Theory of nonlinear dynamic systems
Practice 4

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Numeric precision of ode45

- Sources of errors (in general):
  - Matematical error: from the numeric method, higher for big steps
  - Rounding error: from the computer, higher for a lot of steps
  - Balancing between the 2
  - Matlab ode solvers use adaptive stepsize (depending on the „speed of change")

- Absolute tolerance (in Matlab):
  - Default: 1e-6
  - ~solution is important until then
- Relative tolerance (in Matlab):
  - Default: 1e-3
  - ~number of correct digits

Numeric precision of ode45

- Example:
  - Solve the damped pendulum equation with Max_Time=1500 for
    - b=1.98
    - b=2.02
  - What kind of fixed point you expect (form analytical solution)?
  - What does the numeric solution show?
  - How could you agree the two?

Van der Pol oscillator

- An electronic oscillator is an electronic circuit that produces a periodic, oscillating electric signal, often a sine wave or a square wave

$$\frac{d^2u}{dt^2} + \omega_0^2 u = 0$$

- Solution of DE:

$$u(t) = U_0 \cos(\omega_0 t + \varphi_0)$$

constant amplitude sinusoidal signal, where U_o is the amplitude , ω_o is the frequency and ς_o is the phase.

- Challenge: circuit implementation

Van der Pol oscillator

- Problems:

 - Dependence on the initial conditions (after turning of the circuit the amplitude might change)

 - In reality a perfect structure is needed to sinusoidal solution and constant amplitude

 - Constant amplitude oscillator cannot be constructed with linear elements only

 Conclusion:

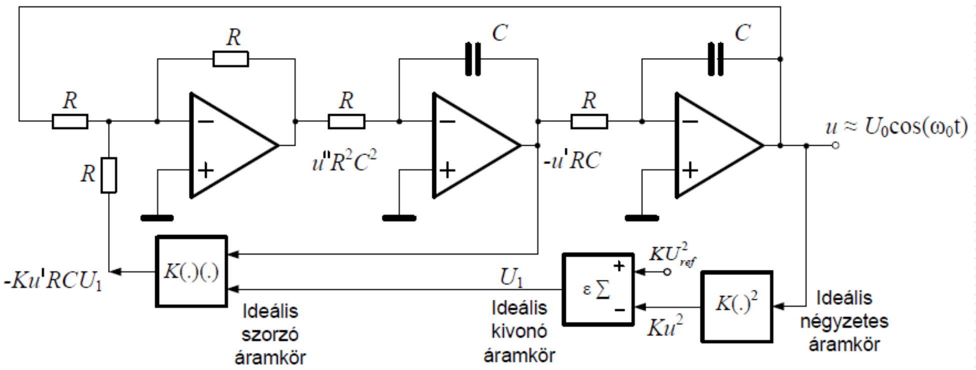
 - We must ensure constant and stable frequency /amplitude vibration.

 - The system has to reach the constant frequency /amplitude in case of any initial conditions.

Van der Pol oscillator

- Solution:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$



A Van der Pol oszcillátor kapcsolási rajza.

Van der Pol oscillator

- Test the effects of the μ parameter!
  - $\mu = 0:0.1:5$
  - $\mu = 5:10:105$
  - Try to calculate the differential equation for $\mu = 500$.
    - What do you experience? What could be the solution?

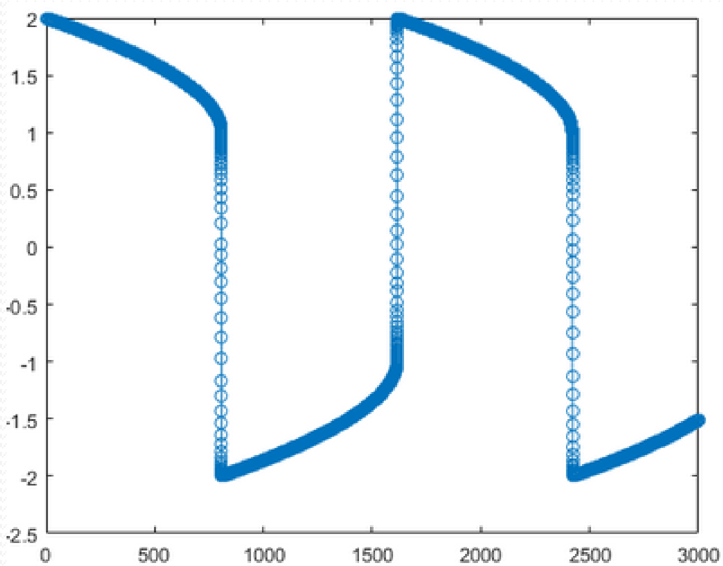
Van der Pol oscillator

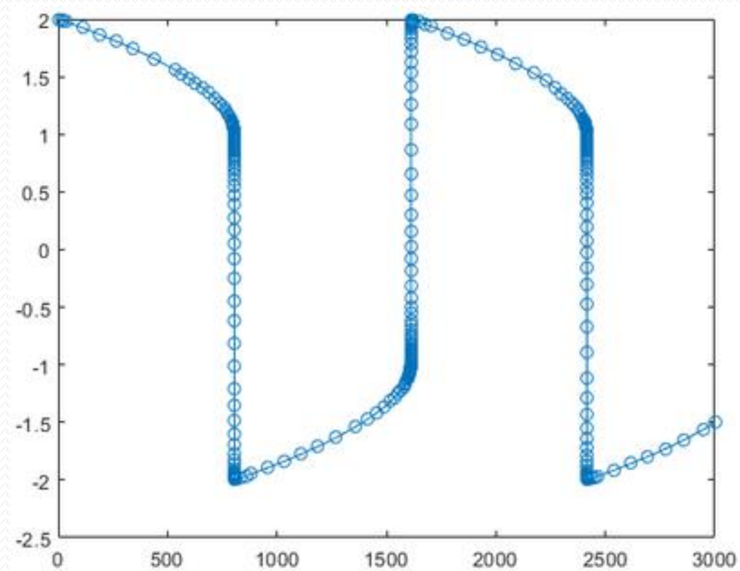
ode45

- cannot handle so stiff problems,
- slows down
- good for most „average” problems
- 1st guess

ode15s

- for stiff problems
- much smaller time steps for much steeper changes, much bigger in „slower changing” regions





- different methods with differnt heuristics
- they combine diffent order ode solvers
- for more details, see Numeric methods 2 class

Explicit Euler

- Def.:

General task:

$$\dot{x} = f(x), x(0) = x_0 \in \mathbb{R}^d$$
$$\varphi: [0, h_0] \times \mathbb{R}^d \to \mathbb{R}^d$$

General method:

$$x_{k+1} = \varphi(h, x_k), k = 0,1,2, \ldots \leftrightarrow \mathrm{X} = \varphi(h, x)$$

φ_E explicit Euler method

$$\mathrm{X} = \varphi_E(h, x), \text{ where } X = x + hf(x)$$

Implicit Euler

- Def.:

General task:

$$\dot{x} = f(x), x(0) = x_0 \in \mathbb{R}^d$$
$$\varphi: [0, h_0] \times \mathbb{R}^d \to \mathbb{R}^d$$

General method:

$$x_{k+1} = \varphi(h, x_k), k = 0,1,2, \ldots \leftrightarrow \mathrm{X} = \varphi(h, x)$$

φ_I implicit Euler method

$$\mathrm{X} = \varphi_I(h, x), \text{ where } X = x + hf(X)$$

Semi-implicit Euler

- Def.:

Newton equation of $V: \mathbb{R} \to \mathbb{R}$ *potential force field*

$$\ddot{x} + V'(x) = 0 \leftrightarrow (PN)\begin{cases} \dot{x} = y \\ \dot{y} = -V'(x) \end{cases}$$

φ_S semi – implicit Euler method

$$\binom{X}{Y} = \varphi_S\left(h, \binom{x}{y}\right), \text{ where} \begin{cases} X = x + hy \\ Y = y - hV'(x + hy) \end{cases}$$

$$\begin{cases} \frac{X-x}{h} = y \\ \frac{Y-y}{h} = -V'(X) \end{cases} \leftrightarrow \begin{cases} X = x + hy \\ Y = y - hV'(x + hy) \end{cases}$$

Semi-implicit Euler

- Suppl.:

The exact solutions of $\Phi: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \left(t, \binom{x}{y}\right) \to \Phi\left(t, \binom{x}{y}\right)$

conserv:

- **energy** of $\frac{y^2}{2} + V(x) = \frac{{y_0}^2}{2} + V(x_0)$
- **area** of *dx dy* at $\mathbb{R}^2$ phase portrait

This special method conservs the *dx dy* area, according to the det (J)≡ 1, where $J = \frac{\partial(X,Y)}{\partial(x,y)}$.

Comparation of numerical methods:

#	method	h	T	$\frac{y_k^2}{2} + V(x_k) if\ 0 \le kh \le T$
1	Φ_E	0.001	100	monotonically increasing between 1 and 1.068
2	Φ_E	0.001	1000	monotonically increasing between 1 and 1.7
3	Φ_I	0.001	100	monotonically decreasing between 1 and 0.934
4	Φ_I	0.001	1000	monotonically decreasing between 1 and 0.46
5	Φ_S	0.1	10000	oscillation between 0.957 and 1.045
6	Φ_V	0.1	10000	oscillation between 0.998 and 1

- Always take care of the physics of the problem
- Energy
- => choose problem specific numerical methods

Euler mehods

- Try different numeric methods (EE, IE, SE) on the prefect string equation!
- Compare the results with ode45 and the analytical solution!

Thank you for your attention!