# Nonlinear Dynamical Systems *for students of engineering, informatics, and bionics*

(Revised and shortened version of the 2013 Hungarian original)
(The introductory seven pages (i-vii) are in Hungarian)

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# Bevezetés

Egli é scritto in lingua matematica — *Galilei*

A Galilei utáni nemzedék számára már magától értetődő tény, hogy a természet könyve a matematika nyelvén íródott. Ezért fogalmazhatta meg[1] Newton: hasznos — más, korábbi fordításokban helyénvaló — dolog differenciálegyenleteket megoldani.

Differenciálegyenletek megoldása a szó teljes értelmében azok kvantitatív és kvalitatív vizsgálatát jelenti. Az intuíciót a vizsgálandó egyenlet konkrét fizikai, műszaki, biológiai vagy éppen közgazdaságtani jelentése alapvetően meghatározza. Az intuíció másik forrása az egyre növekvő számítógépi–szimulációs tapasztalat. A már idézett V.I. Arnold híres megállapítása szerint a matematika a természettudományoknak az az ága, amelyben a kísérletezés olcsó. A számítógéppel kapott eredmények — too much progress, too much promise — mit sem érnek a megfelelő interpretáció nélkül. A keretet a matematika, mint a természet– és a műszaki tudományok univerzális nyelve jelenti.

A differenciálegyenletek, a számítógép és a nem–„*in silico*" kísérletek kapcsolatát S. Luzzatto és J.D. Murray szavai jól jellemzik: „Simulation of (continuous time) dynamical systems is often taken for granted in the sciences and engineering because methods for solving initial value problems of ordinary differential are one of a small number of basic numerical algorithms in toolkits for scientific computation. Modeling is seen as the hard part; simulating the models the easy part. Nonetheless, this process seldom

[1] *Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa.* A XVII. századi latin mondatot nem könnyű modern nyelvekre fordítani, hiszen a differenciál– és integrálszámítás hőskorának többek között az adekvát szaknyelv megteremtése is hosszan elhúzódó feladata volt. Newton és Leibniz eredeti megfogalmazásai (és eredeti, egymásétól egyébként nagyon különböző jelölései) közül ma csak keveset használunk. Az angol nyelvű szakirodalomban a V.I. Arnold *Geometric Methods in the Theory of Ordinary Differential Equations, Springer, Berlin, 1983)* könyvében szereplő fordítás a leginkább elterjedt: *„In contemporary mathematical language, this means: It is useful to solve differential equations".*

answers all of the questions we ask about a model. Dynamical Systems Theory provides mathematical foundations for going much farther, but additional numerical methods are needed to connect the mathematics and the models." (S.L.) valamint „It is premature to say one mechanism is a best model until further experimental information is available." (J.D.M.).

A jegyzet kontextusában a differenciálegyenletek és a dinamikus rendszerek jelentése jól fedi egymást. Megírásával a Pázmány Péter Katolikus Egyetem Információs Technológiai és Bionikai Karán rendszeresen tartott két kurzusom hallgatói részére kívántam segítséget adni, a „Tér–időbeli jelek, modellek és számítógépek" és a „Dinamikai modellek a biológiában" tágabb matematikai környezetének bemutatásával.

Amióta a Práter utcában (PPKE ITK) és a Lágymányosi utcában (SZTAKI) dolgozom, megértettem, hogy a *Big Data* korában az adatok csoportosítása és szűrése elsőrendűen fontos feladattá vált az alkalmazott matematikai analízis egésze szempontjából. Az adatbányászat és az adatfeldolgozás nem az én kenyerem. Konkrét lépéseket, mint oktató, nem tudok tenni ezekbe az irányokba, de azt elhatároztam, hogy a következő években — Ottlik Géza „Iskola a határon" című regényének non est volentis neque currentis (sem azé aki akarja, sem azé aki fut (utána)) mottója erre is vonatkozik — az általam oktatott matematika tárgyakat szeretném az *Observational Mathematics* irányába vinni.

A jegyzetnek, tudom, sok hiányossága és bizonyára jónéhány hibája is van. Minden visszajelzést, kritikai megjegyzést[2] előre is köszönök.

A magyarázatok nemegyszer „fecsegő" hangja a szemléletességet próbálja növelni és a szaknyelvet a köznyelvhez közelíteni. A szemléletességet ezzel együtt elsősorban az ábrák és a kísérő animációk közvetítik.

[2]Komoly hiányosságnak érzem, hogy a fizikai mértékegységeket és a paraméterek konkrét értékeit szinte soha nem tüntettem fel. A mérnök–kollégákkal való célirányos konzultáció sokat segített volna ebben, de nem futotta rá az időmből. Mentségemre szolgál az is, hogy a matematikusok a matematika saját objektumait szokták vizsgálni, és általában nem számokkal, hanem betűkkel számolnak.

Amit legjobban sajnálok, az az, hogy nem tudtam megírni a relaxációs oszcillációk matematikájáról szóló fejezetet (jóllehet tudom, hogy relaxációs oszcillációkkal mind az informatikus–mérnök, mind a bionikus hallgatók több szaktárgyban is gyakran találkoznak). Erre sem volt időm, pedig nagyon szerettem volna. A lineáris analízis rész tárgyalásából hiányzik számos, a Frobenius–Perron tételekre és a Markov–láncokra történő utalás. A parciális egyenletek és az idősorok alapján történő, szinte egyenletek nélküli számolás részletes tárgyalása fel sem merült bennem, az összehasonlíthatatlanul nagyobb falat lett volna.

Az élményt, amit Galilei érzett, amikor először pillantotta meg a Jupiter holdjait, vagy amit Haydn érzett, amikor első alkalommal nézett Herschel távcsövébe egy júniusi éjszakán, még töredékesen sem tudom újra–élni. Mégis, örömmel írom ide — találjon visszhangra az Olvasóban! — Wigner Jenő *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* dolgozatának befejező mondatát: „The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.”[3]

Budapest, 2013 június 30.

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[3]Ráadásként álljon itt még egy idézet, Th. Merton huszadik századi amerikai szerzetes–költő (fiatal korában ismert jazz–zenész) egy esszéjéből: „There is a logic of language and a logic of mathematics. The former is supple and lifelike, it follows our experience. The latter is abstract and rigid, more ideal. The latter is perfectly necessary, perfectly reliable: the former is only sometimes reliable and hardly ever systematic. But the logic of mathematics achieves necessity at the expense of living truth, it is less real than the other, although more certain. It achieves certainty by a flight from the concrete into abstraction. Doubtless, to an idealist, this would seem to be a more perfect reality. I am not an idealist.” A szövegrész világosan utal a matematikai modell–alkotás egyik legfőbb nehézségére — egyszerre kell a köznyelvet, legalább egy természet– vagy műszaki tudomány, valamint a matematika nyelvét használnunk — de egyúttal a mottóul választott Galilei idézet kommentárjának is tekinthető.

Köszönetnyilvánítás:

KÖSZÖNET a közvetett numerikus tapasztalatokért

valamennyi Práter utcai gyakorlatvezetőnek, akikkel együtt dolgoztam:
Balogh Ádám, Gelencsér András, Goda Márton, Hartdégen Márton, Horváth András, Indig Balázs, Juhász János, Keömley–Horváth Bence, Lakatos Péter, Ligeti Balázs, Mohácsi Máté, Reguly István, Szabó Ágnes, Szélig Ádám

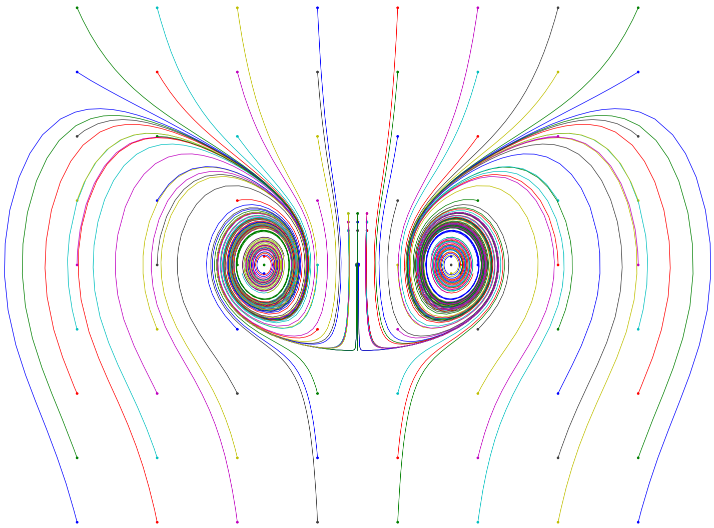
azoknak a fiatal kollégáknak, akikkel más formában dolgoztam együtt:
Bánhelyi Balázs (Szeged), Csikja Rudolf (BME), Koller Miklós (PPKE), Simkó Marcell (PPKE), Stubendek Attila (PPKE), Tornai Gábor (PPKE)

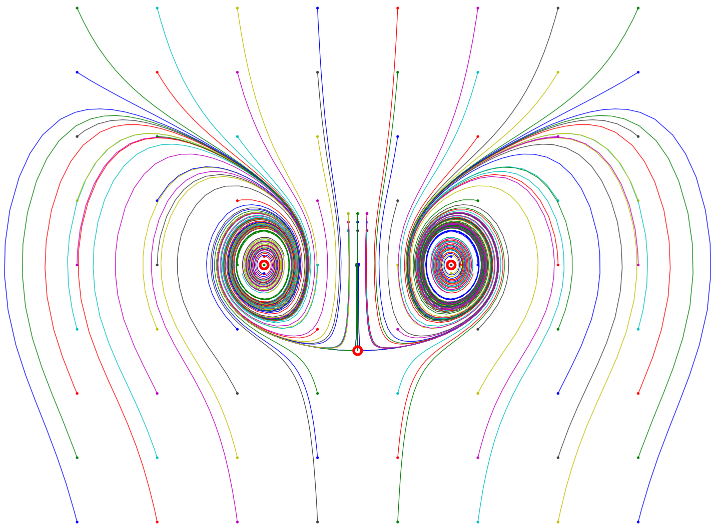
KÖSZÖNET azoknak, akiktől egyet s mást *mind a számítógépi, mind az alkalmazott matematika területén* megtanultam (nem rajtuk múlott, hogy nem többet):
Csendes Tibor (Szeged), Ercsey–Ravasz Mária (Kolozsvár), Galántai Aurél (Óbuda), Hatvani László (Szeged), Horváth Róbert (BME), Hujter Mihály (BME), Karsai János (Szeged), Máté László (BME), Nagy Zoltán (SZTAKI), Roska Tamás (PPKE), Simon L. Péter (ELTE), Stoyan Gisbert (ELTE), Tóth János (BME)

Köszönet az ábrákért és az animációkért, amelyeket *Balogh Ádám* kísérletező kedvvel, időről–időre nekem is meglepetéseket okozva készített el. Az ábrák és az animációk nemcsak illusztrálják a szöveget, hanem esetről esetre annak lényegét fejezik ki. Köszönöm *Kiss Márton* lektor segítőkész és gondos munkáját. A jegyzet végső formába öntéséhez a LATEX titkait jól ismerő *Koller Miklós* nyújtott időt és fáradtságot nem kímélő, nélkülözhetetlen segítséget, amelyért nagyon hálás vagyok. Külön köszönet *Nyékyné Gaizler Judit* prodékán asszonynak és *Simonovits András*–nak a jegyzet írásával kapcsolatos baráti figyelmükért és tanácsaikért.

Ez a jegyzet a 2013–as első változat rövidítése és átdolgozása.

COBRAS OR PEACOCK–BUTTERFLIES?





NONE OF THEM – PLEASE SEE THE 2D FAMILY OF EQUATIONS (??) PARAMETRIZED BY $\mu$.

## A jegyzet szerkezete, szerkesztése

*a Pólya György féle spirális elvet követi:* megerősítő ismétlések, egyre több részlet egyre gazdagabb kibontásával.

A legfontosabb kérdéskörök

- a kvalitatív–geometriai elmélet elemei
- a diszkretizált/közelítő és a pontos megoldás viszonya
- a linearizálás módszere
- a káosz

már az első fejezetben megjelennek. Az itteni tárgyalásmód teljesen szemléletes, és nem lépi túl az LRC–kör, a rugó– valamint az inga/hajóhinta–egyenlet által felkínált kereteket. Az első fejezet négy függeléke — MATLAB, lineáris algebra, lineáris analízis, közönséges differenciálegyenletek egyensúlyi helyzeteinek osztályozása a síkon — ismétlés jellegű. (Az ötödik függelék a függelékek szokásos stílusát követi).[4]

A 41 sorszámozott Tétel mindegyikét igyekeztem érthetővé tenni, de közülük csak alig néhánynak írtam le a bizonyítását. A bizonyítások egy része hibabecslési–perturbációs technikákat mutat be, közöttük az implicit függvény tétel két alkalmazását, a fennmaradók az iterált függvényrendszereken alapuló képtömörítés határértéktételéhez, illetve a kombinatorikus káosz egydimenziós, intervallum–leképezésekre vonatkozó változatához vezetnek el.

A legfontosabb alfejezetek sorszáma: 2.2, 2.15, 3.7, 3.8 — a konkrét példák sokféleségén keresztül ezek mutatják be a dinamikus rendszerek fogalomkörének és a kísérő számítógépes módszerek alkalmazhatóságának távlatait.

[4] Az Olvasók egy része számára a $\ll$ és a $\gg$ jelölések szokatlanok lehetnek: $0 < \varepsilon \ll 1$, illetve $\Omega \gg 1$ az elegendően/nagyon kicsiny, illetve az elegendően/nagyon nagy pozitív számokat jelentik.

# Tartalomjegyzék

# 1. fejezet

# Introductory examples

## 1.1. Nonlinear and linear oscillators

We present four basic ordinary differential equations of second order. Linear damping is allowed but, for the time being, we assume that there is no outer forcing.

We shall consider

- oscillatory circuits in electronics
  - Van der Pol equation
  - the standard linear RLC equation
- oscillators in mechanics
  - the pendulum equation
  - the standard linear spring equation

An RLC circuit is an electrical circuit consisting of a resistor, an inductor, and a capacitor connected in series. The resistor can be linear or nonlinear.

The currents through the resistor, through the inductor, and through the capacitor are the same. Thus we can speak about the current (as a function of time $t \in \mathbb{R}$) and see that

$$I_L = I_R = I_C = I\,.$$

By Kirchhoff's loop rule, the respective voltages $V_L$, $V_R$, $V_C$ (also as functions of $t$) satisfy

$$V_L + V_R + V_C = 0\,.$$

The basic formulas are

$V_L = L\dot{I}_L$ (Lenz's rule for a linear inductor e.g. a linear coil),

$V_R = f(I_R)$ (Ohm's rule for a current–driven resistor), e.g.

$V_R = RI_R$ (Ohm's rule for a linear resistor)

$V_R = p\big(-I_R + \frac{I_R^3}{3}\big)$ (Ohm's rule for a nonlinear resistor of van der Pol type)

$V_C = \frac{1}{C}Q = \frac{1}{C}\int_{-\infty}^{t} I_C(s)\,ds$ (rule for a linear capacitor).

Here linear inductance $L>0$, linear resistance $R\geq 0$, linear capacitance $C>0$ are constants. Also parameter $p\geq 0$ is a constant but charge $Q$ is a function of $t$. Newton–Leibniz formula in defining $V_C$ implies that $\frac{d}{dt}Q = \dot{Q} = I_C$.

### 1.1.1. Van der Pol circuit

By using the previous considerations, the integro–differential form of the equation for Van der Pol's circuit is

$$L\dot{I} + p\big(-I + \frac{I^3}{3}\big) + \frac{1}{C}\int_{-\infty}^{t} I(s)\,ds = 0\,. \tag{1.1}$$

Multiplication by $C$ and differentiation with respect to time $t$ give that

$$CL\ddot{I} + Cp(-1+I^2)\dot{I} + I = 0\,. \tag{1.2}$$

The normal form of van der Pol equation

$$\ddot{x} - \mu(x^2-1)\dot{x} + x = 0 \tag{1.3}$$

is obtained by LINEAR TIME SCALING $I(t) = x(at) = x(\tau)$, $at=\tau$, *a simple method to reduce the number of parameters.* The key is to find the „best choice" for parameter $a$. Since

$$\dot{I} = a\,x' \quad \text{and} \quad \ddot{I} = a^2\,x' \quad \text{by the chain rule} \quad \frac{d}{dt}I = \frac{d}{d\tau}x\cdot\frac{d}{dt}\tau = x'\cdot a\,,$$

Thus equation (1.2) goes over into equation $CLa^2x'' + Cpa(-1+x^2)x' + x = 0$ where $x' = \frac{d}{d\tau}x$. By letting $CLa^2 = 1$, the coefficient of $x''$ is taken for 1. We conclude that $a = \frac{1}{\sqrt{CL}}$ and set $\mu = Cpa = p\sqrt{\frac{C}{L}}$. Replacing $\tau$ by $t$, we arrive at (1.3).

Finally, the second order differential equation (1.3) is written to a system of two first order differential equations, one for $x$ and one for $y$ where $y$ is defined by $y=\dot{x}$ and $\dot{y}$ is taken from $\dot{y} - \mu(x^2-1)y + x = 0$.

All in all, van der Pol equation simplifies to the pair of the two normal forms

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0 \quad \Leftrightarrow \quad \left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= \mu(1-x^2)y - x\end{aligned}\right\} \quad \text{where} \quad \mu\geq 0\,. \tag{1.4}$$

*Symmetries of the van der Pol system.* A planar system of differential equations

$$\left.\begin{array}{l}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{array}\right\}, \quad \begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}\times\mathbb{R} = \mathbb{R}^2 \tag{1.5}$$

can have different types of symmetries. The simplest symmetry is probably reflection symmetry with respect to the origin. The algebraic definition is to require that

$$f(-x,-y) = -f(x,y) \quad \text{and} \quad g(-x,-y) = -g(x,y) \qquad \text{for each} \quad x,y\in\mathbb{R}\,.$$

In other words, replacing $\binom{x}{y}$ by $\binom{-x}{-y}$ (and thus $\binom{\dot{x}}{\dot{y}}$ by $\binom{-\dot{x}}{-\dot{y}}$), system (1.5) is not allowed to change. In fact, $-\dot{x} = f(-x,-y)$ is replaced by $-\dot{x} = -f(x,y)$ and $-\dot{y} = y(-x,-y)$ is replaced by $-\dot{y} = -g(x,y)$. Canceling the negative signs on both sides, we arrive back at system (1.5) again.

The geometric meaning is that—reflecting $\mathbb{R}^2$ with respect to the origin—the vector field

$$\begin{pmatrix}f(x,y)\\ g(x,y)\end{pmatrix} \in \mathbb{R}\times\mathbb{R} = \mathbb{R}^2 \quad \text{based at} \quad \begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}\times\mathbb{R} = \mathbb{R}^2$$

goes over into itself. In other words, the vector field is invariant under the linear transformation defined by matrix $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. The same invariance holds true for the solution trajectories as well (also for numerical trajectories provided by computer simulations through the points $\binom{x_0}{y_0)}$ and $\binom{-x_0}{-y_0)}$, with $x_0, y_0 \in \mathbb{R}$ arbitrarily given.

No doubt the previous two paragraphs are a bit too abstract. However, You should try to visualize them for $f(x,y) = y$, $g(x,y) = \mu(1-x^2)y - x$ if $\mu = \frac{1}{2}$ and $x = x_0 = \pm\sqrt{2}$, $y = y_0 = \pm\sqrt{2}$ (four vectors attached to four points). This can be made by hand or, more conveniently, by the computer. **Then You will understand.** (Maybe it is better to have a look into the next Chapter first. Observe the interplay between algebra and geometry, between formulas and figures.)

For each $\mu \geq 0$, the planar van der Pol system in (1.4) is symmetric with respect to the origin. For $\mu = 0$, also rotational symmetry about the origin holds true.

*Global solution geometry of the van der Pol equation.*

**1.1. Theorem** *For each $\mu > 0$, the planar van der Pol system in (1.4) has a unique, asymptotically stable periodic orbit $\Gamma_\mu$ (which is reflectionally symmetric with respect to the origin). With $t \to \infty$, all trajectories of the pointed plane $\mathbb{R}^2 \setminus \{\binom{0}{0}\}$ are attracted to $\Gamma_\mu$. Both the point at infinity and the equilibrium point at the origin are repelling. If $\mu \to 0^+$ then $\Gamma_\mu \to \Gamma_0$ where $\Gamma_0$ is the circle of radius 2 about the origin, representing the solution $x(t) = 2\cos(t)$, $y(t) = -2\sin(t)$ of the $\mu = 0$ van der Pol system $\dot{x} = y$, $\dot{y} = -x$ with initial condition $x(0) = 2$, $y(0) = 0$.*

A Van der Pol egyenlet szimmetriája egyébként *a belső időváltozó és a paraméter szerinti szimmetriát* is jelent. Valóban, az

$$x(t) = z(\tau) = z(-t)\ ,\quad y(t) = -w(\tau) = -w(-t)\ ,\quad \tau = -t\ ,\quad \nu = -\mu$$

transzformációk együttes hatása a (??) Van der Pol egyenlet síkbeli rendszerré átírt változatára

$$\left.\begin{array}{l}\dot{x} = y\\ \dot{y} = \mu(1-x^2)y - x\end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l}\dot{z} = w\\ \dot{w} = \nu(1-z^2)w - z\,,\end{array}\right.$$

ami egyúttal azt is magyarázza, miért szokás a Van der Pol egyenletet csupán a $\mu$ paraméter $\mu \geq 0$ értékeire vizsgálni.

### 1.1.2. The standard linear RLC circuit

By using the introductory considerations in this Chapter, the differential equation of the linear RLC circuit can be written in the form

$$L\dot{I} + RI + \frac{1}{C}Q = 0\,. \tag{1.6}$$

Multiplication by $C$ and property $I = \dot{Q}$ give that

$$CL\ddot{Q} + CR\dot{Q} + Q = 0 \quad \Leftrightarrow \quad \left.\begin{array}{l}\dot{Q} = I\\ \dot{I} = -\frac{1}{CL}Q - \frac{R}{L}I\end{array}\right\}. \tag{1.7}$$

Argueing as for the van der Pol equation, linear time scaling $Q(t) = x(at) = x(\tau)$, $at = \tau$ leads to

$$\ddot{x} + b\dot{x} + x = 0 \quad \Leftrightarrow \quad \left.\begin{array}{l}\dot{x} = y\\ \dot{y} = -x - by\end{array}\right\}, \tag{1.8}$$

the pair of the two normal forms of the RLC circuit equation with damping coefficient $b = R\sqrt{\frac{C}{L}} \geq 0$ (and $a = \frac{1}{\sqrt{CL}}$).

*Energy considerations.* As the sum of the energy stored in the inductor and in the capacitor at time $t$, the total energy of the linear RLC circuit is

$$E(t) = \frac{1}{2}LI^2(t) + \frac{1}{2C}Q^2(t)\,.$$

Applying (1.6), we obtain via time differentiation that

$$\dot{E}(t) = LI\dot{I} + \frac{1}{C}Q\dot{Q} = I\left(L\dot{I} + \frac{1}{C}Q\right) = -RI^2 \leq 0\,. \tag{1.9}$$

Thus the total energy dissipates in the linear resistor and satisfies (though $I^2(t)=0$ is possible for isolated time instances far from each other) $E(t)\to 0^+$ as $t\to\infty$ whenever $R>0$. Moreover, apart from the equilibrium point at the origin, the energy is strictly decreasing along the trajectories of the $Q$-$I$ planar system (1.7) with $R>0$.

On the other hand, $R=0$ implies conservation of energy by $\dot{E}(t)=0$ for all $t$. In other words, the total energy of an LC circuit is kept fixed. In particular, the trajectories of the $Q$-$I$ planar system (1.7) with $R=0$ stay on energy level curves, i.e., on ellipses of equations $\frac{1}{2}LI^2+\frac{1}{2C}Q^2=\mathcal{C}$ ($\mathcal{C}\geq 0$, $\mathcal{C}$ fixed) of the $Q$-$I$ plane.

Returning to the van der Pol equation, we see that $-\mu(1-x^2)$ in (1.4) can be interpreted as a negative damping $b<0$ in (1.8) if $|x|<1$. Thus, at least for certain time intervals, the nonlinear resistor works as a source of energy in the circuit. Any physical realization of the current–driven resistor $V_R=f(I_R)$ with characteristic $f(I)=I-\frac{I^3}{3}$, $I\in[I_{min},I_{max}]$ is an active element of the circuit and contains something like a battery.[1]

### 1.1.3. The pendulum equation

The pendulum we have in mind consists of a mass $m$ attached to a rigid weightless rod of length $\ell$. The force of gravity is $F_{grav}=mg$, directed downward. The point of suspension $S$ is fixed and the pendulum is allowed to make full turns in a fixed vertical plane. (Thus, it is rather a dizzy swing–boat in an amusement park than a pendulum kept in a human hand.) Linear damping is allowed, too. It is clear that the mass swings along a vertical circle of radius $\ell$ (or along an arc thereof) centered at the fixed point $S$.

The differential equation of the pendulum will be derived as an application of Newton second law in the tangential direction of the circle we mentioned. Thus the vectorial form of Newton second law can be avoided. The unknown of the pendulum equation will be $\theta\in\mathbb{R}$, the angle between the rod and the vertical at the suspension point $S$ measured with counterclockwise orientation.

We learned from Newton that the product of mass $m$ and the tangential acceleration $\ell\ddot{\theta}$ equals to the tangential component of the force acting on the mass at the angular position $\theta$. The force in question is the sum of the tangential component of the gravitational force $F_{grav}$ as well as the damping force which is proportional to the tangential velocity $\ell\dot{\theta}$ pointing back to the vertical along the tangent of the circle. Thus the second law of Newton (connecting mass, acceleration, and resultant force) $m\overrightarrow{a}=\sum_i\overrightarrow{F}_i$ can be

[1] Given any bounded closed interval $[I_{min},I_{max}]\subset\mathbb{R}$, any continuous function $f:[I_{min},I_{max}]\to\mathbb{R}$ and any $\varepsilon>0$, one can build a current–driven resistor $V_R=f_\varepsilon(I_R)$ with a continuous characteristic $f_\varepsilon:[I_{min},I_{max}]\to\mathbb{R}$ such that $|f_\varepsilon(I)-f(I)|<\varepsilon$ for each $I\in[I_{min},I_{max}]$. This is one of the basic theoretical results in electrical circuit theory. Unfortunately, the more complicated $f$, the more complicated the device one has to build. Within the classical van der Pol 1925 circuit, the core of the nonlinear resistor was a tunnel diode designated by van der Pol. In order to build such a nonlinear resistor today, electrical engineers prefer to use operational amplifiers and analog multipliers instead.

written in the form

$$m\ell\ddot{\theta} = -mg\sin(\theta) - \beta\ell\dot{\theta}\,. \tag{1.10}$$

Dividing by $mg$ and arranging all terms to the left–hand side of the equation, we obtain that

$$\frac{\ell}{g}\ddot{\theta} + \frac{\beta\ell}{mg}\dot{\theta} + \sin(\theta) = 0\,. \tag{1.11}$$

Hence LINEAR TIME SCALING $\theta(t) = x(at) = x(\tau)$, $at = \tau$ leads to

$$\ddot{x} + b\dot{x} + \sin(x) = 0 \quad \Leftrightarrow \quad \left.\begin{array}{l} \dot{x} = y \\ \dot{y} = -\sin(x) - by \end{array}\right\}, \tag{1.12}$$

the pair of the two normal forms of the pendulum equation with damping coefficient $b = \frac{\beta}{m}\sqrt{\frac{\ell}{g}} \geq 0$ (and $a = \sqrt{\frac{g}{\ell}}$).

*Energy considerations.* As the sum of the kinetic energy (energy of motion) and the potential energy at time $t$, the total energy of the pendulum can be written as

$$E(t) = \frac{1}{2}m(\ell\dot{\theta}(t))^2 + mg\ell\big(1 - \cos(\theta(t))\big)\,,$$

Applying (1.10), we obtain via time differentiation that

$$\dot{E}(t) = m\ell^2\dot{\theta}\ddot{\theta} + mg\ell\sin(\theta)\dot{\theta} = \ell\dot{\theta}\big(m\ell\ddot{\theta} + mg\sin(\theta)\big) = -\ell\beta mg\dot{\theta}^2 \leq 0\,. \tag{1.13}$$

Thus the total energy dissipates (by the aerodynamical friction due to the motion of the mass) through the air and satisfies (though $\dot{\theta}^2(t) = 0$ is possible for isolated time instances far from each other) $E(t) \to 0^+$ as $t \to \infty$ whenever $\beta > 0$. Moreover, apart from equilibrium points, the energy is strictly decreasing along the trajectories of the $\theta$-$\dot{\theta}$ planar system (1.10) with $\beta > 0$.

On the other hand, $\beta = 0$ implies conservation of energy by $\dot{E}(t) = 0$ for all $t$. In other words, the total energy of the pendulum is kept fixed if aerodynamical friction is absent. In particular, the trajectories of the $\theta$-$\dot{\theta}$ planar system (1.10) with $\beta = 0$ stay on energy level curves, i.e., on curves of equations $\frac{1}{2}m(\ell\dot{\theta})^2 + mg\ell\big(1 - \cos(\theta)\big) = \mathcal{C}$ ($\mathcal{C} \geq 0$, $\mathcal{C}$ fixed) of the $\theta$-$\dot{\theta}$ plane. The sharpness of inequality $\mathcal{C} \geq 0$ is a consequence of the fact that the minimum possible energy is zero, attained in the lower equilibrium position of the pendulum $\theta = 0$, $\dot{\theta} = 0$.

For the normal form equation of the pendulum (1.12) obtained by linear time scaling $\theta(t) = x(at) = x(\tau)$, $at = \tau$, the total energy simplifies to

$$E(x,y) = \frac{1}{2}\dot{x}^2 + \big(1 - \cos(x)\big) = \frac{1}{2}y^2 + \big(1 - \cos(x)\big)\,.$$

### 1.1.4. The standard linear spring equation

The spring we have in mind consists of a rectangular mass $m$ attached to a weightless spring of constant $k$. The mass can move in the right–left direction of a horizontal plane. Thus our spring is horizontal. The force of gravity is $F_{grav} = mg$, directed downward, and the friction coefficient is $\beta$. The position of the mass is zero at the point where the spring is unstretched. Position, velocity, and acceleration of the mass are denoted by $x$, $v = \dot{x}$, $a = \ddot{x}$. They are measured by rightward orientation. Damping is assumed to be linear.

If the spring is stretched, the restoring force is leftward and can be written as $F_{rest} = = -kv - \beta mg$. In view of Newton second law, the equation of motion is

$$m\ddot{x} = -\beta mg\dot{x} - kx \tag{1.14}$$

or, equivalently,

$$m\ddot{x} + \beta mg\dot{x} + kx = 0\,. \tag{1.15}$$

Recall the linear RLC circuit equation (1.7) in the form

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0$$

and observe the complete analogy between the damped linear spring equation (1.15) and the linear RLC circuit equation recalled. It is the right moment to emphasize **the enormous power of mathematics**. From the view–point of mathematical abstraction, the damped linear spring and the linear RLC circuit are exactly the same. Also the mechanical energy $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ and the electrical energy $\frac{1}{2}LI^2 + \frac{1}{2C}Q^2$ as well as the underlying mechanical and electrical oscillations are pairwise identified.

Hence LINEAR TIME SCALING leads to

$$\ddot{x} + b\dot{x} + x = 0 \quad \Leftrightarrow \quad \left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - by \end{aligned}\right\}, \tag{1.16}$$

the pair of the two normal forms for the damped linear spring equation with coefficient $b = \beta g\sqrt{\frac{m}{k}} \geq 0$ (and $a = \sqrt{\frac{k}{m}}$).

*Elementary aspects of linearization.* Since $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots \approx x$ for $x$ small, it is clear that (1.16) is the linear version of (1.12). More precisely, $\binom{x_0}{y_0} = \binom{0}{0} \in \mathbb{R}^2$ is an equilibrium point of system $\dot{x} = y$, $\dot{y} = -\sin(x) - by$ and thus

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) - by \end{aligned}\right\} \quad \Rightarrow \quad \left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - by \end{aligned}\right\} \text{ by linearization at equilibrium } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2.$$

Obviously, $\binom{x_0}{y_0} = \binom{0}{0}$ is not the only equilibrium point of system $\dot{x} = y$, $\dot{y} = -\sin(x) - by$. For example, $\binom{x_0}{y_0} = \binom{\pi}{0} \in \mathbb{R}^2$ is an equilibrium point, too.

As already observed, the point with coordinates $\binom{x_0}{y_0}=\binom{0}{0}\in\mathbb{R}^2$ corresponds to the lower equilibrium position within the physical space of the pendulum. This is downward rest position, the position of the motionless pendulum hanging downward, a stable position. Similarly, the point with coordinates $\binom{x_0}{y_0}=\binom{\pi}{0}\in\mathbb{R}^2$ corresponds to the upper equilibrium position within the physical space of the pendulum. This is straight–up position, the position of the motionless pendulum (or boat–swing) standing upright, a highly unstable position with great balance.

System $\dot{x}=y$, $\dot{y}=-\sin(x)-by$ can be linearized at equilibrium $\binom{\pi}{0}\in\mathbb{R}^2$ as well. The simplest way to do this is to start with $(x-\pi)$ instead $x$ and to recall that—in a small vicinity of the point $x_0=\pi$—the graph of the nonlinear function $\psi(x)=\sin(x)$ is linearly approximated by the graph of its tangent line

$$L^{\psi}_{x_0}(x)=\psi(x_0)+\frac{1}{1!}\psi'(x_0)\big(x-x_0\big)=\sin(x_0)+\cos(x_0)\,\big(x-x_0\big)=0+(-1)\big(x-\pi\big)=\pi-x$$

at the point $x_0=\pi$. Thus

$$\left.\begin{array}{l}\frac{d}{dt}(x-\pi)=y\\ \frac{d}{dt}\,y=-\sin(x-\pi)-by\approx(\pi-x)-by\end{array}\right\}\quad\text{valid for}\quad |x-\pi|\ \text{small}.$$

Introducing local coordinate $w=x-\pi$ near $x_0=\pi$, we conclude that $\dot{w}=y$, $\dot{y}=w-by$. Following the tradition of „abusing” notation, i.e., writing $x$ instead of $w$ in the sequel, we obtain that

$$\left.\begin{array}{l}\dot{x}=y\\ \dot{y}=-\sin(x)-by\end{array}\right\}\quad\Rightarrow\quad\left.\begin{array}{l}\dot{x}=y\\ \dot{y}=x-by\end{array}\right\}\quad\text{by linearization at equilibrium}\quad\begin{pmatrix}\pi\\0\end{pmatrix}\in\mathbb{R}^2.$$

Summarizing the observations above, we pass to the matrix notation of linear homogeneous systems of ordinary differential equations. The linearized systems in local coordinates near equilibria $\binom{0}{0}\in\mathbb{R}^2$ and $\binom{\pi}{0}\in\mathbb{R}^2$ are

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0&1\\-1&-b\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0&1\\1&-b\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\tag{1.17}$$

respectively.

*A general formula for linearization* (what at first reading seems to be a bit difficult). Please read the last paragraph of this Subsection to understand.

**1.2. Megjegyzés** *Starting from an equilibrium point $\binom{x_0}{y_0}\in\mathbb{R}^2$, i.e., from a constant solution (a solution satisfying $\binom{x(t)}{y(t)}=\binom{x_0}{y_0}$ for all time) of the planar system of differential equations (1.5), we see that*

$$\left.\begin{array}{l}\dot{x}=f(x,y)\\ \dot{y}=g(x,y)\end{array}\right\}\quad\Rightarrow\quad\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=J_{x_0,y_0}\begin{pmatrix}x\\y\end{pmatrix}\quad\textit{by linearization at}\quad\begin{pmatrix}x_0\\y_0\end{pmatrix}\in\mathbb{R}^2.\tag{1.18}$$

*Here $J_{x_0,y_0}$ is the $2\times 2$ Jacobian matrix of first order partial derivatives evaluated at equilibrium $\binom{x_0}{y_0}\in\mathbb{R}^2$ and (1.18) is understood in local coordinates near the equilibrium. All in all,*

$$J_{x_0,y_0}=\begin{pmatrix} f'_x(x_0,y_0) & f'_y(x_0,y_0)\\ g'_x(x_0,y_0) & g'_y(x_0,y_0)\end{pmatrix}$$

*and thus the linearized system at equilibrium $\binom{x_0}{y_0}\in\mathbb{R}^2$ can be written as*

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\begin{pmatrix} f'_x(x_0,y_0) & f'_y(x_0,y_0)\\ g'_x(x_0,y_0) & g'_y(x_0,y_0)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

The natural way of understanding a general formula is to consider at least two different special cases. Derive (1.17) via computing the respective Jacobian matrices for the $2\times 2$ system in (1.12). **Then You will understand.**

### 1.1.5. The pendulum equation as an example for a Hamiltonian system with potential

The undamped pendulum equation (the $b=0$ special case of equation (1.12))

$$\ddot x+\sin(x)=0 \quad\Leftrightarrow\quad \left.\begin{array}{l}\dot x=y\\ \dot y=-\sin(x)\end{array}\right\} \tag{1.19}$$

belongs to the class of two–dimensional Hamiltonian systems with potential

$$\ddot x+V'(x)=0 \quad\Leftrightarrow\quad \left\{\begin{array}{l}\dot x=y\\ \dot y=-V'(x)\end{array}\right. \quad\Leftrightarrow\quad \begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\begin{pmatrix}y\\ -V'(x)\end{pmatrix}. \tag{1.20}$$

The free constant in defining $V$ is usually chosen in such a way that the minimum of the potential function $V:\mathbb{R}\to\mathbb{R}$ be zero. In system (1.19) we set $V(x)=1-\cos(x)$ with derivative $V'(x)=\cos(x)$ and $\min_{-\infty<x<\infty}V(x)=V(0)=0$.

*Conservation of energy.* In agreement with the last line of the subsection entitled *The pendulum equation*, the total energies are

$$E(x,y)=\frac{1}{2}y^2+\big(1-\cos(x)\big)\qquad\text{and}\qquad E(x,y)=\frac{1}{2}y^2+V(x),$$

respectively. Recall that the total energy at time $t$ is $E(t)=E(x(t),y(t))$. As for the undamped pendulum equation, we apply the chain rule and obtain that

$$\dot E(t)=\frac{d}{dt}E(x(t),y(t))=\underline{grad}\,E(x,y)\cdot\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=(E'_x,E'_y)\cdot\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$$

$$= E'_x\,\dot{x} + E'_y\,\dot{y} = V'(x)\,y + y\big(-V'(x)\big) = 0$$

for all $t$. Hence $E(t)$ is constant. In other words, we have energy conservation. The sum of the kinetic energy (energy of motion) $\frac{1}{2}y^2$ and the potential energy $V(x)$ is kept fixed along the trajectories. For $x_0, y_0 \in \mathbb{R}$ arbitrarily chosen, the trajectory $\binom{x(t)}{y(t)} \subset \mathbb{R}^2$ through $\binom{x_0}{y_0} = \binom{x(t_0)}{y(t_0)} \in \mathbb{R}^2$ stays on the energy level curve $E(x,y) = E(x_0,y_0) = \mathcal{C}_0$ forever.

Moreover, the gradient, row vector $\underline{grad}\,E(x_0,y_0)$ $\big($ the normal vector to the level curve $E(x,y) = \mathcal{C}_0$ through $\binom{x_0}{y_0} \in \mathbb{R}^2$ $\big)$ and the tangent, column vector $\binom{\dot{x}(t_0)}{\dot{y}(t_0)}$ to the trajectory at time $t_0$ $\big($ at the time the trajectory passes through $\binom{x_0}{y_0} \in \mathbb{R}^2$ $\big)$ are perpendicular. This is the geometric meaning of the zero scalar product $\cdot$ we met in computing $\dot{E}(t)$ for $t = t_0$.

*Conservation of area.* Nevertheless, the energy is not the only quantity preserved by the dynamics of (1.19) and (1.20).

In order to understand the concept of an area–preserving differential equation, we return to system (1.5), the general form of autonomous[2] ordinary differential equations in two dimension. Consider a trajectory

$$\begin{pmatrix} x_{t_0,x_0,y_0}(t) \\ y_{t_0,x_0,y_0}(t) \end{pmatrix} \quad \text{with subscripts referring to initial condition} \quad \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The solution operator $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by letting

$$\Phi\left(t, t_0, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) = \begin{pmatrix} x_{t_0,x_0,y_0}(t) \\ y_{t_0,x_0,y_0}(t) \end{pmatrix} \qquad \text{whenever} \quad t, t_0 \in \mathbb{R} \;\; \text{and} \;\; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2.$$

Consider a bounded region $\Omega \subset \mathbb{R}^2$ with area $\mathrm{vol}\big(\Omega\big)$ and introduce

$$\Phi(t, t_0, \Omega) = \bigcup \left\{ \Phi\left(t, t_0, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \Omega \right\},$$

the image of $\Omega$ along the trajectories after traveling time $t - t_0$, i.e., at time $t$. For brevity, we write $\Omega(t - t_0) = \Phi(t, t_0, \Omega)$ and observe that $\Omega(0) = \Phi(t_0, t_0, \Omega) = \Omega$. Finally, we say that system (1.5) is area–preserving if, by definition,

$$\mathrm{vol}\big(\Omega(t - t_0)\big) = \mathrm{vol}\big(\Omega\big) \qquad \text{for each} \quad t, t_0 \in \mathbb{R} \;\; \text{and (bounded, measurable)} \;\; \Omega \subset \mathbb{R}^2.$$

By a classical result of Liouville, system (1.5) is area–preserving if and only if

$$\mathrm{div}\begin{pmatrix} f \\ g \end{pmatrix}(x_0, y_0) = \mathrm{trace}\big(J_{x_0,y_0}\big) = f'_x(x_0,y_0) + g'_y(x_0,y_0) = 0 \qquad \text{for each} \;\; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2.$$

[2]The term autonomous means simply that there is no explicit time $t$ on the right–hand side of (1.5). In other words, the vector field $\binom{f(x,y)}{g(x,y)}$ does not depend on the particular time instances.

As for the two–dimensional Hamiltonian systems with potential (1.20), we can directly check if the trace of the Jacobian matrix is zero. Now this is trivial because both $f'_x(x_0, y_0)$ and $g'_y(x_0, y_0)$ are identically zero.

*What is essential here.* Now we are in a position to summarize the conservation properties of the differential equations (1.19) and (1.20):

- the total energy $E(x, y) = \frac{1}{2}y^2 + V(x)$ is preserved along individual trajectories
- traveling the same time along trajectories, area of regions is preserved

Both assertions above remain valid for two–dimensional Hamiltonian systems, i.e., two–dimensional autonomous ordinary differential equations of the form

$$\left.\begin{aligned} \dot{x} &= H'_y(x, y) \\ \dot{y} &= -H'_x(x, y) \end{aligned}\right\} \quad \text{where} \quad H = H(x, y) \quad \text{is an arbitrary} \quad C^2 \quad \text{function}, \tag{1.21}$$

the so–called Hamiltonian function $\big($ chosen for the total energy $E(x, y) = \frac{1}{2}y^2 + V(x)$ in (1.20) $\big)$.

*What is relevant here for biology.* Differential equations investigated so far are only toy models for real–world applications.

On one hand, the coupled system

$$m_n \ddot{\underline{r}}_n + \frac{\partial}{\partial \underline{r}_n} V\big(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\big) = 0 \ , \quad n = 1,2,\ldots,N \tag{1.22}$$

is based on (1.20) and serves as the fundamental mass–spring model[3] of molecular dynamics. Her $m_n > 0$, $\underline{r}_n$, and $\dot{\underline{r}}_n$ are the mass, position, and velocity of the $n$-th atom in a molecule, respectively and $V : \mathbb{R}^{3N} \to \mathbb{R}$ is the potential energy. Unfortunately, $V$ can hardly be specified and $N$, e.g. in protein dynamics, is usually too large. But all this does not diminish the role of system (1.22) and of its various versions in pharmaceutical industry and research.

On the other hand, for $\mu > 0$ large, van der Pol equation[4] (1.3) is closely related to the Fitzhugh–Nagumo neuron model. In fact, applying the famous Lienard transformation

$$y = x - \frac{x^3}{3} - \frac{\dot{x}}{\mu},$$

[3] One may think of the ball-and-stick molecular models in chemistry where the balls are joints and the sticks can be made shorter and longer by hidden elastic springs. Observe also the similarity between ball-and-stick molecular models and images in cryo electron microscopy whereby biological samples, such as purified proteins, are embedded in vitreous ice preserving their native structures.

[4] It is worth mentioning here that van der Pol himself came across system (1.3) as he was building electronic circuit models of the human heart. The asymptotically stable periodic orbit $\Gamma_\mu$, $\mu > 0$ in Theorem 1.1. can be interpreted both as a simple mathematical description of normal heart beat and as an electric signal generated by an (oversimplified, toy) cardiac pacemaker.

we see that

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0 \quad \Leftrightarrow \quad \left.\begin{array}{l} \dot{x} = \mu\big(x - \frac{x^3}{3} - y\big) \\ \dot{y} = \frac{1}{\mu}\,x \end{array}\right\} \quad \text{where} \quad \mu \gg 1\,.$$

By letting $\varepsilon = \mu^{-2}$ and $a = \mu$, LINEAR TIME SCALING $x(t) = v(at)$, $y(t) = w(at)$ yields that van der Pol equation (1.3) goes over into case $a = b = I = 0$ of the famous Fitzhugh–Nagumo system

$$\left.\begin{array}{l} \dot{v} = v - \frac{v^3}{3} - w + I \\ \dot{w} = \varepsilon(v + a - bw) \end{array}\right\} \quad \text{where} \quad 0 < \varepsilon \ll 1\,, \tag{1.23}$$

modeling spike generation in a neuron stimulated by an external constant input current. Wave propagation in a nerve fiber can be described by the solutions of the mixed system

$$\left.\begin{array}{l} \dot{v} = v - \frac{v^3}{3} - w + I + v''_{xx} \\ \dot{w} = \varepsilon(v + a - bw) \end{array}\right\} \quad \text{where} \quad x \quad \text{is the spatial variable and} \quad 0 < \varepsilon \ll 1\,, \tag{1.24}$$

coupling a parabolic, partial and an ordinary differential equation. It is worth noting that (1.24) is a simplified version of the most famous Hodgkin–Huxley system modeling nerve cell excitability and explaining of how voltage–gated -ion channels give rise to propagating action potentials along the squid giant axon. For simulations in huge neuron networks, already coupled systems of the type (1.23) are too complicated. One works usually with integrate–and–fire models instead.

Nevertheless, toy models are demonstrative examples. They trigger understanding.

### 1.1.6. Simple one–species and two–species models in population dynamics

*The rise of public interest in population dynamics.* The sudden collapse of the Californian fishing industry in 1952 put marine population dynamics in the focus of public opinion and led to significant government spending on research in this field.[5]

[5]The seasonal catch of sardines along the coast of California dropped by 98% within two years (from 353088 tons in 1950/51 to 5711 tons in 1952/53). Most scholars agree that the catastrophe was caused by the coincidence of several factors but the role and relative importance of the individual factors including those of the Pacific Decadal Oscillation have remained highly debated. It is much better to speak about a complex and still unknown interplay of reasons. For many years after the disaster, overfishing was erroneously considered as the main culprit. — Data is taken from *The Collapse of the California Sardine Fishery – What Have We Learned?* by John Radovich, in Resource Management and Environmental Uncertainty (M.H. Glantz and J.D. Thompson, eds.), San Francisco, Wiley, 1981.

However, marine population dynamics started much earlier. In 1926 the biologist D'Ancona completed a statistical study[6] of the numbers of each species sold on the fish markets of three ports: Fiume, Trieste, and Venice. He observed that the highest percentages of predators occurred during and just after the war, when fishing was drastically curtailed. For example, the yearly percentages of predator species in the Fiume catch[7] are shown in the following table:

Percentages of predators in the Fiume fish catch

| 1914 | 1915 | 1916 | 1917 | 1918 | 1919 | 1920 | 1921 | 1922 | 1923 |
|---|---|---|---|---|---|---|---|---|---|
| 12 | 21 | 22 | 21 | 36 | 27 | 16 | 16 | 15 | 11 |

D'Ancona concluded that the predator–prey balance was at its natural state during the war, and that intense fishing before and after the war disturbed this natural balance — to the harm of predators. He was puzzled by the result. The proportions within the total population and within the "harvested" population turned out to be different. Having no biological or ecological explanation for this phenomenon, D'Ancona asked his father–in–law, the mathematician Volterra if he could come up with a mathematical model that might explain what was going on.

Volterra established the following, admittedly simplistic system of differential equations

$$\left.\begin{aligned} \dot{x} &= -ax + bxy \\ \dot{y} &= cy - dxy \end{aligned}\right\} . \tag{1.25}$$

Here $a, b, c, d > 0$ are constants while $x = x(t) \geq 0$ and $y = y(t) \geq 0$ stays for the biomass of predators and preys, respectively. The logic behind is quite simple. The predator species cannot live on its own whereas the prey species can. In fact, equation $\dot{x} = -ax$ yields exponential extinction, equation $\dot{y} = cy$ yields exponential overpopulation. The interaction between the two species is beneficial to the predator but has a negative impact on the prey. The quadratic terms in system (1.25) take account of predator–prey encounters (the probability of which can be assumed to be proportional to $xy$).

The particular form of system (1.25) remains ungrounded. In contrast to physics, *there are no "first principles" in biology*. Any supporting argumentation moves in the realm of parallels. However, system (1.25) is good enough to explain the phenomenon observed by D'Ancona. The only positive equilibrium of (1.25) is $\binom{c/d}{a/b}$. Now we pass to

[6]It is a must to mention here the fundamental role of biology in the development of statistics. Actually, both William S. Gosset (1876-1937; he is best known under the pseudonym "Student") and Ronald A. Fisher (1890-1962), arguably the two most important founding fathers of statistics, were heavily confronted with practical problems due to the unavoidable irregularity of biological material. For many years, they worked for the Guinness brewery in Dublin and for the Rothamsted Agricultural Experiment Station, respectively.

[7]numerical data and explanation are taken from the Duke Mathematics Department webpage https://studylib.net/doc/5846135/

the modified system $\dot{x}=x(-a+by)-hx$, $\dot{y}=y(c-dx)-hy$ with "harvesting" and see that the equilibrium goes over into $\binom{(c-h)/d}{(a+h)/b}$. The more percentage of both species is taken out from the sea, the lower the proportion between predators and preys in the catch. In fact, $\frac{c-h}{a+h}$ is a decreasing function of $h$ (considered only for $0 \le h \le c$). Thus system (1.25) captures the right intuition and the same is true for the modification due to fishing. As a qualitative model, system (1.25) is acceptable.

*Modeling aspects in single–species population dynamics* The oldest differential equation (Malthus, 1798) in population dynamics is

$$\dot{u} = \gamma u \quad \text{where} \quad \text{growth rate} = \text{birth rate} - \text{death rate} \quad , \quad \gamma = \beta - \mu > 0\,. \tag{1.26}$$

The main objections are that equation (1.26) tacitly assumes unlimited food supply and implies unlimited growth. (For a limited amount of time—e.g., for a mass of bacteria growing in a Petri dish—equation (1.26) is justified.) The objections are resolved (Verhulst, 1836) by equation

$$\dot{u} = ru\left(1 - \frac{u}{K}\right) \qquad \text{with constants} \quad r, K > 0\,. \tag{1.27}$$

The idea behind is that population change is proportional to the population size and also to the size of the remaining resources $K-u \geq 0$. For $0 < u(0) = u_0 < K$, the solution is strictly increasing and satisfies $u(t) \to K^-$ as $t \to \infty$. This explains why $K$ is termed as the carrying capacity (of the environment) and suggests that (1.27) is a good model for single–species populations. Note also that the quadratic term $u^2$ has some resemblance to system (1.25) and can be interpreted as intraspecies competition.

However, solutions to equation (1.27) cannot be fitted to real data without substantial difficulties. Thus, (1.27) can be accepted as a relevant qualitative model but it is quantitatively not acceptable[8] in general. For equation $\dot{u} = f(u)$ with any continouosly differentiable functions satisfying $f(0) = f(K) = 0$, $f'(0) > 0$, and $f(u) > 0$ whenever $0 < u < K$, all solutions grow exponentially at low size and saturate towards the carrying capacity at high size. A standard choice is to take $f(u) = \beta u e^{-bu} - \mu u$ with some $b > 0$, assuming that the birth rate is exponentially decreasing with the size of the population. Equation (1.26) can be generalized by letting $\beta = \beta(t)$, $\mu = \mu(t)$, $K = K(t)$ (seasonal fluctuations) or $\beta = \beta(t,u)$, $\mu = \mu(t,u)$, too.

Incorporating aspects of time delay in equation (1.26) leads to a new type of models. For example, consider equation

$$\dot{u}(t) = \beta u(t-\tau) - \mu u(t) \qquad \text{with time delay} \quad \tau > 0\,. \tag{1.28}$$

[8]It is worth mentioning here that equation (1.27) is an overall good model for the autocatalytic chemical reaction $A+B \rightleftharpoons 2B$. Indeed, $\dot{A} = -k_+AB + k_-B^2$ and $\dot{B} = k_+AB - k_-B^2$ with reaction rate constants $k_+, k_- > 0$. Since $\dot{A}+\dot{B} = 0$, we have that $A+B$ equals to $c > 0$, a constant. We arrive at a reformulation of (1.27), equation $\dot{B} = ck_+B - (k_+ + k_-)B^2$. What really matters, solutions are in almost full accordance with experimental data.

The logic behind is that juvenile members of the population are unable to produce offspring. Of course equation (1.28) can be seriously criticized. For $\beta-\mu>0$, the characteristic equation $\lambda+\mu=\beta e^{-\lambda\tau}$ has a unique real solution $\lambda_0>0$ (and countably infinite complex solutions). It follows by direct substitution that $u(t)=e^{\lambda_0 t}u_0$ solves (1.28) and $u(t)= = u_0>0$. As for equation (1.26), we arrived at the annoying existence of unbounded solutions.[9]

Now we assume that the species is internally structured by age in the sense that the total population at time $t\geq 0$ of its members between age $a_1\geq 0$ and $a_2\geq a_1$ equals to $\int_{a_1}^{a_2}\rho(t,a)\,da$ where $\rho(\cdot,a)$ is an age–dependent density function. In the first attempt of a mathematical analysis, the dynamics of the population is governed by

$$\text{the birth law}\qquad \rho(0,t)=\int_0^\infty \beta(a)\rho(a,t)\,da\ ,\quad t>0 \tag{1.29}$$

plus

$$\text{the aging law}\qquad \rho'_a(a,t)+\rho'_t(a,t)=-\mu(a)\rho(a,t)\ ,\quad t,a\geq 0\,, \tag{1.30}$$

a first order linear hyperbolic partial equation[10] with boundary condition (1.29) and initial condition $\rho(0,a)=\varphi(a)$, $a\geq 0$. Here, of course, $\beta(a)$ and $\mu(a)$ stay for the age–dependent birth rate and death rate, respectively. Nonlinearity appears most easily via making $\beta(a)$ and $\mu(a)$ to depend on the total population $\int_0^\infty \rho(a,t)\,da$ as well.

Spatial diffusion (e.g., one–dimensional and with constant coefficient $D>0$) can be added and makes equation (1.27) to

$$u'_t=Du''_{xx}+ru\left(1-\frac{u}{K}\right)\ ,\quad t>0 \text{ and } x\in\mathbb{R}\,, \tag{1.31}$$

a famous (Fisher, 1937) nonlinear parabolic partial differential equation.

Discrete versions of the models above are at hand. In scientific computing, they are automatically created by various discretization methods. The arguments we applied in deriving equations (1.26)–(1.31) work in various discrete settings, too.

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[9]Let us mention here two biologically relevant delay equations

$$\dot u(t)=ru(t)\left(1-\frac{u(t-\tau)}{K}\right)\qquad \text{with constants}\quad r,\tau,K>0$$

and

$$\dot u(t)=\beta u(t-\tau)e^{-bu(t-\tau)}-\mu u(t)\qquad \text{with constants}\quad \beta,\tau,b,\mu>0$$

of great historical interest. The first one (Hutchinson, 1948) is (1.27) modified via assuming that the population adapts only with a delay to the remaining resources. The second one (Nicholson, 1954) is a nonlinear version of (1.28). They were established to explain large, periodic oscillations in water–flea and blowfly populations, respectively. For $\tau$ large enough, the global behavior of the first equation is pretty similar to the dynamics described in Theorem 1.1.. The second equation is known to have bounded chaotic oscillations, too.

[10]On the left–hand side of the transport equation (1.30) we have $\lim_{h\to 0^+}\frac{\rho(a+h,t+a)-\rho(a,t)}{h}$.

If we are given four equidistant age groups, then a linear dynamics can be introduced by letting

$$\begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \\ x_4^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{pmatrix} \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{pmatrix} \qquad \text{for} \quad k = 0,1,2,\dots. \tag{1.32}$$

Observe the similarity between (1.29) and the first coordinate of the linear recursion (1.32). The remaining coordinates correspond to (1.30). Clearly $\beta_1, \beta_2, \beta_3 \geq 0$ and $\sigma_1, \sigma_2, \sigma_3 \in (0,1)$ can be interpreted as age–specific birth rates and age–specific survival rates, respectively. Note that $1-\sigma_1, 1-\sigma_2, 1-\sigma_3$ represent age–specific death rates. Linear population models of the type (1.32) go back in Europe to the 1202 book Liber Abaci by Fibonacci (long before the introduction of matrices) but the Fibonacci numbers themselves appear already in the Chanda?sastra (c. 3rd/2nd century B.C.) by Pingala, an early Indian music theorist.

*A somewhat naive analysis of a two–species competition model*