Theory of nonlinear dynamic systems
Practice 8

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Logistic mapping

$$f(x) = \mu x(1 - x)$$

where

$$0 \leq \mu \leq 4$$

and

$$0 \leq x \leq 1$$

- Biological meaning:

$$\dot{x} = bx - cx^2$$

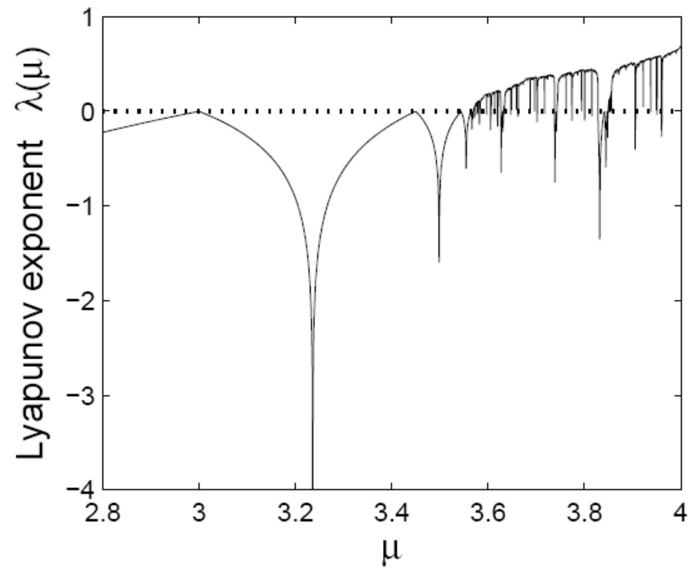
where, b is the birth rate and cx is the death rate in a population.

Order in the chaos

- Ljapunov exponent: measuring the chaos

$$\lambda \approx \frac{1}{n} \ln \left[\frac{f^n(x_0+\epsilon) - f^n(x_0)}{\epsilon} \right] \approx \frac{1}{n} \ln \left[\left. \frac{df^n(x)}{dx} \right|_{x=x_0} \right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

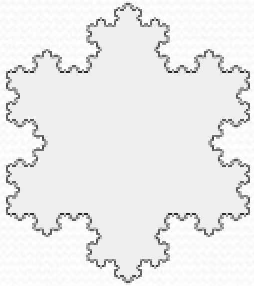


- Positive values indicate chaos
- For μ=4 it approximates ln2
- These finding are not dependent to the initial condition

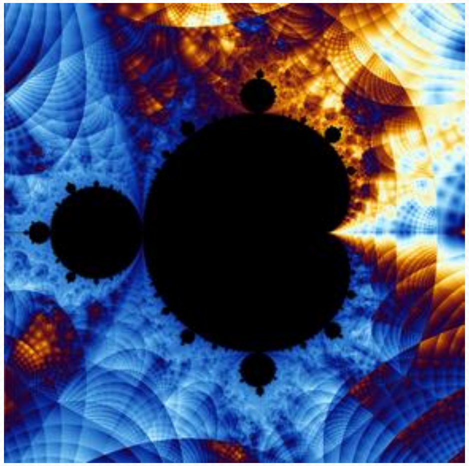
Introduction– fractals I.

- What does fractal mean?
  - Some kind of shape…
- It is self-similar
  (smaller parts are resemble to the original shape)
- Its mathematical description is usually simply
- The borders of fractals are „infinitely creased”

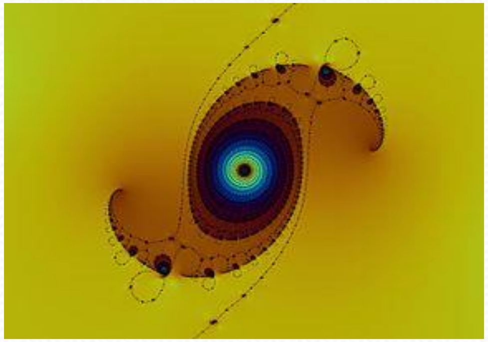
Introduction– fractals II.



- Examples:

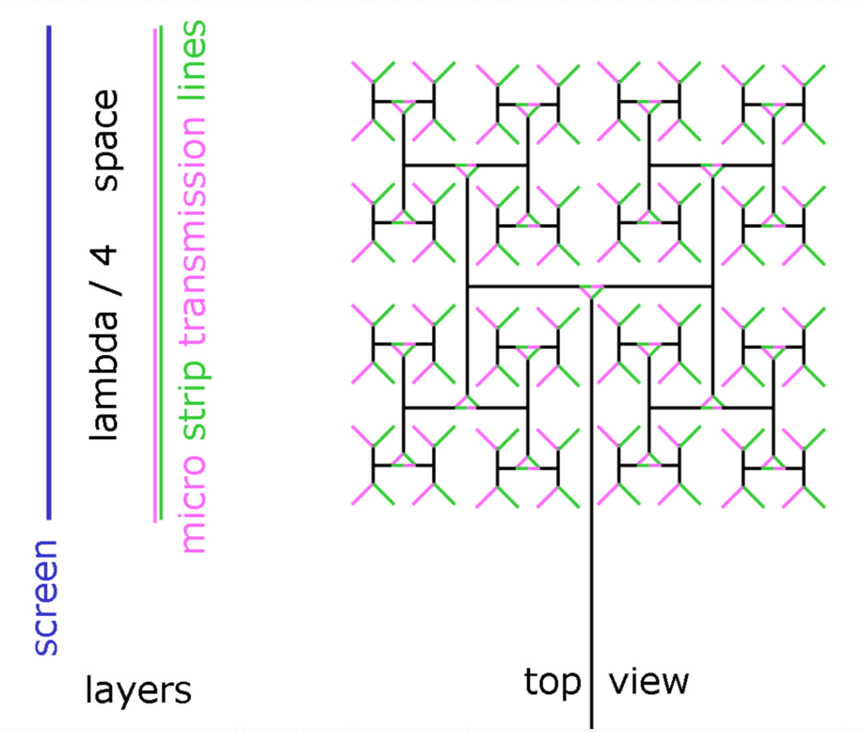


  - Koch-curve
  - Mandelbrot-set
  - Julia-set
  - Fern shape



Introduction– fractals III.

- Why are they useful?
  - Simple mathematical equation
  - Low memory requirement => graphical applications
  - „natural shapes"
- Bases of compression algorithms
- Tool development: e.g.: antenna
  - maximizing the surface/volume ratio



Cantor set

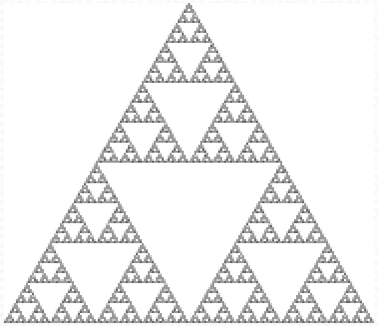
- formula for Cantor set construction:

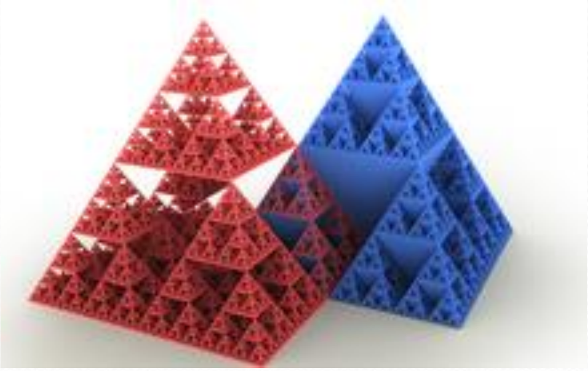
$$F_1 = \frac{1}{3}x$$

$$F_2 = \frac{2}{3} + \frac{1}{3}x$$

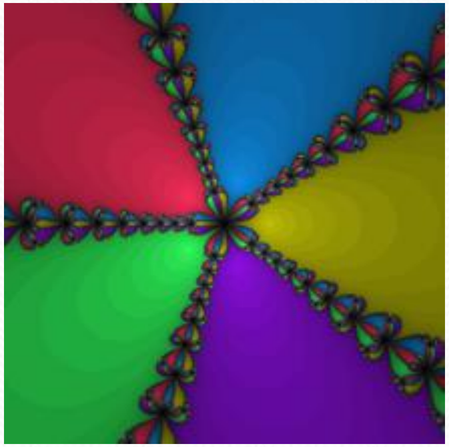
Sierpinski triangle

- Similar to the Cantor set, but here we cut the middle ¼ of a triangle
- Method:
  - 1. Start with an equilateral triangle.
  - 2. Subdivide it into four smaller congruent equilateral triangles and remove the central one.
  - 3. Repeat step 2 with each of the remaining smaller triangles
- The area tends to zero.





Newton's method



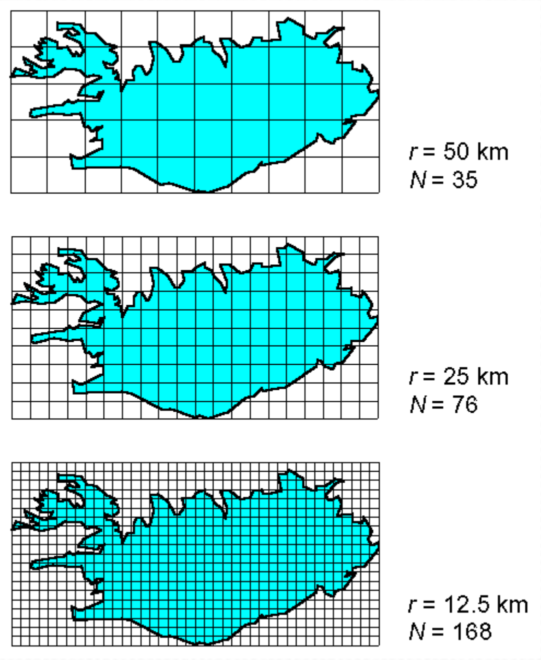
- For finding the roots of equations
- A generalization of Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- Equations to try: e.g.: $\mathbf{x^3+1}$, $\mathbf{x^3-1}$, $\mathbf{x^3-2x+2}$
- The colors show the sets of initial points (in the complex plain) from which the system converge to the same root
- It can „get stucked" in special cases
- Fractal develops around the roots based on the fate of the initial points around the roots

Fractal dimension

- Dim=log(number of self-similar pieces in the next iteration)/log(magnification factor)
- E.g.:
  - Dim_{Cantor}=log(2)/log(3)
  - Dim_{Siepr}=log(3)/log(2)
- Box-counting dimension:
  - ε :resolution
  - n=1/ε :magnification
  - b :number of boxes with the feature (boundary)



$$\mathrm{Dim}_{\mathrm{Box}} = \lim_{n\to\infty}(\log(b * \varepsilon)/\log n)$$

Thank you for your attention!