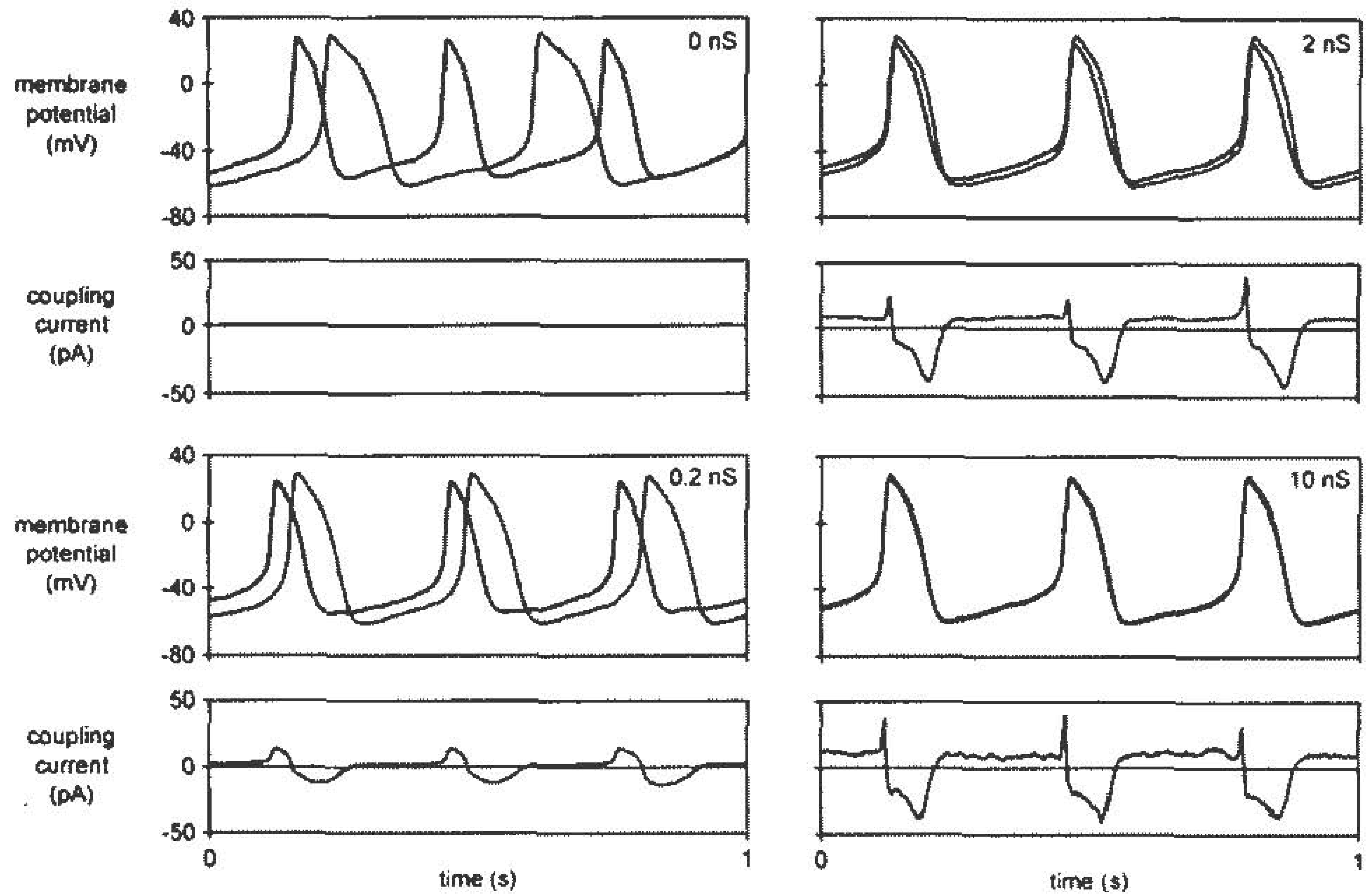


Fig. 4. Mutual synchronization of two single isolated rabbit sinoatrial node cells studied with the coupling clamp technique. Action potentials are shown in blue and red, and the intercellular coupling current in green. The coupling conductance is increased from 0 nS (uncoupled conditions) to 10 nS. Note that a coupling conductance as small as 0.2 nS is sufficient for frequency entrainment. At 10 nS, the action potentials show full waveform entrainment

(E.E. Verheijek et al., “Pacemaker synchronization of electrically coupled rabbit sinoatrial node cells,” *J. Gen. Physiol.,* vol. 11 I , pp. 95-112, January 1998)

The concept of "Synchronization"

- In a classical context, synchronization (from Greek: syn = the same, common and: chronos = time) means adjustment of rhythms of <u>self-sustained periodic oscillators</u> due to their weak interaction (coupling); this adjustment can be described in terms of phase locking and frequency entrainment [1].

(1) If you have two vibrating objects with the same natural frequency or corresponding harmonic, they will both have a forced vibration effect on each other. This process, given time, normally leads to a condition where both objects synchronize. Of interest, both oscillators do not, necessarily, must have exactly the same natural frequency. If there is enough "coupling" between the oscillators, they will sometime "lock-in" with one another at a slightly shifted frequency: the frequencies become equal or entrained. The onset of a certain relationship between the phases of these oscillators is often termed phase locking.

What is a self-sustained periodic oscillator ?

1. The oscillator is an active system. It contains an internal source of energy that is transformed into oscillatory behavior. Being isolated, it continues to generate the same rhythm until the source of energy expires. It is described as an autonomous dynamical system.

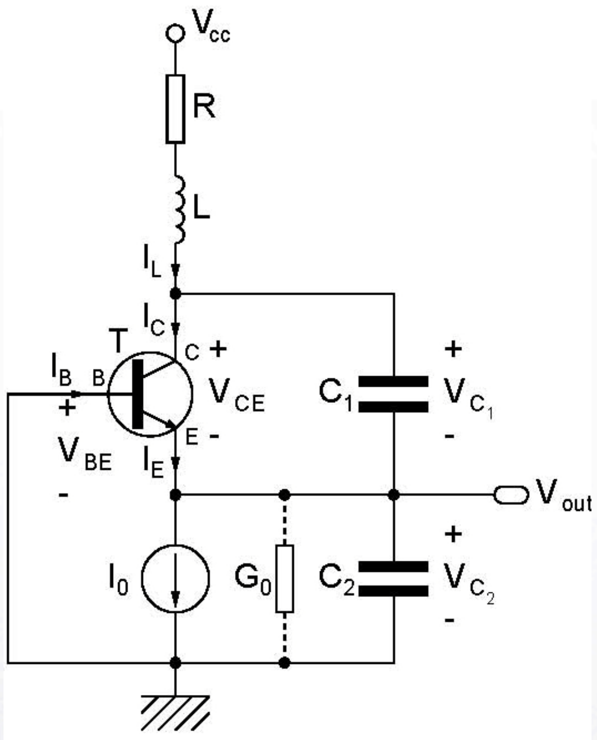
2. The form of the oscillation is determined by the parameters of the system and does not depend on initial conditions.

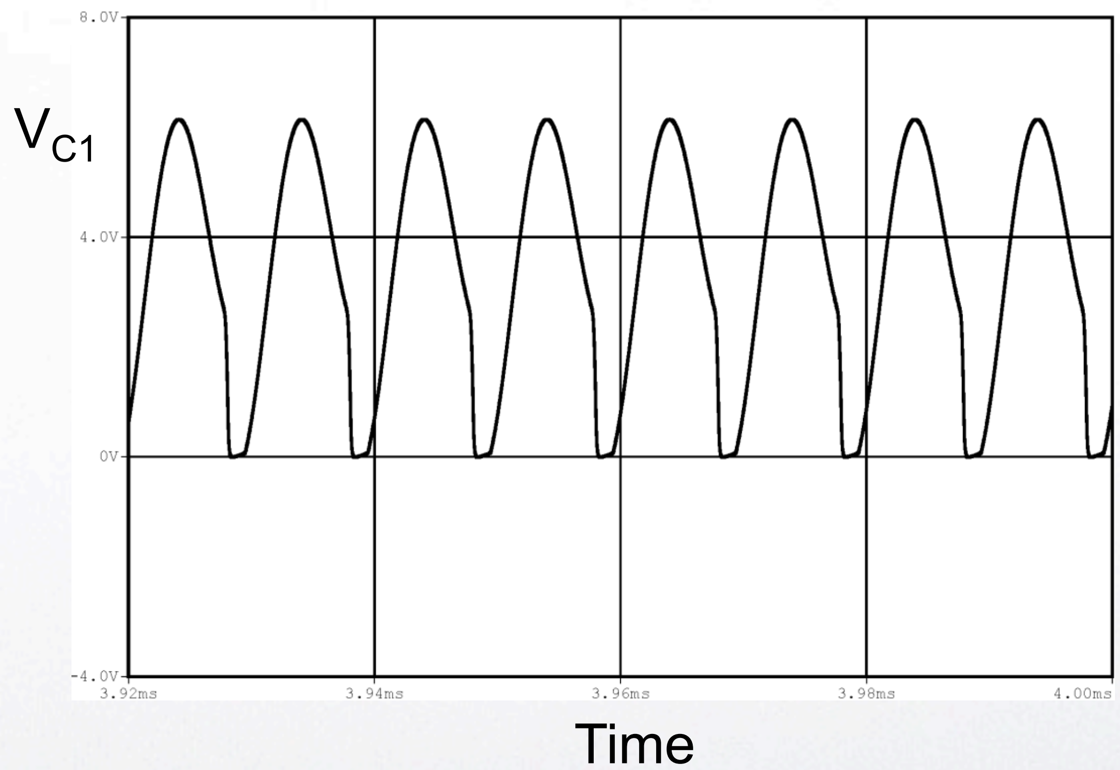
3. The oscillation is stable to (small) perturbations.

The above properties are characteristic of nonlinear oscillators

Electronic nonlinear circuits

Example: Colpitts oscillator





Period: 10.1 μs

Electronic nonlinear circuits

Example: Two identical coupled Van der Pol oscillators

The coupled oscillators synchronize: two different interpretations

for the phases $\varphi_i = \tan^{-1}\left(\frac{y_i}{x_i}\right)$

$$\left|\varphi_2(t) - \varphi_1(t)\right| \xrightarrow[t\to+\infty]{} 0$$

phase synchronization

generalization: to non identical systems

limitation: to systems where a phase can be defined → rhythmic behavior

for the states

$$\left|\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}(t) - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}(t)\right| \xrightarrow[t\to+\infty]{} 0$$

complete (or identical) synchronization

generalization: to systems with any behavior

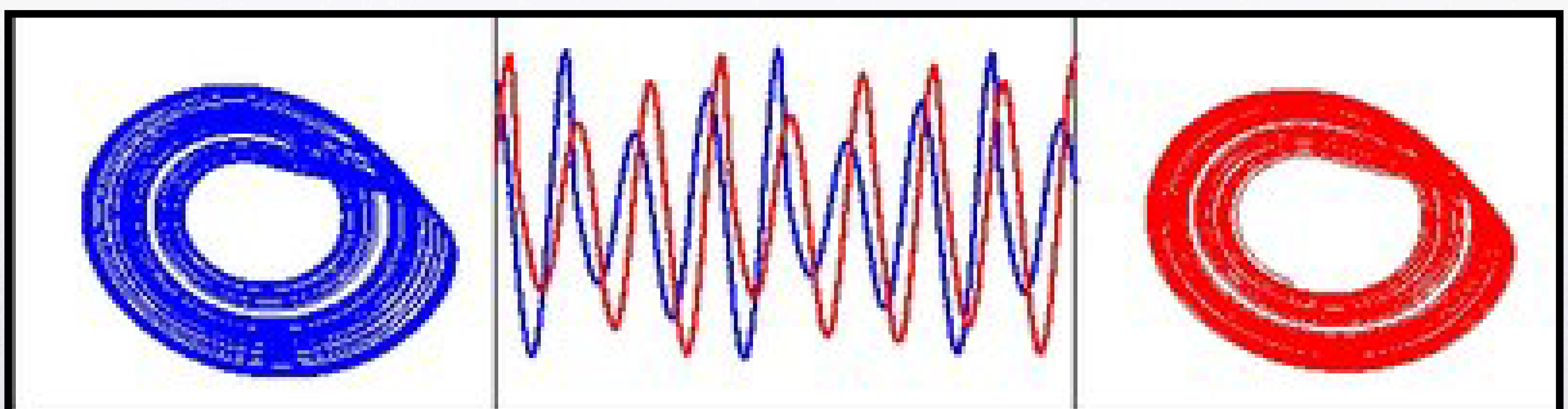
limitation: to identical or approximately identical systems

Synchronization in chaotic oscillators

Modern concept covers also **chaotic systems**; in this case one distinguishes between different forms of synchronization (complete, lag, generalized, phase, imperfect), the most notable being complete (or identical) and phase synchronization [*].

Example: Phase synchronization of two coupled chaotic oscillators

[*] S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, The Synchronization of Chaotic Systems, Physics Reports 366, pp. 1-101, 2002



Synchronization: why ?

- Synchronization phenomena are pervasive in biology and are related to several central issues of neuroscience [1].
- Synchronization may allow distant sites in the brain to communicate and cooperate with each other. For example, synchronization between areas of the visual cortex and parietal cortex, and between areas of the parietal and motor cortex was observed during the visual-motor integration task in awake cats [2].

[1] W. Singer and C. M. Gray, “Visual features integration and the temporal correlation hypothesis,” *Annual Rev. Neurosci.*, vol. 18, pp. 555–586, 1995.

[2] P. R. Roelfsema, A. K. Engel, P. Knig, and W. Singer, “Visuomotor integration is associated with zero time-lag synchronization among cortical areas,” *Nature*, vol. 385, pp. 157–161, 1997

Synchronization: why ?

- Direct participation of synchrony in a cognitive task was experimentally demonstrated in humans [3].
- Synchronization may help protect interconnected neurons from the influence of random perturbations (intrinsic neuronal noise) which affect all neurons in the nervous system [4].

[3] E. Rodriguez, N. George, J.-P. Lachaux, J. Martinerie, B. Renault, and F. J. Varela, "Perception's shadow: Long distance synchronization of human brain activity," *Nature*, vol. 397, pp. 430–433, 1999.

[4] N.Tabareau, J-J. Slotine, Q. Pham, "How synchronization protects from noise", PLoS Computational Biology, pp. 1-9, Vol. 6, N. 1, 2010

Synchronization: why ?

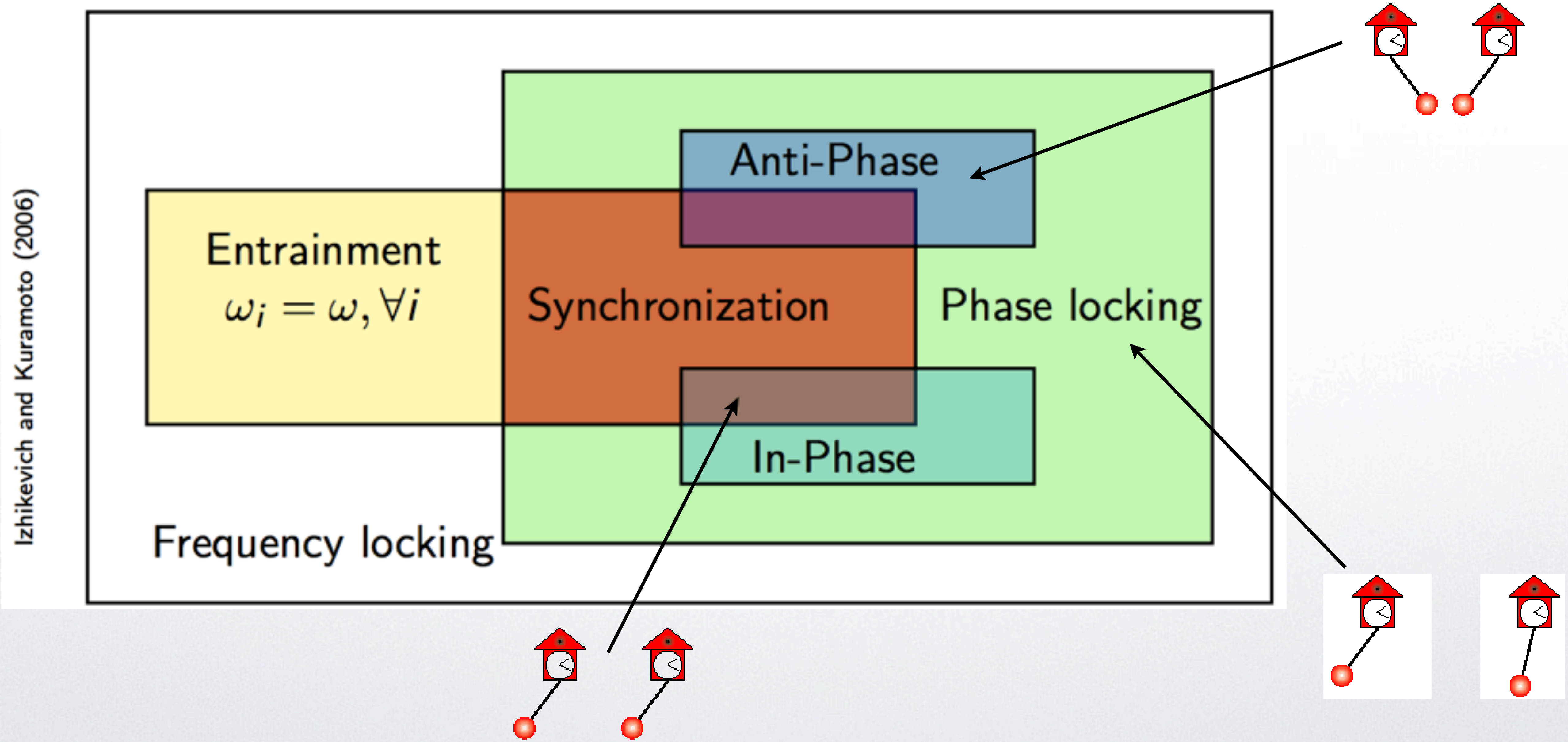
- Spiking neurons, like any other physical, chemical, or biological oscillators, can synchronize and exhibit collective behavior that is not intrinsic to any individual neuron.
- Partial synchrony in cortical networks is believed to generate various brain oscillations, such as the alpha and gamma EEG (electroencephalography) rhythms. However, increased synchrony may result in pathological types of activity, such as epilepsy.
- Coordinated synchrony is needed for locomotion and swim pattern generation in fish. Depending on the circumstances, synchrony can be good or bad, and **it is important to know what factors contribute to synchrony and how to control it.**

(Extracted from: E. M. Izhikevich , Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting, Ch. 10, MIT Press, Cambridge, MA, USA, 2007)

Take home messagge

- Synchronization properties are influenced by the general properties of the oscillatory network: complex systems can be more or less prone to synchronize due to their specific features.
- Synchronization requires knowledge of both **nonlinear dynamics** and of **complex systems**.

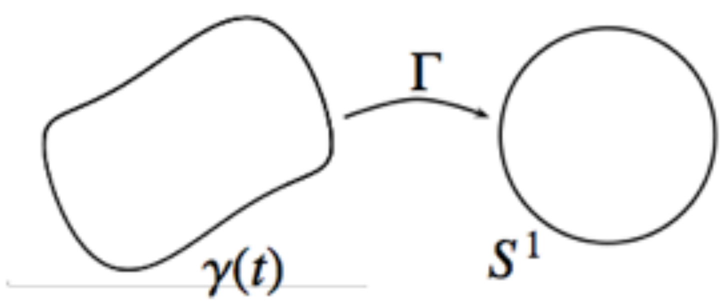
The dynamics of coupled periodic oscillators: strong synchronization



Oscillatory networks

Single oscillator

$\dot{X}_i = F_i(X_i) \quad X_i \in \boldsymbol{R}^m, \quad F_i : \ \boldsymbol{R}^m \to \boldsymbol{R}^m, \quad (i = 1, 2, ..., n)$
has at least one hyperbolic T_i-periodic solution $\gamma_i(t) : \ \boldsymbol{R} \to \boldsymbol{R}^m$



$\theta_i(t) = \omega_i \ t, \ \theta_i \in S^1 = [0, \ 2\pi[\ , \ \omega_i = 2\frac{\pi}{T_i}$

Weakly Connected Oscillatory Networks ($\varepsilon \ll 1$)

$\dot{X}_i = F_i(X_i) \ + \ \varepsilon \ G_i(\boldsymbol{X}), \quad \boldsymbol{X} = [X_1', \ldots X_n']', \quad G_i : \ \boldsymbol{R}^{m \times n} \to \boldsymbol{R}^m$

$$\theta_i(t) = \omega_i \ t + \phi_i(\epsilon t)$$

Oscillatory networks: Global dynamic behaviour

Weakly Connected Oscillatory Networks ($\varepsilon \ll 1$)

$$\dot{X}_i = F_i(X_i) + \varepsilon\, G_i(\boldsymbol{X}), \quad \boldsymbol{X} = [X_1', \ldots X_n']', \quad G_i : \boldsymbol{R}^{m\times n} \rightarrow \boldsymbol{R}^m$$

$$\theta_i(t) = \omega_i\, t + \phi_i(\epsilon t)$$

- **Time–domain techniques** do not allow to identify all the limit cycles (either stable or unstable).
  - It would require to consider *infinitely many* initial conditions.
  - Unstable limit cycles cannot be detected through simulation.
- By means of **Spectral techniques** (*Describing Function* and *Harmonic Balance*), the computation of all the limit cycles is reduced to a non-differential algebraic problem.
- Such methods are not suitable for characterizing the global dynamic behavior of complex networks with a large number of attractors.

Oscillatory networks: Malkin Theorem

Weakly Connected Oscillatory Networks ($\varepsilon \ll 1$)

$$\dot{X}_i = F_i(X_i) + \varepsilon\, G_i(\boldsymbol{X}), \quad \boldsymbol{X} = [X_1', \dots X_n']', \quad G_i : \boldsymbol{R}^{m\times n} \to \boldsymbol{R}^m$$

$$\theta_i(t) = \omega_i\, t + \phi_i(\epsilon t)$$

Phase deviation equation

$$\dot{\phi}_i = \frac{\omega}{T}\int_0^T Q_i'(t)\, G_i\left[\gamma\left(t + \frac{\phi - \phi_i}{\omega}\right)\right] dt,$$

$$T = m.c.m.(T_1, \dots, T_n)$$

$$\gamma\left(t + \frac{\phi - \phi_i}{\omega}\right) = \left[\gamma_1'\left(t + \frac{\phi_1 - \phi_i}{\omega_1}\right), \dots, \gamma_n'\left(t + \frac{\phi_n - \phi_i}{\omega_n}\right)\right]'$$

$$\dot{Q}_i(t) = -[DF_i(\gamma_i(t))]' Q_i(t), \quad Q_i'(0) F_i(\gamma_i(0)) = 1$$

Joint application of the DF and MT

1. The periodic trajectories $\gamma_i(t)$ of the uncoupled oscillators are approximated through the *describing function technique*.
2. Once the approximation of $\gamma_i(t)$ is known, a first harmonic approximation of $Q_i(t)$ is computed, by exploiting the linear adjoint problem and the normalization condition.
3. The approximated phase deviation equation is derived by analytically computing the integral expression given by the *Malkin's Theorem*.

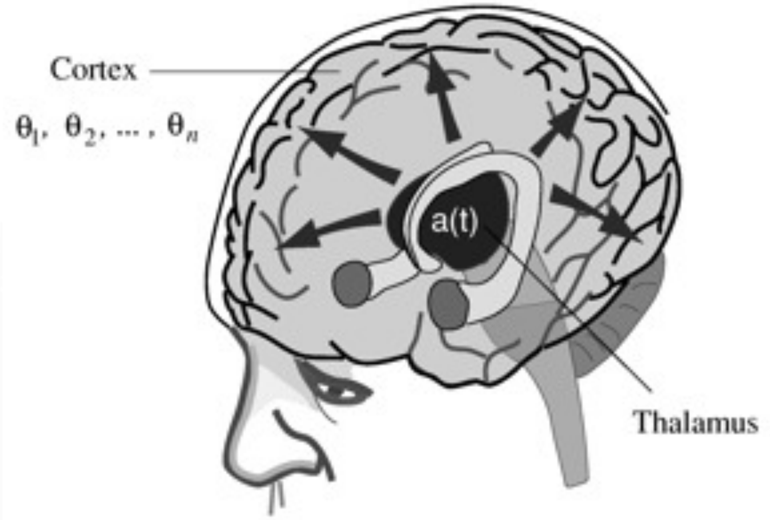
The phase equation is analyzed in order to determine the total number of stationary solutions (equilibrium points) and their stability properties. They correspond to the total number of limit cycles of the original weakly connected network.

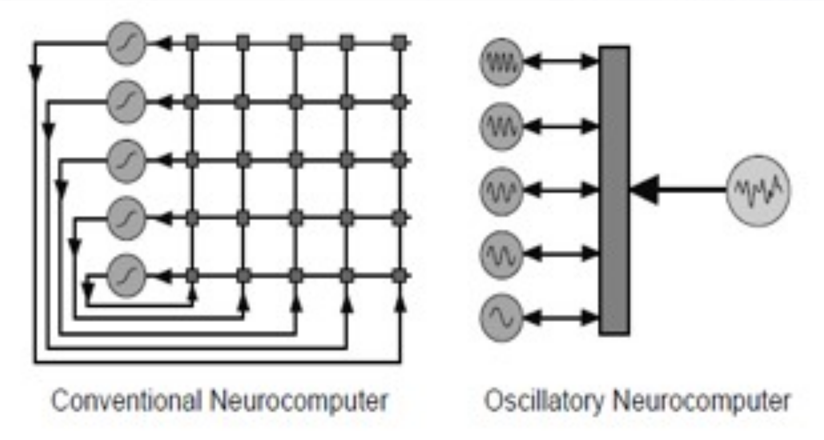
Applications

- Synchronous states can be exploited for dynamic pattern recognition and to realize associative and dynamic memories. By means of a simple learning algorithm, the phase-deviation equation is designed in such a way that given sets of patterns can be stored and recalled. In particular, two models of WCONs have been proposed as examples of associative and dynamic memories.
- Spiral waves are the most universal form of patterns arising in dissipative media of oscillatory and excitable nature. By focusing on oscillatory networks, whose cells admit of a Lur'e description and are linearly connected through weak couplings, the occurrence of spiral waves has been studied.

Oscillatory model of neurocomputing

- Oscillations experimentally observed in visual cortex after stimulus
- Synchronized oscillations observed in parts of the brain not geometrically close
- Synchronized oscillations is linked to association
- Can we build an image recognition system from coupled oscillators?





Hoppensteadt and Izhikevich, Phys Rev L, VOLUME 82, NUMBER 14, April 5, 1999

Oscillatory associative memories

International Journal of Bifurcation and Chaos, Vol. 17, No. 12 (2007) 4365–4379

WEAKLY CONNECTED OSCILLATORY NETWORK MODELS FOR ASSOCIATIVE AND DYNAMIC MEMORIES

FERNANDO CORINTO, MICHELE BONNIN
and MARCO GILLI

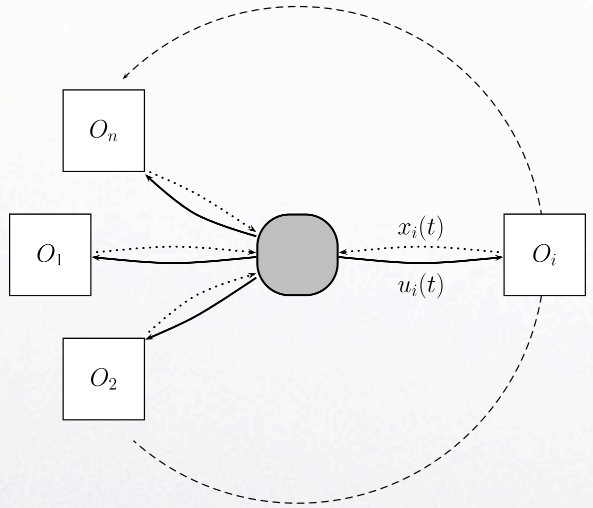
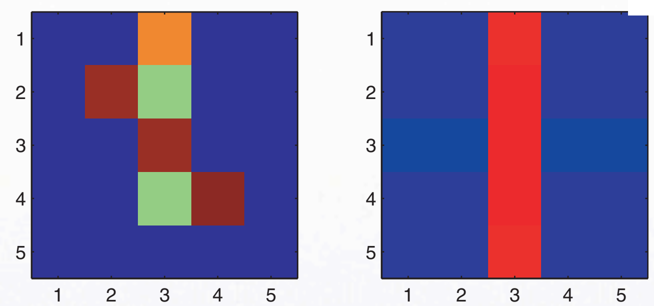
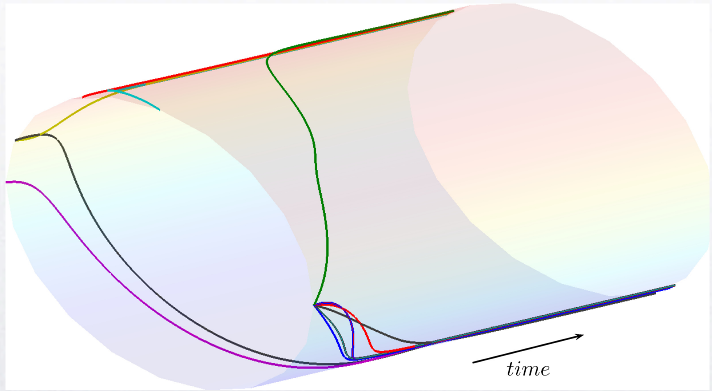
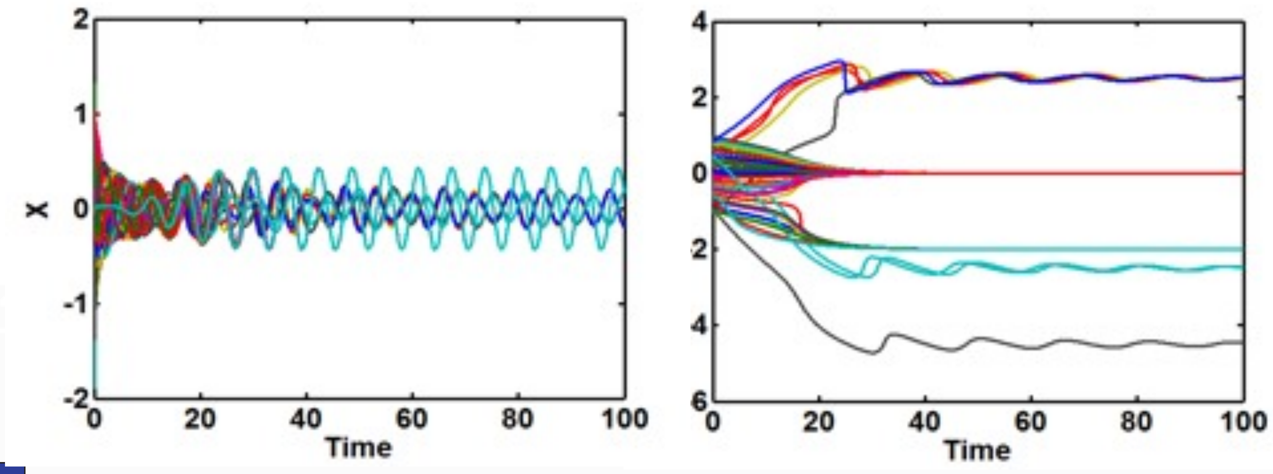
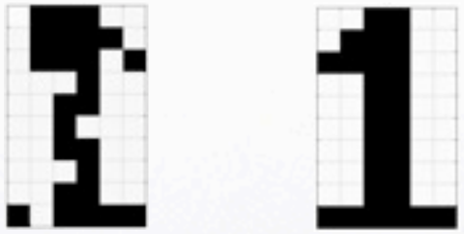


Fig. 1. Weakly connected oscillatory network having a star topology.









All oscillators are phase locked.
Degree of matching remains above a threshold.
Thus a better discrimination of matching patterns.

Oscillatory associative memories

Can oscillatory associative memories outperform "static" associative memories?

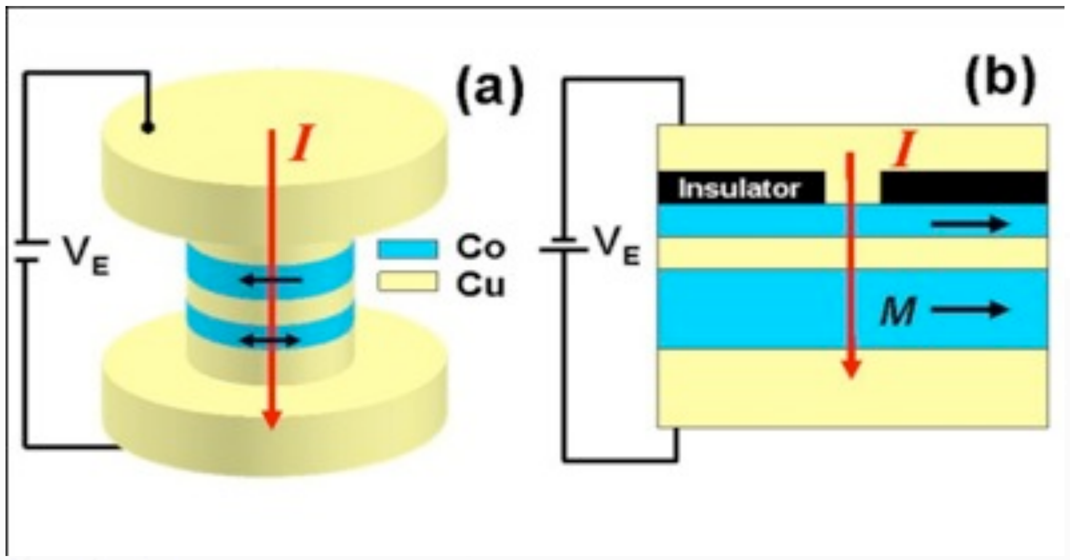
- Goal: find classes of problem solved only by oscillatory networks
- no restrictions about the architecture of the networks

Oscillatory associative memories

Can oscillatory associative memories outperform "static" associative memories?

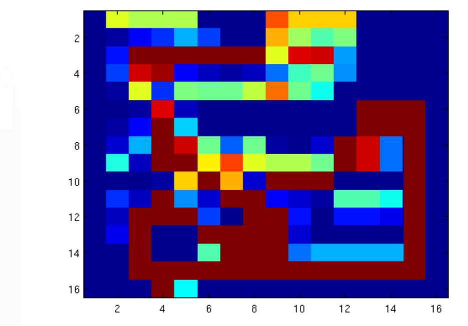
- Goal: conceive non-boolean spatio-temporal algorithms to solve a classical problem in a more efficient (in terms of speed, power, ...) way
- consider physical constrains

Spin-Torque Oscillatory arrays

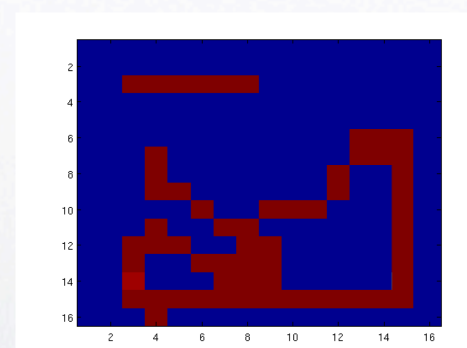


pattern recognition tasks

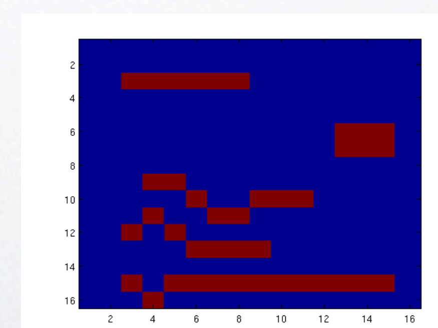
(a) Input



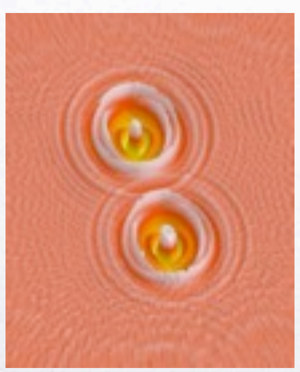
(b) Output



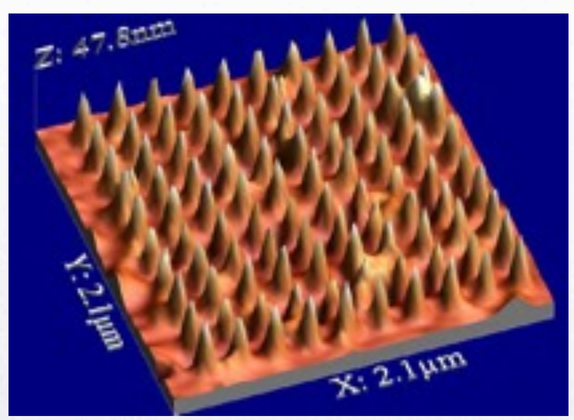
(c) Thresholded output



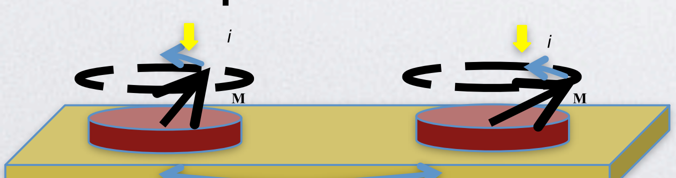
(d) Horizontal edge detection



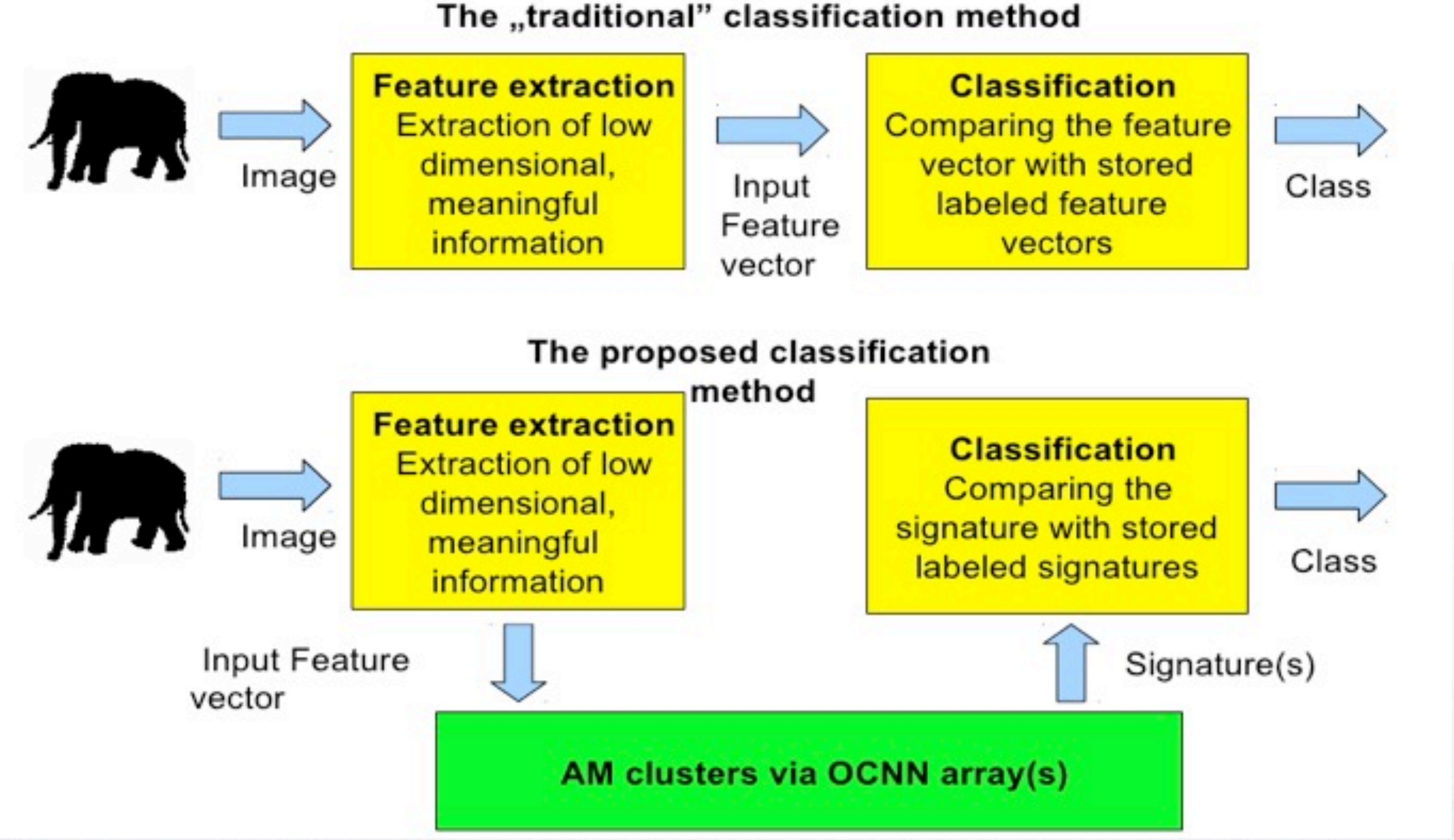
coupled STOs



array of STOs



Spin-Torque Oscillatory arrays



The addition of O-CNN arrays can enhance the computational power of the architecture and increase the detection rate.
The OCNN array can transform the input feature vector in a way which helps classification.

Conclusions and Perspectives

- Simulation with real-life data: Images taken by a mobile robot has been used for classification with similar results.
- Boundary conditions /lateral input/: with side input, changing the boundary conditions the properties of the array can be changed, this can be used to increase the computational strength (programmability) of the array
- OCNN array with different spin oscillators: The usage of two different dynamics in one network would increase the possible outcomes of one OCNN array
- Transient based computation: using the evolution of the phase shift to determine extra properties about the input vectors
- Synchronization requires knowledge of both **nonlinear dynamics** and of complex systems.

Nonlinear analysis tools

- Differential or integral equations represent suitable mathematical models of physical systems.
- Approximate analytical tools are required for studying (analysis and design) nonlinear dynamical systems describing electrical circuits, mechanical and biological systems, ...
  - Tools for detecting oscillations

Nonlinear systems/circuits: limit cycles

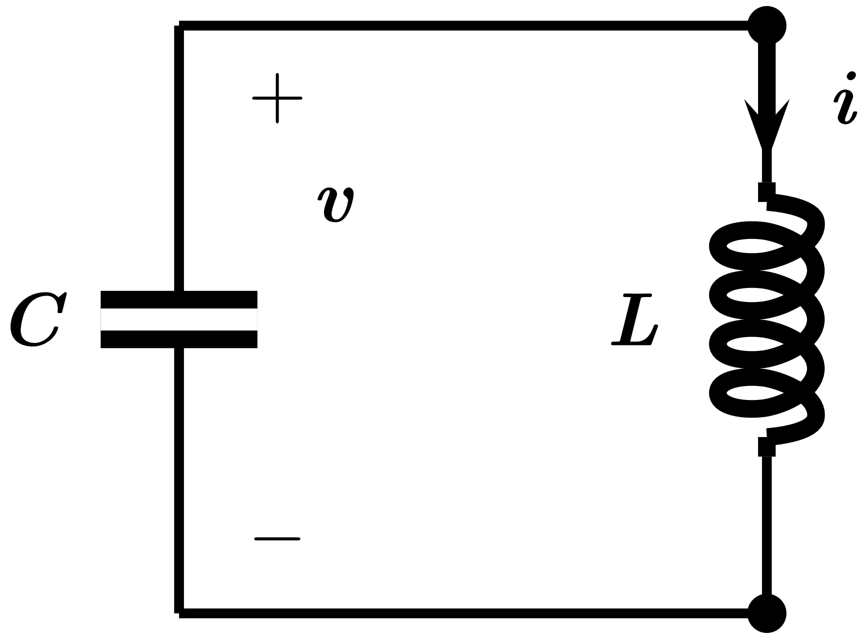
$$\frac{d\,x}{d\,t} = \boldsymbol{f}(x,t) \qquad x \in R^n, \;\; t \in R^+$$

*D*efinition: A non-constant solution $x(t) = \boldsymbol{\Phi}(t, x_0)$ is said to be periodic if there exists T such that:

$$\forall\, t: \quad \boldsymbol{\Phi}(t+T, x_0) = \boldsymbol{\Phi}(t, x_0)$$

The image of $\boldsymbol{\Phi}(t, x_0)$ in the state-space (or phase-space) R^n is called periodic trajectory or limit cycle of period T.

Linear systems/circuits: limit cycles



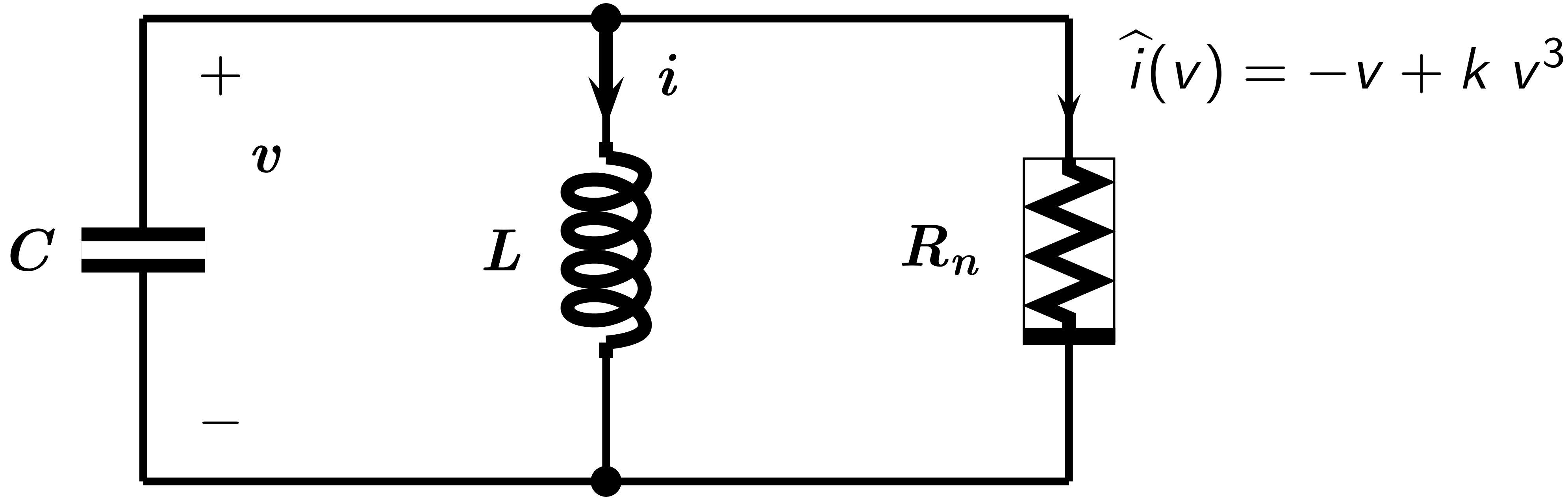
$$\frac{dv}{dt} = -\frac{i}{C}$$

$$\frac{di}{dt} = \frac{v}{L}$$

Eigenvalues: $\lambda_{12} = \pm j\omega \qquad \omega = \frac{1}{\sqrt{LC}}$

The circuit presents infinitely many non-isolated cycles with the same frequency. The cycle amplitude depends on the initial conditions.

Nonlinear systems/circuits: limit cycles



$$
\begin{aligned}
\frac{dv}{dt} &= -\frac{1}{C}\,[i + \hat{\imath}(v)] \\
\frac{di}{dt} &= \frac{v}{L}
\end{aligned}
$$

The circuit presents a single limit cycle, that attracts all the trajectories.

Equilibrium point analysis

Equilibrium point $\overline{x} = (0, 0)$

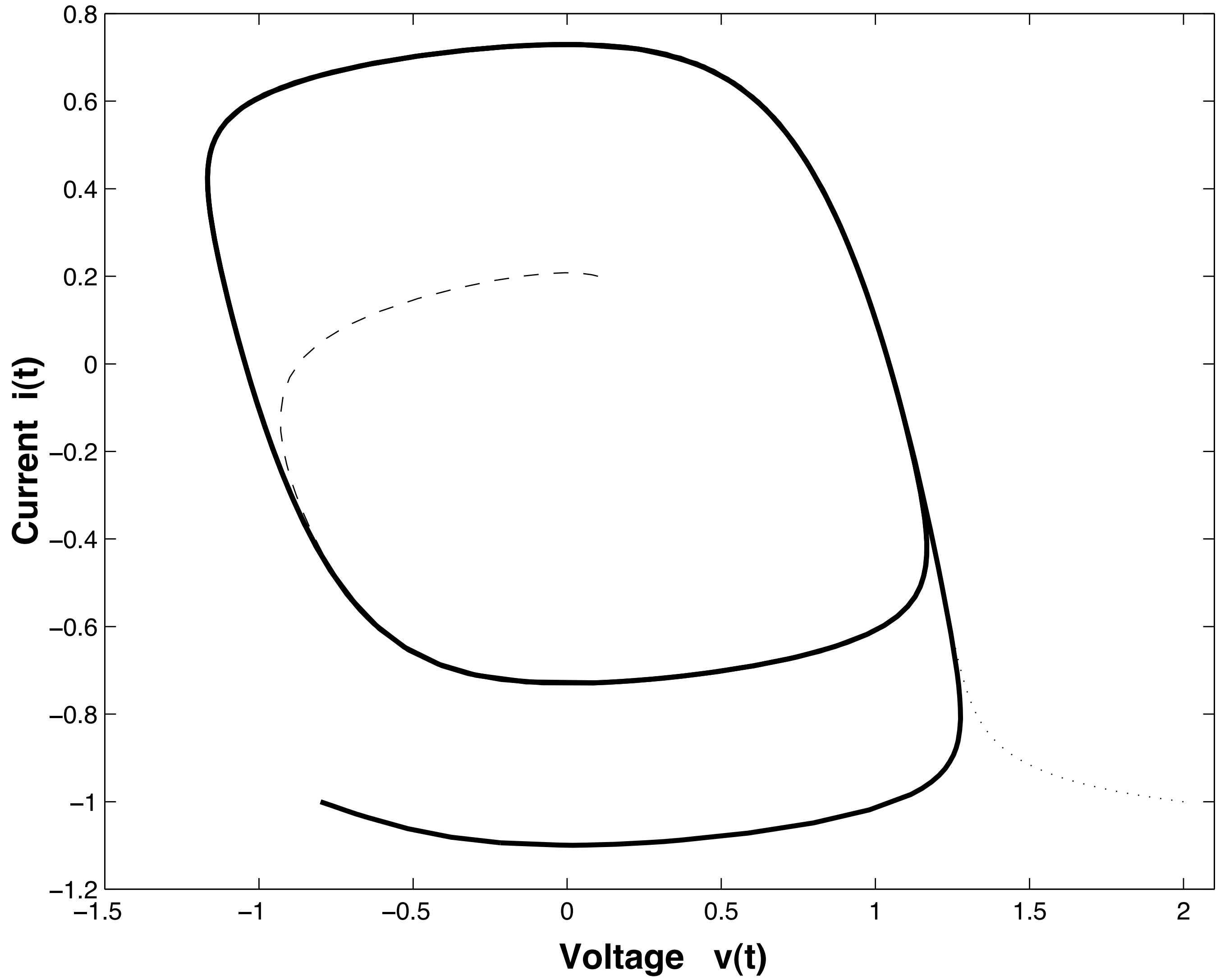
$$\boldsymbol{J} = \begin{pmatrix} \frac{1}{C} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{pmatrix}$$

$$\lambda_{12} = \frac{1}{2C} \pm \frac{1}{2LC}\sqrt{L^2 - 4LC}$$

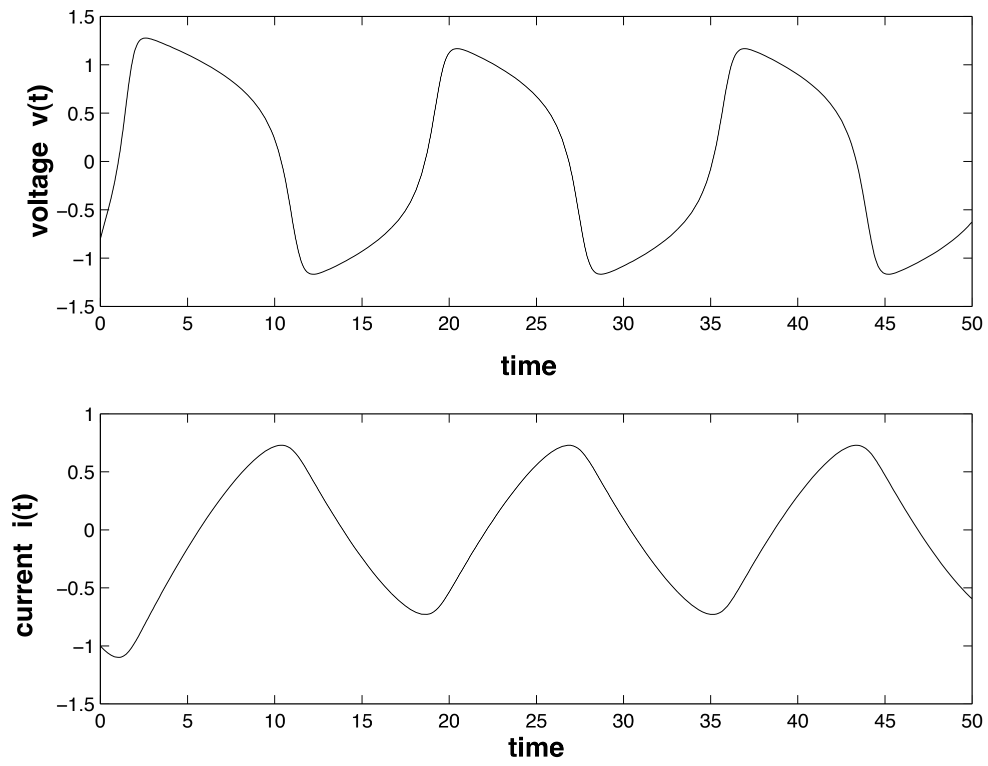
$$\begin{cases} L < 4C & \text{unstable focus} \\ L > 4C & \text{unstable node} \end{cases}$$

Limit cycle

$$L = \frac{9}{2} \quad C = 1 \quad k = 1$$



Time waveforms



- In most circumstances it is not possible to derive explicitly the limit cycle. Numerical simulations are useful to 'discover' limit cycles.
- Theorems may be used to prove the existence of a limit cycle in planar systems (Liénard's Theorem and Bendixson's Theorem).

Computation of limit cycles

- Determination of all the periodic limit cycles (either stable and unstable) and their stability properties (Floquet's multipliers – FMs)
  - In large scale dynamical systems the sole numerical simulation does not allow to identify all the limit cycles (either stable and unstable)
    - It would require to consider infinitely many initial conditions
    - Unstable limit cycles cannot be detected through simulation
  - By means of Spectral Techniques, the computation of all the limit cycles is reduced to non-differential (sometimes algebraic) problem.
    - Harmonic Balance Technique
    - Describing Function Technique

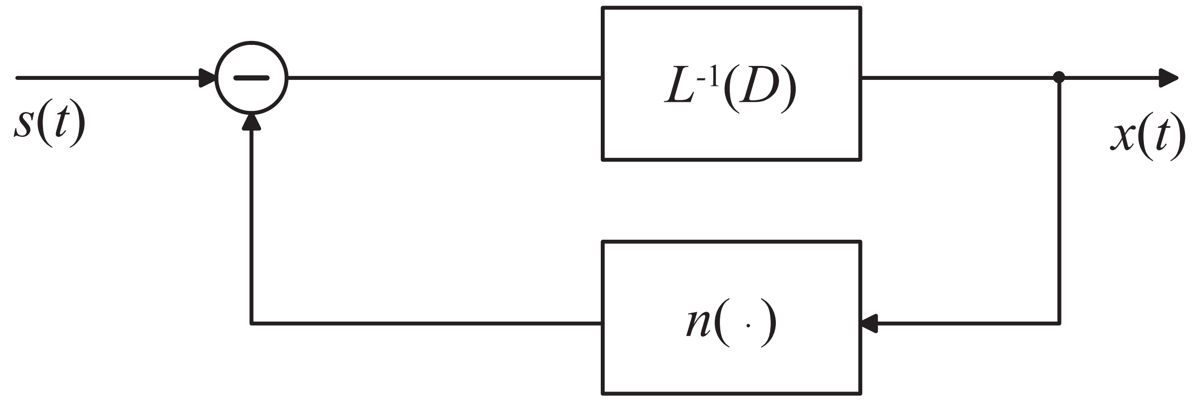
Computation of limit cycles: Time-domain methods

- If the system possesses a stable cycle γ, we can try to find it by numerical integration (simulation). If the initial point for the integration belongs to the basin of attraction of γ, the computed orbit will converge to γ in forward time. Such a trick will fail to locate a saddle cycle, even if we reverse time.
- there exist different time domain methods especially to directly locate periodic orbits even if they are saddle or unstable cycles. The problem of finding the steady state is converted into a boundary-value problem, to which the standard approaches, such as shooting methods and finite-difference methods, can be applied.
- $\boldsymbol{\Phi}(t_0 + T, x_0) = \boldsymbol{\Phi}(t_0, x_0) = x_0$, where the minimum cycle period T is usually unknown. An extra phase condition has to be added in order to 'select' a solution among all those corresponding to the cycle.

Computation of limit cycles: Perturbation methods

- Applicable to nonlinear ODEs that have periodic solutions and a small parameter
  - Method of averaging
  - Regular perturbation
  - Multiple scales
  - Picard iteration

Spectral methods for Lur'e systems



$$L(D)x(t) + \boldsymbol{n}[x(t)] = s(t), \qquad x(t) \in R$$

If the systems admits of a periodic solution of period T, then $x(t)$ can be expanded through the Fourier series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega t) + B_k \sin(k\omega t) \qquad \omega = \frac{2\pi}{T}$$

Examples

Third order oscillator

$$L(D) = \frac{D^3 + (1+\alpha)D^2 + \beta D + \alpha\beta}{\alpha\left(D^2 + D + \beta\right)} \qquad \boldsymbol{n}(x) = -\frac{8}{7}x + \frac{4}{63}x^3$$

Second order oscillator

$$L(D) = \frac{LCD^2 - LD + 1}{kLD} \qquad \boldsymbol{n}(x) = x^3$$

The harmonic balance (HB) technique

1. The state is represented through a **finite** (**N**) number of harmonics

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)$$

2. The term $L(D)x(t)$ yields:

$$\begin{aligned} L(D)x(t) &= L(0)A_0 + \sum_{k=1}^{N}\{\boldsymbol{Re}[L(jk\omega)]A_k + \boldsymbol{Im}[L(jk\omega)]B_k\}\cos(k\omega t) \\ &+ \sum_{k=1}^{N}\{\boldsymbol{Re}[L(jk\omega)]B_k - \boldsymbol{Im}[L(jk\omega)]A_k\}\sin(k\omega t) \end{aligned}$$

The harmonic balance (HB) technique

3. The term $\boldsymbol{n}[x(t)]$ yields (by truncating the series to N harmonics):

$$\boldsymbol{n}[x(t)] = C_0 + \sum_{k=1}^{N} C_k \cos(k\omega t) + D_k \sin(k\omega t)$$

$$\begin{aligned}
C_0 &= \frac{1}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)\right] dt \\
C_k &= \frac{2}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)\right] \cos(k\omega t)\, dt \\
D_k &= \frac{2}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)\right] \sin(k\omega t)\, dt
\end{aligned}$$

The harmonic balance (HB) technique

3. The term $s(t)$ yields (by truncating the series to N harmonics):

$$s(t) = P_0 + \sum_{k=1}^{N} P_k \cos(k\omega t) + Q_k \sin(k\omega t)$$

$$\begin{aligned} P_0 &= \frac{1}{T}\int_0^T s(t)\, dt \\ P_k &= \frac{2}{T}\int_0^T s(t) \cos(k\omega t)\, dt \\ Q_k &= \frac{2}{T}\int_0^T s(t)\, \sin(k\omega t)\, dt \end{aligned}$$

The harmonic balance (HB) technique

4. A set of $2N + 1$ **nonlinear equations** is obtained, by equating the coefficients of the constant term and of the harmonics $\cos(k\omega t)$, $\sin(k\omega t)$

$$
\begin{array}{lcccll}
L(0)A_0 & + & C_0(A_0, ..., B_N) & = & P_0 & \\
\boldsymbol{Re}[L(jk\omega)]A_k - \boldsymbol{Im}[L(jk\omega)]B_k & + & C_k(A_0, ..., B_N) & = & P_k & 1 \leq k \leq N \\
\boldsymbol{Im}[L(jk\omega)]A_k + \boldsymbol{Re}[L(jk\omega)]B_k & + & D_k(A_0, ..., B_N) & = & Q_k & 1 \leq k \leq N
\end{array}
$$

5. **Autonomous systems**: the term A_1 is assumed to be equal to zero (i.e. the phase of the first harmonic of $x(t)$ is arbitrarily fixed); since ω is unknown, the system has an equal number $[(2N + 1)]$ of equations and unknowns.

The describing function (DF) technique

- The state is represented through a **single** harmonic (with amplitude B and frequency ω) and a bias (A)

$$x(t) = A_0 + B_1 \sin(\omega t) = A + B \sin(\omega t), \quad (A, B, \omega \textit{ unknowns})$$

- The nonlinear term $\boldsymbol{n}[x(t)]$ is approximated up to the first harmonic

$$\boldsymbol{n}[x(t)] = N_0 + N_1 \sin(\omega t)$$

$$N_0 = \frac{1}{T} \int_0^T \boldsymbol{n}\left[A + B \sin(\omega t)\right] \, dt$$

$$N_1 = \frac{2}{T} \int_0^T \boldsymbol{n}\left[A + B \sin(\omega t)\right] \, \sin(\omega t) \, dt$$

$$\frac{2}{T} \int_0^T \boldsymbol{n}\left[A + B \sin(\omega t)\right] \, \cos(\omega t) \, dt = 0 \;\; (single - valued \; functions)$$

The describing function (DF) technique

- By substituting in $L(D)x(t) + n[x(t)] = 0$ we obtain the following nonlinear algebraic system:

$$
\begin{aligned}
L(0)A + N_0(A, B) &= 0 \\
B\,\boldsymbol{Re}\{L(j\omega)\} + N_1(A, B) &= 0 \\
\boldsymbol{Im}\{L(j\omega)\} &= 0
\end{aligned}
$$

- The conditions under which the describing function technique yields rigorous results are rather restrictive (*Mees et al.*). However in most cases it works (*Gilli et al.*).
- Limit cycle stability and bifurcations can be studied through approximate methods based on the DF technique (see (*Genesio and Gilli*) for the extension to large scale dynamical systems).

Computation of limit cycles: Examples

- Duffing's equation
- Van der Pol circuit
- Chua's circuit

Efficient HB implementations

1. Consider the time samples vectors

$$y(t) = L(D)x(t)$$

$$\underline{y} = [y(t_1), ..., y(t_{2N}),\ y(t_{2N+1})]'$$

$$\underline{x} = [x(t_1), ..., x(t_{2N}),\ x(t_{2N+1})]'$$

$$\underline{s} = [s(t_1), ..., s(t_{2N}),\ s(t_{2N+1})]'$$

$$t_p = \frac{T}{2N+1}p \qquad\qquad p = 1, \ldots, 2N+1$$

Efficient HB implementations

2. Impose that the HB equation be satisfied for $t = t_p$

$$y(t_p) + \boldsymbol{n}[x(t_p)] = s(t_p), \qquad p = 1, \ldots, 2N + 1$$

that in vector notation yields

$$\underline{y} + \boldsymbol{n}[\underline{x}] = \underline{s}$$

with

$$\boldsymbol{n}[\underline{x}] = [\ \boldsymbol{n}[x(t_1)],\ \boldsymbol{n}[x(t_2)], \ldots, \boldsymbol{n}[x(t_{2N+1})]\ ]'$$

Efficient HB implementations

$$\underline{x} = \Gamma^{-1}\underline{X}, \qquad \underline{X} = [A_0,\ A_1, ..., A_N,\ B_1, ..., B_N]'$$

$$\Gamma^{-1} = \begin{bmatrix} 1 & \gamma^c_{1,1} & \gamma^s_{1,1} & \cdots & \gamma^c_{1,N} & \gamma^s_{1,N} \\ 1 & \gamma^c_{2,1} & \gamma^s_{2,1} & \cdots & \gamma^c_{2,N} & \gamma^s_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \gamma^c_{2N+1,1} & \gamma^s_{2N+1,1} & \cdots & \gamma^c_{2N+1,N} & \gamma^s_{2N+1,N} \end{bmatrix}$$

$$\gamma^c_{p,q} = \cos(q\omega t_p) = \cos\left(\frac{q2\pi p}{2N+1}\right)$$

$$\gamma^s_{p,q} = \sin(q\omega t_p) = \sin\left(\frac{q2\pi p}{2N+1}\right)$$

Efficient HB implementations

$$\underline{y} = \Gamma^{-1}\,\Omega(\omega)\,\underline{X}$$

$$\Omega(\omega) = \begin{bmatrix} L(0) & 0 & 0 & \dots & 0 & 0 \\ 0 & R_1 & I_1 & \dots & 0 & 0 \\ 0 & -I_1 & R_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R_N & I_N \\ 0 & 0 & 0 & \dots & -I_N & R_N \end{bmatrix}$$

$$R_k = \boldsymbol{Re}\{L(jk\omega)\}, \quad I_k = \boldsymbol{Im}\{L(jk\omega)\}$$

$$\boldsymbol{n}[\underline{x}] = \boldsymbol{n}[\Gamma^{-1}\underline{X}]$$

Efficient HB implementations

$$\underline{y} + \boldsymbol{n}[\underline{x}] = \underline{s}$$

$$\Downarrow$$

$$\Gamma^{-1}\,\Omega(\omega)\,X + \boldsymbol{n}[\Gamma^{-1}\underline{X}] = \underline{s}$$

$$\Downarrow$$

$$\Omega(\omega)\,\underline{X} + \Gamma\,\boldsymbol{n}[\Gamma^{-1}\underline{X}] = \Gamma\,\underline{s}$$

The $2N + 1$ equations in the $2N + 1$ unknowns $\underline{X}$ can be solved without performing any integrals.

Limit cycle stability

- Limit cycles may present the same stability characteristics of equilibrium points: they may be stable, unstable or behave as saddles.
- The stability of limit cycles is studied through the *Poincarè map*, that reduces the stability property of a limit cycle to those of a nonlinear discrete system.
  - Example: Piece-wise linear Van der Pol oscillators
- The stability can also be studied through spectral techniques.

Limit cycle stability

- Autonomous systems:

$$\frac{dx(t)}{dt} = f(x)$$

- Limit cycle solution of period T detected through HB

$$\gamma(t+T) = \gamma(t) = \mathbf{A_0} + \sum_{k=1}^{N} [\mathbf{A_k} \cos(k\omega t) + \mathbf{B_k} \sin(k\omega t)]$$

$\gamma(t), \mathbf{A_k}, \mathbf{B_k} \in R^n$

- Variational equation:

$$\frac{d\widetilde{x}(t)}{dt} = P(t)\widetilde{x}(t), \quad (x(t) = \widetilde{x}(t) + \gamma(t))$$

$P(t) = Df[\gamma(t)]$ is the Jacobian evaluated at the limit cycle

Limit cycle stability

- Floquet multipliers μ_i (eigenvalues of the Poincarè map) can be numerically evaluated as the eigenvalues of:

$$exp[P(t_M)\Delta]exp[P(t_{M-1})\Delta]\dots exp[P(t_1)\Delta]$$

$t_p = p\frac{T}{M} = p\Delta$

- One Floquet multipliers is always unitary, the other ones determine the stability characteristics of the limit cycle.

Limit cycles in arrays of oscillators: Applications

1. Real time image processing.
   1. Multiscale analysis and image pre-processing (PDE based methods).
   2. Medical applications: mammography, echo-cardiography.
   3. Stereoscopic vision.
2. Bio-inspired models.
   1. Visual system: retina model.
   2. Tactile system.
   3. Audio system.
   4. Talamo-cortical system (networks of weakly coupled oscillators): associative memories.
3. *CNN Universal Machine*: analog algorithms.

References

1. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer-verlag New York Inc., 1983.
2. A. H. Nayfeh, *Perturbation Methods*, John Wiley, New York, 1973.
3. A. I. Mees, *Dynamics of Feedback Systems*, John Wiley, New York, 1981.
4. M. Gilli, F. Corinto, and P. Checco, "Periodic oscillations and bifurcations in cellular nonlinear networks," *IEEE Trans. Circuits Syst. I*, vol. 51, pp. 948-962, 2004.