Functional analysis

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Theme 1

1.1 Metric space

Definition: In general a metric space (M, d) is defined to be a set M together with a function $d : M \times M \to \mathbb{R}$ called a metric satisfying four conditions:

1. $d(x, y) \geq 0 \quad \forall x, y \in M$ (nonnegativity)
2. $d(x, y) = 0 \Leftrightarrow x = y$ (nondegeneracy)
3. $d(x, y) = d(y, x) \quad \forall x, y \in M$ (symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in M$ (triangle inequality)

1.1.1 E.g: discrete metric

Example: Let M be any set and define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

This is called the discrete metric.

1.2 Normed space

Definition: A normed linear space $(V, \|\cdot\|)$ is a linear space V together with a function $\|\cdot\| : V \to \mathbb{R}$ called a norm satisfying four conditions:

1. $\|v\| \geq 0 \quad \forall v \in V$ (nonnegativity)
2. $\|v\| = 0 \Leftrightarrow v = 0$ (nondegeneracy)
3. $\|\lambda v\| = |\lambda| \cdot \|v\| \quad \forall v \in V, \lambda \in \mathbb{R}$ (multiplicativity)
4. $\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V$ (triangle inequality)

1.3 Metric in a normed space (P)

Theorem: Norms always give rise to metrics. Specifically, if $(V, \|\cdot\|)$ is a normed space and $d(v, w)$ is defined by

$$d(v, w) = \|v - w\|,$$

then d is a metric on V.

Proof: Let's assume that $d(v, w) = \|v - w\|$. Let's see if the metric's conditions still stand:

1. $d(v, w) = \|v - w\| \geq 0 \quad \forall v, w \in V$. This condition still stands, because of the nonnegativity property of the norm.
2. $d(v, w) = \|v - w\| = 0 \Leftrightarrow v = w$. This condition still stands, because of the nondegeneracy property of the norm.
3. $d(v, w) = \|v - w\| = \|(-1)(w - v)\| = |(-1)| \cdot \|w - v\| = \|w - v\| = d(w, v) \quad \forall v, w \in V$. This condition still stands due to the multiplicativity property of the norm.
4. We need to prove, that $d(v, w) \leq d(v, z) + d(z, w)$. That means $\|v - w\| \leq \|v - z\| + \|z - w\|$. Let's begin from the norm's triangle inequality property:

   $$\|x + y\| \leq \|x\| + \|y\|.$$

   Let $x = v - z$, and $y = z - w$:

   $$\|v - z + z - w\| \leq \|v - z\| + \|z - w\|.$$

   Thus we have proven the triangle inequality property of the metric space.

1.4 Special norms in $\mathbb{R}^n$

Example: $V = \mathbb{R}^n$ with $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ for $x = (x_1, x_2, ..., x_n)$. This is the ususal, standard, or Euclidean norm on $\mathbb{R}^n$.

Example: $V = \mathbb{R}^n$ with $\|x\|_1 = |x_1| + |x_2| + ... + |x_n|$. This sequence is related to the l^1 sequence space.

Example: $V = \mathbb{R}^n$ with $\|x\|_\infty = \max(|x_1|, |x_2|, ..., |x_n|)$. This sequence is related to the l^∞ sequence space.

1.5 Inner product space

Definition: An inner product space $(V, \langle \cdot, \cdot \rangle)$ is a lonear space V together with a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ called an inner product satisfying five conditions:

1. $\langle v, v \rangle \geq 0 \quad \forall v \in V$ (nonnegativity)
2. $\langle v, v \rangle = 0 \Leftrightarrow v = 0$ (nondegeneracy)
3. $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle \quad \forall v, w \in V, \lambda \in \mathbb{R}$ (multiplicity)
4. $\langle v, w \rangle = \langle w, v \rangle \quad \forall v, w \in V$ (symmetry)
5. $\langle v, w + u \rangle = \langle v, w \rangle + \langle v, u \rangle \quad \forall u, v, w \in V$ (distibutivity)

If $\mathbb{R}$ is replaced by $\mathbb{C}$ everywhere in this definition, and the symmetry property is replaced by the Hermitian symmetry property:

$$\langle v, w \rangle = \overline{\langle w, v \rangle} \quad \forall v, w \in V,$$

where the bar indicates the complex conjugate, we get a complex inner product space.

1.6 Relation between inner product and norm (P)

Theorem: Inner products always give rise to norms. Specifically, if $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and $\|v\|$ is defined by

$$\|v\| = \sqrt{\langle v, v \rangle},$$

then $\|\cdot\|$ is a norm on V.

Proof: Let's assume that $\|v\| = \sqrt{\langle v, v \rangle}$. Let's see if the norm's conditions still stand:

1. $\|v\| = \sqrt{\langle v, v \rangle} \geq 0 \quad \forall v \in V$. This condition still stands, besause of the nonnegativity property of the inner product, and because of the laws of square roots.
2. $\|v\| = \sqrt{\langle v, v \rangle} = 0 \Leftrightarrow v = 0$. This can only happen if $\langle v, v \rangle = 0$ due to the square roots. And this condition will still stand because of the nondegeneracy property of the inner product.
3. $\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v \rangle} = \sqrt{\lambda^2 \langle v, v \rangle} = |\lambda| \sqrt{\langle v, v \rangle} = |\lambda| \cdot \|v\|$. This property obviously still stands.
4. $\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle$. So $\|v + w\|^2 \leq \|v\|^2 + \|w\|^2$. Because this is true for the square of the original condition, we will accept, that this is true for that inequality aswell.

Theme 2

2.1 Sequence spaces: $c_0, l^\infty, l^p, p > 1$

Sequence spaces are linear spaces whose elements are sequences. The elements can be sequences of real or complex numbers. Addition and scalar multiplication are defined pointwise. There are many sequence spaces; we discuss some of them.

Definition: c_0 is the collection of all sequences that converge to 0.

Definition: l^∞ is the collection of all bounded sequences $\{x_n\}_{n=1}^\infty$.

Notice that $c_0 \subset l^\infty$. Both of these collections become normed linear spaces with norm defined by

$$\|(x_n)\|_\infty = \sup(|x_n| \big| 1 \leq n < \infty).$$

Definition: The space l^p $1 \leq p < \infty$ consists of all sequences $\{x_n\}_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty |x_n|^p \leq \infty.$$

With norm defined by

$$\|\{x_n\}\|_p = \Big(\sum_{n=1}^\infty |x_n|^p\Big),$$

l^p becomes a normed linear space.

2.2 Relation of l^p and l^q (P)

Observation: Let's assume that $p > q$. In this case

$$l^\infty \supset c_0 \supset l^1 \supset l^2 \supset ... \supset l^p \supset ... \supset l^q \supset ...$$

Proof: The first three is trivial. l^2 is a subset of l^1 because l^1 also has the squares of itself in it aswell.

Following this pattern, we get, that if $p < q$, then all of l^q's elements are also found in l^p. Because any sequence that is convergent, when taken to the power of p will also be convergent, when we take it to the power of q, if $q > p$.

2.3 Function spaces: $C([a, b])$ with possible norms

These are linear spaces consisting of functions. As with sequence spaces, addition and multiplication are defined pontwise. The scalars, again, can be taken to be either real or complex.

Definition: Let

$$V = \{f : [a, b] \to \mathbb{R} \big| \exists B \geq 0 \text{ such that } |f(x)| \leq B \forall x \in [a, b]\}$$

This is a linear space.

Definition: The colletction

$$\{f : [a, b] \to \mathbb{R} \big| f \text{ is continuous}\}$$

is a subspace of V. This subspace of continuous functions on a closed and bounded interval is a very important space in analysis; it if most often denoted by $(C([a, b]), \|\cdot\|_\infty)$, or just $C([a, b])$.

2.3.1 Possible norms

With norm defined by

$$\|f\|_\infty = \sup(|f(x)| \big| x \in [a,b]),$$

both V and $C([a,b])$ become normed linear spaces.

Also $C([a,b])$ becomes an inner product space, with the inner product

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx.$$

This means, a possible norm is

$$\|f\|_2 = \left(\int_a^b f^2(x)\right)^{\frac{1}{2}}.$$

Theme 3

3.1 The topology of metric spaces

Definition: Let (M, d) be a metric space. The r-ball centered at x is the set

$$B_r(x) = \{y \in M \big| d(x,y) < r\}$$

for any choice of $x \in M$ and $r > 0$.

3.1.1 Open set

Definition: A point x is an interior point of E if there exists an $r > 0$ such that $B_r(x) \subseteq E$.

Definition: E is open if every point of E is an interior point.

3.1.2 Closed set

Definition: A point $x \in M$ is a limit point of a set $E \subseteq M$ if every open ball $B_r(x)$ contains a point $y \neq x$, $y \in E$.

Definition: E is closed if every limit point of E is in E.

3.1.3 Properties (P)

3.2 Convergence of a sequence in a metric space

Definition: Let (M, d) be a metric space, and $(x_n) \in M$ sequence. We say that (x_n) is convergent, and the limit of (x_n) is x, if $\forall \epsilon > 0$ $\exists N$ index, such that $\forall n \geq N$, $d(x_n, x) < \epsilon$.

3.3 Continuity of functions between metric spaces

Definition: Let (M, d_M), and (N, d_N) be metric spaces, and $f : M \to N$. f is continuous in the point $x_0 \in M$, if $\forall \epsilon > 0$ $\exists \delta > 0$, such that if $d_M(x, x_0) < \delta$, then $d_N(f(x), f(x_0)) < \epsilon$.

Theme 4

4.1 Compact sets

Definition: A collection of sets is called a cover of E if E is contained in the union of the sets in the collection.

Definition: If each set in a cover of E is open, the cover is called an open cover.

Definition: If the union of a subcollection of the cover still contains E, the subcollection is referred to as a subcover.

Definition: E is compact, if every open cover of E contains finite subcover.

Definition: E is sequentially compact if every sequence of E contains a convergent subsequence, whose limit is an element of E.

4.1.1 Examples in finite and infinite dimension

Example: Every set with finite elements is compact. Proof idea: chose the subcovers, so they dont overlap. This way there will always be finite subcovers.

Example: $[a, b]$, where $a \leq b \in \mathbb{R}$ is compact. Proof: Heine-Borel thm.

Example: $(0, 1]$ is not compact. Proof idea: Let a cover be defined by: $r_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)$, and $G_n = B_{r_n}\left(\frac{1}{n}\right)$. This way $(0, 1] \subset \bigcup G_n$, and we cant chose finite subcovers to cover the interval.

4.1.2 Properties (P)

Statement: Every $E \subset M$ compact set is closed.

Statement: Every $E \subset M$ compact set is bounded.

Theorem: A subset of a metric space is compact if and only if it is sequentially compact.

Proof: Indirect proof. Let M be a compact set, and Let's say, that there is a sequence, which has no convergent subsequences. Let the different points in this sequence be $y_k, k \in \mathbb{N}$. The $\bigcup\{y_n\}$ is a closed set, because it contains all of the limit points - it ha none. We can cover $\{y_k\}$, with non-overlapping open balls. $M \setminus \bigcup\{y_k\}$ is an open set, which is an open cover of M together with the previous cover of $\{y_k\}$ - and we cant choose finite subcovers. This is a contradiction.

4.2 Compact sets finite dimension (Heine-Borel theorem) (P)

Theorem: A subset E of $\mathbb{R}^n$ is compact if and olny if it is closed and bounded.

Proof: The Bolzano-Weierstrass theorem says, that every bounded sequence contains convergent subsequences. So we can select a convergent subsequence from an n-dimensional point sequence. The limit of this convergent subsequence is a limit point of the set, which is in the set, because the set is closed.

4.3 The unit ball in $C([0,b])$ is not compact

Let's have a look at the ball in $C([0,b])$, whitch has a radius of b:

$$B_1(0) = \{f \in C([0,b]) : \max|f(x)| \leq 1\}.$$

This set is closed, and bounded, but not compact. We give an $(f_n) \subset C([0,b])$ sequence, where $\|f_n\| = 1$ for every n, but has no convergent subsequence:

$$f_(x) = \begin{cases} 1, & \text{if } x = \frac{b}{n} \\ 0, & \text{if } x \leq \frac{b}{n+1} \text{ or } x \geq \frac{b}{n-1} \\ \text{linear}, & \text{if } x \in \left(\frac{b}{n+1}, \frac{b}{n}\right) \\ \text{linear}, & \text{if } x \in \left(\frac{b}{n}, \frac{b}{n-1}\right) \end{cases}$$

We can easily see that this sequence has no convergent subsequences, so the ball is not compact.

Theme 5

5.1 Completeness of a metric space

Definition: $(x_n) \subset M$ is a Cauchy sequence, if $\forall \epsilon > 0 \ \exists N$ index, such that

$$d(x_n, x_m) < \epsilon \quad \forall n, m \geq N.$$

Definition: An M metric space is called complete , if every Cauchy sequence is convergent in M converges to a point of M.

5.2 Banach space

Definition: A complete normed linear space is called a Banach space.

5.3 Hilbert space

Definition: A complete inner product space is called a Hilbert space.

5.4 $C([a, b])$ is complete - and it is not complete, depending on the norm (P)

Statement: $(C([a, b]), \|\cdot\|_\infty)$ is complete.

Proof: Let $(f_n) \subset C([a, b])$ a Cauchy sequence. This means, that $\forall \epsilon > 0 \ \exists N$ such that in the case of $\forall n, m \geq N$

$$\|f_n - f_m\|_\infty = \max_{x \in [q,b]} |f_(x) - f_m(x)| < \epsilon \Rightarrow |f_n(x) - f_m(x)| < \epsilon$$

This means, that for a fixed x the $(f_n(x))$ sequence is a Cauchy sequence, so it is convergent aswell: $\lim f_n(x) = f(x)$. So f_n is not only convergent, but it converges evenly. This is why $f \in C([a, b])$, because an evenly convergent sequence of continous functions has a limit of a continuous function.

Statement: $(C([a, b]), \|\cdot\|_2)$ is not complete.

Proof: Let

$$f_n = \begin{cases} 0, & x < \frac{1}{2} - \frac{1}{n} \\ 1, & x > \frac{1}{2} \\ \text{linear}, & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} \end{cases}$$

We can see that

$$\lim_{n,m \to \infty} \int_a^b (f_n(x) - f_m(x))^2 dx = 0.$$

This means that (f_n) is a cauchy sequence in $C^2([a, b])$. The pointwise limit of the function is:

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1, & x \geq \frac{1}{2} \\ 0, & x < \frac{1}{2} \end{cases}$$

f is not continuous, thus we proved, that $C^2([a, b])$ is not a complete space.

Theme 6

6.1 Measurable space

Let X be an arbitrary set, 2^X is the set of all the subsets of X. Let $\mathcal{R} \subset 2^X$, a set of subsets of X.

Definition: $\mathcal{R}$ is an algebra, if it satisfies the following conditions:

1. $X \in \mathcal{R}$,
2. $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$,
3. $A, B \in \mathcal{R} \Rightarrow A \backslash B \in \mathcal{R}$.

Definition: $\mathcal{R}$ is a σ-algebra if

$$A_k \in \mathcal{R}, \quad k = 1, 2, ... \quad \bigcup_{k=1}^{\infty} A_k \in \mathcal{R}.$$

Definition: If $\mathcal{R} \subset 2^X$ is a σ-algebra, then $(X, \mathcal{R})$ is a measurable space.

6.2 Measure

A measure is a σ-additive function over a σ-algebra: $\mu : \mathcal{R} \to \mathbb{R}^+ \cup \{+\infty\}$. The σ-additive property means that if we look at countable disjoint sets, whose union is in $\mathcal{R}$:

$$A_1, A_2, ... \in \mathcal{R} \quad A_i \cap A_j = \emptyset, \text{ if } i \neq j, \quad \bigcup_{k=1}^{\infty} A_k \in \mathcal{R},$$

then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Also $\mu(\emptyset) = 0$.

6.2.1 E.g.: Counting measure

Example: X is arbitrary, $\mathcal{R} = 2^X$, $A \in \mathcal{R}$

$$\mu(A) = \begin{cases} |A|, & \text{if it is finite} \\ +\infty, & \text{if the set is infinite} \end{cases}$$

6.2.2 E.g.: Probability measure

X is finite or countable, $X = \{x_1, x_2, ..., x_n, ...\}$. $\mathcal{R} = 2^X$. We also have $p_1, p_2, ..., p_n, ... \geq 0$ numbers, so that $\sum_{k=1}^{\infty} p_k = 1$. We give the measure like so:

$$A \subset X : \quad \mu(A) = \sum_{x_i \in A} p_i.$$

6.3 Introduction of the Lebesgue measure on $\mathbb{R}$

Let $X = \mathbb{R}$. We define the measure, and the measurable sets step-by-step.

1. Let $\mathcal{I}$ be the set of finite interval sets:

   $$\mathcal{I} = \{x : a \leq x \leq b\}, \quad a, b \in \mathbb{R},$$

   where we could have $<$ instead of $\leq$. The measure on $\mathcal{I}$ is the length: $m(\mathcal{I}) = b - a$.

2. We extend the measure to the $\mathcal{E}$ simple sets.

$$\mathcal{E} = \{A \subset \mathbb{R} : \quad A = \bigcup_{k=1}^{n} \mathcal{I}_k, \quad \mathcal{I}_k \in \mathcal{I} \text{ are disjoint}\}.$$

$$m(A) = \sum_{k=1}^{n} m(\mathcal{I}_k).$$

3. We define an outer measure on $2^{\mathbb{R}}$. $m^* : 2^{\mathbb{R}} \to \mathbb{R}^+ \cup \{+\infty\}$. If $A \subset \mathbb{R}$ arbitrary set, then we definte the outer measure of A, $m^*(A)$ like so:

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} m(\mathcal{I}_k) : \quad A \subset \bigcup_{k=1}^{\infty} \mathcal{I}_k, \quad \mathcal{I}_k \in \mathcal{I} \right\}.$$

4. $\exists \mathcal{M}$ σ-algebra, which contains the simple sets:

$$\mathcal{E} \subset \mathcal{M} \subset 2^{\mathbb{R}},$$

and $m^*|_{\mathcal{M}}$: σ-additive. So the restriction of m^* on $\mathcal{M}$ is a measure, which is called Lebesgue measure.

Definition: $\mathcal{M}$'s elements are the measurable sets inside $\mathbb{R}$. m^*'s restriction on $\mathcal{M}$ is the Lebesgue measure.

Theme 7

7.1 Properties of Lebesgue-measurable sets

Statement: If $A \in \mathcal{M}$ and $m(A) = 0$, then for every $\epsilon > 0$ we can give finite or countable many intervals: $I_k, k = 1, 2, ...$ such that

$$A \subset \bigcup_{k=1}^{\infty} I_k, \quad \sum_{k=1}^{\infty} m(I_k) < \epsilon.$$

Statement: If $A = \{x\} \in \mathcal{M}$, then $m(A) = 0$. If $A = \{x_1, ..., x_n, ...\} \subset \mathbb{R}$ is countable, then $m(A) = 0$.

7.2 Null sets

Definition: A is a null set, if $m(A) = 0$. We denote the set of these sets with $\mathcal{N}$.

7.3 Cantor set in $[0, 1]$

Example: Cantor set. Well construct the set step-by-step:

1. Let $C_0 = [0, 1]$.
2. Let $C_1 = C_0 \backslash (\frac{1}{3}, \frac{2}{3})$.
3. Let $C_2 = C_1 \backslash (\frac{1}{9}, \frac{2}{9}) \backslash (\frac{7}{9}, \frac{8}{9})$
4. Continue...

Definition: $C = \bigcap_{k=0}^{\infty} C_k$ is the Cantor set.

7.3.1 Properties (P)

Statement: The properties of the Cantor set:

1. C is closed.
2. The cardinality of C is continuum.
3. C is measurable, and $m(C) = 0$.

Proof:

1. C_k is closed, for every k. The intersection of closed sets is closed.
2. We can see, that if we write the elements of the Cantor set in base 3, none of the elements will have 1-s in them. From here we can use the Cantor diagonal method to prove that there are continuum many numbers in the set.
3. We can check the length of the unused interval in the n-th step using the next formula: $\frac{2^n}{3^{n+1}}$. If we add these together, we get:

   $$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 1.$$

   Obviously $C \in \mathcal{N}$.

Theme 8

8.1 Measurable functions

Definition: $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ function is measurable , if for every $a \in \mathbb{R}$, the $\{x : f(x) < a\} \subset \mathbb{R}^n$ set is Lebesgue measurable.

8.1.1 Characterization (P)

8.2 Simple functions

Definition: A real valued function with only a finite number of elements in its range is called a simple function. One type of simple function is the characteristic function, χ_E, of a set $E \subseteq \mathbb{R}^n$. This is defined by

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Every simple function can be written as a finite linear combination of characteristic functions. Specifically, if the range of the simple function s is $\{c_1, ..., c_n\}$, then

$$s(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x),$$

where $E_k = \{x_s(x) = c_k\}$.

8.3 "Almost everywhere", as an equivalence relation (P)

Definition: Two given functions f, g are equal almost everywhere ($f \sim g$, or $f = g$), if

$$m(\{x : f(x) \neq g(x)\}) = 0.$$

Statement: $f = g$ almost everywhere is an equivalence relation.

Proof:

1. It is obviously reflexive.
2. It is also obviously symmetric.
3. Let $f = g$ a.e., and $g = h$ a.e.

   $$m(\{x : f(x) \neq h(x)\}) \leq m(\{x : f(x) \neq g(x)\}) + m(\{x : g(x) \neq h(x)\}) = 0.$$

   So the relation is transitive.

Theme 9

9.1 Lebesgue integral

We define the Lebesgue integral step-by-step

1. step: Let's assume that f is simple:

$$f(x) = \sum_{k=1}^{n} c_k \chi_{E_k}, \quad E_k \in \mathcal{M}, \quad E_k \cap E_j = \emptyset, \quad c_k \in \mathbb{R}.$$

Definition: If $E \in \mathcal{M}$ is measurable, then the integral of the f function from step 1 on the set E by the measure m is

$$\int_E f dm := \sum_{k=1}^{n} c_k m(E \cap E_k).$$

2. step: Let's assume, that $f : [a, b] \to \mathbb{R}^+$ is measurable. Then the integral is defined, as follows:

$$\int_E f dm := \sup \left\{ \int_E s dm : \text{ s is simple, } s(x) \le f(x) a.e. \right\}$$

3. step: $f : [a, b] \to \mathbb{R}$ is an arbitrary measurable function. First we construct this function from the difference of two non-negative functions:

$$f = f_+ - f_-,$$

where

$$f_+(x) = \begin{cases} f(x), & f(x) \ge 0 \\ 0, & \text{else} \end{cases}, \quad f_-(x) = \begin{cases} -f(x), & f(x) < 0 \\ 0, & \text{else} \end{cases}.$$

The integrals of these functions are well defined.

Definition: f is Lebesgue integrable, if at least one of the functions mentioned above, is finite. In this case the integral of f over E with respect to the measure m is

$$\int_E f dm = \int_E f_+ dm - \int_E f_- dm.$$

9.1.1 Properties

Statement: The properties of Lebesgue integral:

1. $f, g \in \mathcal{L}, c \in \mathbb{R}$,

$$\int_E (f+g) dm = \int_E f dm + \int_E g dm, \quad \int_E c \cdot f dm = c \cdot \int_E f dm.$$

2. If $m(E) < \infty$ and $a \le f \le b$ is measurable, then

$$a \cdot m(E) \le \int_E f dm \le b \cdot m(E).$$

(this is also true, when the conditions only happen a.e.)

3. If $f, g \in \mathcal{L}$ and $f(x) \le g(x)$, then

$$\int_E f dm \le \int_E g dm.$$

(this is also true, when the conditions only happen a.e.)

4. If $f \in \mathcal{L}$, then $|f| \in \mathcal{L}$, and

$$\left| \int_E f dm \right| \le \int_E |f| dm.$$

(also true the other way around.)

5. If $m(E) = 0$, then for every measurable function

$$\int_E f dm = 0.$$

6. If $E = E_1 \cup E_2$, where $E_1 \cap E_2 = \emptyset$,

$$\int_E f dm = \int_{E_1} f dm + \int_{E_2} f dm.$$

7. If $f = g$ a.e., then

$$\int_E f dm = \int_E g dm.$$

9.2 Condition of integrability (P)

Theorem: $f : [a, b] \to \mathbb{R}$ bounded, measurable function is Lebesgue integrable.

Proof: If f is bounded, then $\exists M$, such that $|f(x)| \leq M$, so

$$\int_{[a,b]} f dm \leq M(b-a) \Rightarrow \exists \int_{[a,b]} f dm.$$

9.3 Comparison of the Lebesgue integral and the Riemann integral

Theorem: If $f \in \mathcal{R}[a, b]$, then $f \in \mathcal{L}[a, b]$ too, and

$$\int_a^b f(x)dx = \int_{[a,b]} f dm.$$

But these to sets are not the same. We see, that $\mathcal{R}[a, b] \subset \mathcal{L}[a, b]$. A good example is the $f : [0, 1] \to \mathbb{R}$ Dirichlet-function:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

This is obviously not Riemann integrable, but since this function is 0 almost everywhere,

$$\int_{[0,1]} f dm = 0.$$

9.4 Convergence theorems

9.4.1 Lebesgue's Monotone Convergence Theorem

Theorem: Suppose, that $E \in \mathcal{M}$, and that $\{f_k\}_{k=1}^{\infty}$ is a monotone growing sequence of nonnegative, measurable functions:

$$0 \leq f_1(x) \leq f_2(x) \leq ... \quad \text{for almost all } x \in E$$

Let f be defined to be the pointwise limit, $f(x) = lim_{k\to\infty} f_k(x)$ of this sequence. Then f is integrable and

$$\lim_{k\to\infty} \left(\int_E f_k dm \right) = \int_E f dm.$$

9.4.2 Lebesgue's Dominated Convergence Theorem

Assume that $E \in \mathcal{M}$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions, and f is a pointwise limit of this sequence. Further, assume, that there exists a function $g \in \mathcal{L}(\mathbb{R})$ such that $|f(x)| \leq g(x)$ for almost every $x \in E$, and $\forall k$. Then we conclude, that

$$\lim_{k\to\infty} \left(\int_E f_k dm \right) = \int_E f dm.$$

Theme 10

10.1 Lebesgue's $\mathcal{L}^p(R)$ spaces, where $1 \leq p < \infty$

Definition: We define for a real number $1 \leq p < \infty$, and $R = [a, b]$, the Lebesgue space $\mathcal{L}^p$:

$$\mathcal{L}^p(R) = \{f : R \to \mathbb{R}, \quad \int_R |f|^p dm < \infty\}.$$

In $\mathcal{L}^p$ spaces, we consider the functions that are almost everywhere equal, to be the same. In fact $\mathcal{L}^p$ really consists of equivalence classes of functions rather than functions, where

$$f \sim g \Leftrightarrow f(x) = g(x) \quad a.e.$$

We define the p-norm of an element $f \in \mathcal{L}^p$ to be the number

$$\|f\|_p = \left(\int_R |f|^p dm \right)^{\frac{1}{p}}.$$

Theorem: $\mathcal{L}^p$ is a vector space.

10.2 Connection between $\mathcal{L}^p(R)$ and $\mathcal{L}^q(R)$ when $m(R) < \infty$

Let's assume, that $m(R) < \infty$, and $1 \leq p < q \leq \infty$. In this case

$$\mathcal{L}^q \subset \mathcal{L}^p.$$

(Proof idea: Hölders inequality)

Theme 11

11.1 Essential supremum of a real function

Definition: $f : R \to \mathbb{C}$ is essentially bounded, if $\exists M \in \mathbb{R}$ constant, and $\exists E \in \mathcal{M}$ null set, such that

$$|f(x)| \leq M, \quad \forall x \notin E.$$

Definition: If f is essentially bounded, then it's essential supremum is defined as follows:

$$ess \sup f := \inf\{M | \exists E, m(E) = 0 : |f(x)| \leq M, \forall x \notin E\}$$

11.2 The $\mathcal{L}^\infty(R)$ space

Definition: The $\mathcal{L}^\infty(R)$ function space is the collection of essentially bounded functions over R.

$$\mathcal{L}^\infty(R) = \{f : R \to \mathbb{C}, \text{ essentially bounded}\}.$$

This space is a normed vectorspace, with the norm

$$\|f\|_\infty := ess \sup f.$$

11.2.1 Connection with $\mathcal{L}^p(R)$ when $m(R) < \infty$

Let's assume, that $m(R) < \infty$, and $1 \leq p < \infty$. In this case

$$\mathcal{L}^\infty \subset \mathcal{L}^p.$$

(Proof idea: The integral of all essentially bounded functions on a finite interval is finite, but there are not essentially bounded functions, that have a finite integral over a finite interval).

11.3 Riesz theorem on Lebesgue $\mathcal{L}^p$ spaces

Theorem: In the case of $1 \leq p \leq +\infty$, $\mathcal{L}^p(R)$ space is complete. This means that every Cauchy series in R has a limit inside R. Because it is normed, it is a Banach space.

Theme 12

12.1 $\mathcal{L}^2$(R) as a Hilbert space

Statement: An $\mathcal{L}^p$ space is an inner product space if and only if $p = 2$. In this case the inner product defined on $\mathcal{L}^2(\mathcal{X}, \mathcal{R}, \mu)$ is

$$\langle f, g \rangle = \int_{\mathcal{X}} f\bar{g}d\mu.$$

Thus $\mathcal{L}^2$ is a Hilbert space.

12.2 Orthonormal sequence

Definition: Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. We say that v and w in V are orthogonal, if $\langle v, w\rangle = 0$.

Definition: We say that v is normalised if $\|v\| = \sqrt{\langle v, v\rangle} = 1$.

Definition: A sequence $\{v_k\}_{k=1}^{\infty}$ in V is an orthonormal sequence, if $\langle v_k, v_j\rangle = \delta_{kj}, 1 \leq k, j < \infty$, where

$$\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases}$$

12.2.1 Example in $\mathcal{L}^2[-\pi, \pi]$

The sequence

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}}, \quad k = 1, 2, ... \right\}$$

is an orthonormal in the inner product space $\mathcal{L}^2([-\pi, \pi])$.

12.3 Complete ON system

Definition: Let $\{f_k\}_{k=1}^{\infty}$ be an orthonormal sequence in V. If it is the case, that for each $f \in V$ we can find constants c_k (depending on f) such that

$$f = \sum_{k=1}^{\infty} c_k f_k,$$

then we say that the sequence $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence in V.

12.4 Method for orthonormalising (P)

Statement: For every independet system $(f_n) \subset \mathcal{L}^2$, $\exists!(\varphi_n) \subset \mathcal{L}^2$ orthonormal system, such that

$$span(f_n) = span(\varphi_n)$$

Proof: Gram-Schmidt Orthogonalization:
Let

$$\varphi_1 = \frac{f_1}{\|f_1\|}.$$

Determining φ_2, we need $\{\varphi_1, \varphi_2\}$ to be an orthonormal space, and that with the linear combination of these two base vectors, we can construct f_2. We can achieve this by substracting the projection of f_2 on φ_1 from f_2, and then norming this result.

$$\varphi_2 = \frac{f_2 - \langle f_2, \varphi_1\rangle\varphi_1}{\|f_2 - \langle f_2, \varphi_1\rangle\varphi_1\|}.$$

Continuing with this process we get that

$$\varphi_n = \frac{f_n - \sum_{k=1}^{n-1}\langle f_n, \varphi_k\rangle\varphi_k}{\left\|f_n - \sum_{k=1}^{n-1}\langle f_n, \varphi_k\rangle\varphi_k\right\|}.$$

Theme 13

13.1 Legendre polynomials, their construction

Example: Let's consider $\mathcal{L}^2([-1,1])$. If one applies the Gram-Schmidt process to the functions $1, x, x^2, x^3, ...$ one obtains the complete othonormal sequence of Legendre polynomials:

$$P_n(x) = \sqrt{\frac{2n+1}{2}} \cdot \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n.$$

13.2 A complete ON system: Haar functions

Example: Let's give an example in $\mathcal{L}^2([0,1])$. The Haar functions are an orthonormal system in this space. These are not polinomials, they are simple wavelets.Wavelet series are used in signal and image processing and, in some context, are replacing the classical Fourier series.

We define

$$H_{0,0}(x) = 1, \qquad H_{n,k}(x) = \begin{cases} -2^{\frac{n}{2}}, & \frac{k-1}{2^n} \leq x < \frac{k-\frac{1}{2}}{2^n}, \\ 2^{\frac{n}{2}}, & \frac{k-\frac{1}{2}}{2^n} \leq x < \frac{k}{2^n}, \\ 0, & \text{otherwise}, \end{cases}$$

for $n \geq 1$, $1 \leq k \leq 2^n$.

(The sign of the function can be switched, it works that way aswell.)

13.3 Dimension of a vector space

The base of normed, and inner product spaces are vector spaces. We learned in Linear Algebra, that the vector space V is n dimensional, if it has a generator system of n linearly independent vectors.

Definition: We say that the dimension of the vector space V is infinite, if for every $n \in \mathbb{N}$ it has n independent vectors.

13.3.1 Examples

Example: For example

$$dim(l^p) = +\infty, \quad dim(C[a,b]) = +\infty.$$

Theme 14

14.1 Fourier analysis in $\mathcal{L}^2(R)$

Statement: Let $(\varphi_n) \subset \mathcal{L}^2$ be an orthonormed system. Then we can construct every $f \in \mathcal{L}^2$ like so:

$$f = \sum_{k=1}^{\infty} c_k \varphi_k,$$

where $c_k = \langle f, \varphi_k \rangle$. This is the Fourier construction of f with respect to the $\{\varphi_k\}_{k=1}^{\infty}$ orthonormed system.

From here it is obvious, that when we use the

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}}, \quad k = 1, 2, ... \right\}$$

orthonormal system, then the construction of any $f \in \mathcal{L}^2[-\pi, \pi]$ is as follows

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx),$$

and the coefficients:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad k = 0, 1, 2, ...,$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k = 1, 2, ...$$

14.2 Fourier coefficients (P)

Proof: First of all we have to prove, that $c_k = \langle f, \varphi_k \rangle$, and although in finite cases this is obvious, we have to see if this still stands in infinite dimensional spaces. Let

$$s_N = \sum_{k=1}^{N} c_k \varphi_k.$$

Because (ϖ_n) is a complete space, we know that

$$\lim_{N\to\infty} \|f - s_N\| = 0 \Rightarrow \lim_{N\to\infty} \langle f - s_N, \varphi_m \rangle = 0$$

Expanding this inner product we get

$$\lim_{N\to\infty} \langle f - s_N, \varphi_m \rangle = \langle f, \varphi_m \rangle - \lim_{N\to\infty} \langle s_N, \varphi_m \rangle = \langle f, \varphi_m \rangle - \sum_{k=1}^{N} c_k \langle \varphi_k, \varphi_m \rangle = \langle f, \varphi_m \rangle - c_m = 0.$$

Proof: Now we just need to substitute (φ_n), with the respected orthonormal system, in this case the trigonometric system:

$$\varphi_1 = \frac{1}{\sqrt{2\pi}} : \quad \varphi_1 \cdot \langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2\pi}} dx = \frac{1}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(0) dx,$$

$$\varphi_{2n-1} = \frac{\cos(nx)}{\sqrt{\pi}} : \quad \varphi_{2n-1} \cdot \langle f, \frac{\cos(nx)}{\sqrt{\pi}} \rangle = \frac{\cos(nx)}{\sqrt{\pi}} \cdot \int_{-\pi}^{\pi} f(x) \frac{\cos(nx)}{\sqrt{\pi}} dx = \cos(nx) \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx.$$

We can do the same thing to φ_{2n} too.

14.3 Parseval's theorem (P)

Theorem: Suppose that $\{\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence in $\mathcal{L}^2$. Then

$$\|f\|^2 = \sum_{k=1}^{\infty} c_k^2.$$

Proof: In Fourier analysis convergence of an infinite sum shoud be understanded by the norm of $\mathcal{L}^2$, so

$$\lim n \to \infty \left\| \sum_{k=1}^{n} c_k \varphi_k \right\| = \|f\| .$$

Using the orthogonality of the system, the square of the left side is:

$$\left\| \sum_{k=1}^{n} c_k \varphi_k \right\|^2 = \sum_{k=1}^{n} \|c_k \varphi_k\|^2 = \sum_{k=1}^{n} c_k^2.$$

14.4 Riesz-Fisher theorem

Theorem: Let $(d_k) \in l^2$. There exists an $f \in \mathcal{L}^2$ such that

$$\|f\|^2 = \sum_{k=1}^{\infty} d_k^2.$$

Proof: We can easily see that

$$f = \sum_{k=1}^{\infty} d_k \varphi_k$$

is a good solution to the problem.

14.5 Isometry of $\mathcal{L}^2(R)$ and l^2

Statement: $\mathcal{L}^2(R)$ and l^2 are isometricly isomorphic. We can give this isometry - with respect to the arbitrary complete orthonormal system - with the Fourier coefficients:

$$f \leftrightarrow (c_n).$$

Theme 15

15.1 General $\mathcal{L}^2_\rho(R)$ spaces with ρ weighting functions

Let the common domain of the elements of the function space be $R \subset \mathbb{R}$. We are not going to use the classical Lebesgue measure however we define the measure with a weight function. Let $\varrho : R \to \mathbb{R}^+$ be a Lebesgue integrable function, and with this we define the measure of $A \subset R$ like fo:

$$\mu(A) = \int_A \varrho dm.$$

Formally we can write "$d\mu = \varrho dm$", so an integral of f over E with respect to μ is

$$\int_E f d\mu = \int_E f \varrho dm.$$

Definition: We construct $\mathcal{L}^2_\varrho(R)$ space like so:

$$\mathcal{L}^2_\varrho = \{f : R \to \mathbb{R} : \quad \int_R f^2 d\mu = \int_R f^2 \varrho dm < \infty\}.$$

In this space aswell we look at functions that are equal almost everywhere to be the same.

Definition: An inner product of $f, g \in \mathcal{L}^2_\varrho(R)$ is

$$\langle f, g\rangle_\varrho = \int_R f g \varrho dm,$$

therefore the norm is:

$$\|f\|_{\varrho,2} = \left(\int_R |f|^2 \varrho dm\right)^{\frac{1}{2}}.$$

15.2 ON systems of polynomials

We look at the $\{1, x, x^2, x^3, ...\}$ independent system in $\mathcal{L}^2_\varrho$ space.

15.2.1 E.g.: Chebyshev polynomials (P)

$R = [-1, 1]$

Chebyshev polynomials of the first kind

Statement: Let $\varrho_1(x) = \frac{1}{\sqrt{1-x^2}}$. The Chebyshev polinomials of the first kind are written, as follows:

$$T_n(x) = \cos(n \arccos(x)).$$

We calculate the first few:

$$T_0(x) = 1$$

$$T_1(x) = \cos(\arccos(x)) = x$$

$$T_2(x) = \cos(2\arccos(x)) = 2\cos^2(\arccos(x)) - 1 = 2x^2 - 1$$

$$\vdots$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Proof: Let $\arccos(x) = \theta$. Calculating

$$T_{n+1}(x) = \cos((n+1)\theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) = 2cos(n\theta)\cos(\theta) - (\cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta)) =$$

$$= 2xT_n(x) - \cos((n-1)\theta) = 2xT_n(x) - T_{n-1}(x).$$

Chebyshev polinomials of the second kind

Statement: Let $\varrho_2(x) = \sqrt{1-x^2}$. The Chebyshev polinomials of the second kind are written, as follows:

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))}.$$

15.2.2 Eg.: Hermite polynomials

$R = \mathbb{R}$

Statement: Let $\varrho(x) = e^{-x^2}$. The Hermite polinomials are written, as follows:

$$H_n(x) = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

The recursive construction:

$$H_{n+1}(x) = 2xH_n(x) - \frac{d}{dx}H_{n-1}(x).$$

Theme 16

16.1 Abstract linear operators

We start by considering two real (or complex) linear spaces, X and Y over $\mathbb{K}$.

Definition: A $T : X \to Y$ mapping is called a linear operator if for every $x_1, x_2 \in X$, and $\alpha, \beta \in \mathbb{K}$,

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2).$$

We denote $T(x)$ with Tx.

16.2 Continuity

Definition: When X and Y are normed spaces, a linear operator $T : X \to Y$ is continuous in the $x_0 \in X$ point, if $\forall \epsilon > 0 : \exists \delta > 0$, so that when

$$\|x - x_0\|_X < \delta, \quad \Rightarrow \quad \|Tx_0 - Tx\|_Y < \epsilon.$$

16.2.1 Properties (P)

Statement: The continuity of an operator $T : X \to Y$ is equivalent to sequential continuity. This means that T is continuous in the point $x_0 \in X$ if and only if $\forall (x_n) \subset X$:

$$\lim_{n\to\infty} x_n = x_0 \Longrightarrow \lim_{n\to\infty} Tx_n = Tx_0.$$

Theorem: Let X and Y be normed linear spaces. A linear operator $T : X \to Y$ is continuous at every point if it is continuous at a single point.

Proof: Suppose that T is continuous at the point x_0, and let x be any point in X and $\{x_n\}_{n=1}^\infty$ a sequence in X converging to x. Then the sequence $\{x_n - x + x_0\}_{n=1}^\infty$ converges to x_0, and therefore, since T is continuous at x_0, $\{T(x_n - x + x_0)\}_{n=1}^\infty$ converges to Tx_0. Since T is linear,

$$T(x_n - x + x_0) = Tx_n - Tx + Tx_0,$$

and hence $\{Tx_n\}_{n=1}^\infty$ converges to Tx. Since x is chosen arbitrarily, T is continuous on all of X.

16.3 Boundedness, and continuity (P)

Definition: Let X and Y be normed linear spaces. A linear operator $T : X \to Y$ is said to be bounded if there exsists an $M > 0$ such that

$$\|Tx\|_Y \leq M \|x\|_X, \quad \forall x \in X.$$

Theorem: A $T : X \to Y$ linear operator is bounded if and only if it is continuous.

Proof:

1. step Prove that boundedness $\Rightarrow$ continuity.
Let's assume, that T is bounded. $\exists M$ such that $\forall x \in X$, $\|Tx\| \leq M \|x\|$. If we pick a sequence $(x_n) \in X$, so that $\lim_{n\to\infty} x_n = 0$, then

$$\lim_{n\to\infty \|Tx_n\|} \leq M \lim_{n\to\infty} x_n = 0.$$

This means that T is continuous in 0, so it is continuous everywhere.

2. step Prove that continuity ⇒ boundedness.
Let's assume, that T is continuous at $x_0 = 0$. Then for $\epsilon = 1$ there exists a $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow \|Tx\| \leq 1.$$

Let $x \neq 0$ be arbitrary. Then the norm of $y = \delta \frac{x}{\|x\|}$ is

$$\|y\| = \delta \frac{\|x\|}{\|x\|} = \delta.$$

From the continuity of the operator,

$$\|Ty\| \leq 1.$$

Expanding this inequality:

$$\|Ty\| = \frac{\delta}{\|x\|} \|Tx\| \leq 1 \quad \Rightarrow \quad \|Tx\| \leq \frac{1}{\delta} \|x\|.$$

This means we can choose M to be $\frac{1}{\delta}$.

16.4 Operator norm

Definition: In general, the norm of a bounded operator $T : X \to Y$ is defined to be

$$\|T\| = \inf\{M \big| \|Tx\|_Y \leq M \|x\|_X\}.$$

Statement:

$$\|T\| = \inf\left\{M \big| x \neq 0 : \frac{\|Tx\|}{\|x\|} \leq M\right\} = \sup\left\{\frac{\|Tx\|}{\|x\|} \big| x \neq 0\right\} = \sup\left\{\|Tx\| \,\big|\, \|x\| = 1\right\}.$$

Theme 17

17.1 Examples of bounded linear operators in

17.1.1 $\mathbb{R}^n$

Example: Let $X = \mathbb{R}^n$, and $Y = \mathbb{R}^m$. A $T : X \to Y$ operator is linear if and only if $\exists \mathbf{A} \in \mathbb{R}^{m\times n}$ such that $\forall x \in X : \quad Tx = \mathbf{A}x$. The bound of the operator depends on the norm. For example it could be the maximum of the rowsum, or columnsum.

17.1.2 l^2

Example: Let $X = Y = l^2$, and $T : X \to Y$, the left shift operator:

$$T(x_1, x_2, x_3, ...) = (x_2, x_3, ...).$$

The norm in this space:

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}}.$$

More so

$$\|Tx\| = \left(\sum_{k=2}^{\infty} |x_k|^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}} = \|x\|.$$

This operator is bounded. For example $(0, 1, 0, 0, ...) : \|Tx\| = \|x\| = 1$. Thus $\|T\| = 1$.

17.1.3 $C([a, b])$

Let $X = C[a, b]$, $Y = \mathbb{R}$, and $T : X \to Y$, the integral operator, where

$$Tf = \int_a^b f(x)dx.$$

$$\|Tf\| = \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \le \int_a^b \|f\| dx = \|f\| (b - a).$$

This means, that the operator is bounded. Furhtermore in case of $f = c$, $\|Tf\| = \|f\| (b - a)$. Thus $\|T\| = b - a$.

17.2 $\mathcal{B}(X, Y)$ as a normed space

Definition: The collection of all bounded linear operators from X to Y will be denoted by $\mathcal{B}(X, Y)$.

$$\mathcal{B}(X, Y) = \{T : X \to Y, \text{ bounded, linear}\}.$$

This is a normed space with the previously defined norm. Specially, if $X = Y$, then we denote $\mathcal{B}(X, X)$ with $\mathcal{B}(X)$.

$$\mathcal{B}(X) = \{T : X \to X, \text{ bounded, linear}\}.$$

17.3 Completeness

Statement: If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

Theme 18

18.1 Bounded linear operators in a Banach space.

Definition: The collection of all bounded linear operators from X to Y will be denoted by $\mathcal{B}(X,Y)$.

$$\mathcal{B}(X,Y) = \{T : X \to Y, \text{ bounded and linear}\}.$$

This is a normed space, with the norm previously defined. Specially, if $X = Y$, then we denote $\mathcal{B}(\mathcal{X},\mathcal{X})$, with $\mathcal{B}(X)$.

Statement: If Y is a Banach space, then $\mathcal{B}(X,Y)$ is also a Banach space.

18.2 Multiplication of operators

Statement: Operator multiplication is sub-multiplicative, which means, that

$$\|TS\| \leq \|T\| \cdot \|S\|.$$

18.3 $\mathcal{B}(X)$

A special case of $\mathcal{B}(X,Y)$ is when $X = Y$. We denote this special case, by $\mathcal{B}(X)$:

$$\mathcal{B}(X) = \{T : X \to X, \text{ bounded, linear}\}.$$

In finite dimensions we would call these operators linear transformations. If X is a Banach space, then $\mathcal{B}(X)$ is also a Banach space.

Let X be a Banach space. This way, we know, that $\mathcal{B}(X)$ is also a Banach space. The product of $T, S \in \mathcal{B}(X)$ is

$$TS = T \circ S \in \mathcal{B}(X).$$

In this Banach-algebra there exists an identity for multiplication. $I : X \to X$, such that $x \to Ix := x$. In this case obviously

$$TI = IT = T, \quad \forall T \in \mathcal{B}(X).$$

18.4 Inverse of an operator

Definition: An element $T \in \mathcal{B}(X)$ is invertable, if there exists an $S \in \mathcal{B}(X)$ operator such that $TS = ST = I$.

18.5 A condition on the existence of the inverse operator (P)

Theorem: Let X be a Banach space. Suppose, that $T \in \mathcal{B}(X)$ is such that $\|T\| < 1$. Then the operator $I - T$ is invertible in $\mathcal{B}(X)$, and its inverse is given by

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$$

Proof: $I - T : X \to X$ operator means that if $(I - T)x = y$, then $x - Tx = y$. If this operator is invertible, then we can calculate the original x from a fixed y. We rearrange the equation, and it is equivalent with:

$$x = y + Tx.$$

We solve the equation for an arbitrary $y \in X$ with iteration. Let $x_0 \in X$ an arbitrary starting point. The next steps of the iteration:

$$x_1 = y + Tx_0, \quad x_2 = y + Tx_1, \quad \ldots \quad x_n = y + Tx_{n-1}, \quad \ldots$$

This way we get an $(x_n) \subset X$ sequence. Now

$$x_{n+1} - x_n = y + Tx_n - y - Tx_{n-1} = Tx_n - Tx_{n-1} = T(x_n - x_{n-1}) = T^n(x_1 - x_0).$$

Because of the boundedness of the norm

$$\|x_{n+1} - x_n\| \leq \|T^n\| \cdot \|x_1 - x_0\| \leq \|T\|^k \cdot \|x_1 - x_0\|.$$

Knowing that $\|T\| < 1$, we can easily see, that $\|x_{n+1} - x_n\| \to 0$ exponentially. So (x_n) is a Cauchy sequence in X, so it is convergent. Let $x^* = \lim_{n\to\infty} x_n$.

We defined the sequence like so:

$$x_{n+1} = y + Tx_n.$$

Using the continuity we get that

$$X^* = y + Tx^*.$$

From here it is obvious, that

$$(I - T)x^* = y \Rightarrow x^* = (I - T)^{-1}y.$$

Also

$$x_n = y + Tx_{n-1} = y + T(y + Tx_{n-2}) = ... = \sum_{k=0}^{n-1} T^k y.$$

From this, we get

$$x^* = \lim_{n\to\infty} \sum_{k=0}^{n-1} T^k y \Rightarrow (I - T)^{-1} = \sum_{k=0}^{\infty} T_k.$$

18.6 Basic properties

Theorem: Let $T \in \mathcal{B}(X)$ be invertable. Let's assume, that the norm of an $S \in \mathcal{B}(X)$ is $\|S\| < \frac{1}{\|T^{-1}\|}$. This way $T + S$ is also invertable.

Theorem: In $\mathcal{B}(X)$ the set of invertable operators is open.

Theme 19

19.1 Spectrum of a bounded linear operator

Definition: The spectrum of an element $T \in \mathcal{B}(X)$ is defined to be the set of all complex numbers λ, such that $T - \lambda I$ is not invertible. We denote this set by $\sigma(T)$.

19.2 Connection with the eigenvalues

Statement: If X is finite dimensional, then the elements of $\mathcal{B}(X)$ are square matrices. In this case the spectrum of the operator are the eigenvalues. This is not completly true in infinite dimensions.

19.3 Properties of the spectrum (P)

Theorem: The $\sigma(T)$ spectrum of $T \in \mathcal{B}(X)$ is a non-empty set.

Theorem: The $\sigma(T)$ spectrum of $T \in \mathcal{B}(X)$ is a closed set on $\mathbb{C}$.

Proof: We prove, that $\sigma(T)^C$ is open. Let's say, that $\lambda \notin \sigma(T)$. Then $T - \lambda I$ is invertible, so it is part of the open set of invertible operators. This means, that $\exists \epsilon > 0$ such that the ϵ-ball centered at λ is also in $\sigma(T)^C$. This means that $\sigma(T)$ is closed.

Theorem: The $\sigma(T)$ spectrum of $T \in \mathcal{B}(X)$ is a bounded set.

Proof: Let $|\lambda| > \|T\|$. We prove, that in this case λ is not in the spectrum. We can write $T - \lambda I = -\lambda(I - \lambda^{-1}T)$. The norm of $\|[\| \lambda^{-1}T] = |\lambda|^{-1} \|T\| < 1$. So this means that $I - \lambda^{-1}T$ is invertible, so $T - \lambda I$ is invertable. This means than $\lambda \notin \sigma(T)$. So:

$$\sigma(T) \subseteq \{\lambda \big| |\lambda| \leq \|T\|\}.$$

19.3.1 Examples

In finite dimension: Let the matrix be

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5-i \end{bmatrix}.$$

The spectrum of $\mathbf{A}$ consists of the eigenvalues:

$$\sigma(\mathbf{A}) = \{1, 2, 5 - i\}.$$

In infinite dimension: Let $X = l^2$, and $\mathbf{D} = diag(\lambda_n)$, an infinite dimensional diagonal matrix, which is also the operator. We can easily see, that the eigenvalues will be in the spectrum, because $D - \lambda_n I$ is not invertable. $\forall \lambda_n \in \sigma(\mathbf{D})$.

Now let $\lambda \neq \lambda_n$. In this case a potential inverse of $\mathbf{D} - \lambda I$ is $diag(\frac{1}{\lambda_n - \lambda})$. This inverse exists if λ isn't a congestion point of (λ_n). This means that $\sigma(D)$ also contains all of (λ_n)'s congestion points aswell.

Theme 20

20.1 Linear functionals

Definition: If the image of a linear operator is $\mathbb{R}$ or $\mathbb{C}$, then the $T : X \to \mathbb{K}$ operator is called a functional.

20.2 Norm of a bounded linear functional

Statement: The norm of a linear functional is defined as follows:

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\}.$$

20.3 Examples in function spaces

1. The integral operator:
$$f_1(x) = \int_a^b x(t)dt, \quad \|f_1\| = b - a.$$
2. With a fixed $y \in C[q, b]$:
$$f_2(x) = \int_a^b x(t)y(t)dt, \quad \|f_2\| = \int_a^b |y(t)|dt.$$
3. With a fixed $t_0 \in [a, b]$:
$$f_3(x) = \delta(t_0) = x(t_0).$$

20.4 Dual space

The dual space of $(X, \|\cdot\|)$ space is a collection of all bounded, linear functionals on X. We denote this whit $X^* = \mathcal{B}(X, \mathbb{R})$.

20.5 Examples

20.5.1 $\mathbb{R}^n$ with different norms (P)

Example: We already know, that if $F : \mathbb{R}^n \to \mathbb{R}$ is a linear map, than $\exists a \in \mathbb{R}^n$ such that $f(x) = a^T x$. This is why $(\mathbb{R}^n)* = \mathbb{R}^n$. But the inducated norm depends on the original space's norm.

1. Lets start with $(\mathbb{R}^n, \|\cdot\|_2)$:
$$|f(x)| = \left|\sum_{k=1}^n a_k x_k\right| \leq \left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n x_k^2\right)^{\frac{1}{2}} = \|a\|_2 \cdot \|x\|_2 .$$
$|f(a)| = \|a\|_2^2$, $\|f\| = \|a\|_2$.
2. Lets start with $(\mathbb{R}^n, \|\cdot\|_\infty)$:
$$|f(x)| = \left|\sum_{k=1}^n a_k x_k\right| \leq \sum_{k=1}^n |a_k x_k| \leq \max(x_k) \sum_{k=1}^n a_k = \|a\|_1 \cdot \|x\|_\infty .$$
$x_k = sgn(a_k)$, so $\|f\| = \|a\|_1$.
3. In general if p and q are Hölder conjugates, then
$$(R^n, \|\cdot\|_p)^* = (\mathbb{R}^n, \|\cdot\|_q).$$

20.5.2 l^p

Example: With same arguments, we can see that $(l^p)^* = l^q$, if p and q are Hölder conjugates.

Theme 21

21.1 Second dual space

Definition: The second dual space of $(x, \|\cdot\|)$ space, is the dual space of X^*. It is denoted by X^{**}.

21.2 Weak and strong convergence

Definition: An (x_n) sequence in X is weakly convergent to $x_0 \in X$, if

$$\lim_{n\o\infty} f(x_n) = f(x_0), \quad \forall f \in X^*.$$

Definition: Strong convergence is the convergence in the norm, defined in the Topology. The (x_n) sequence is strongly convergent and the limit is x_0 if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

21.2.1 Their connection (P)

Statement: If the $(x_n \subset X)$ sequence is strongly convergent, it is also weekly convergent.

Proof: Let's suppose, that (x_n) is strongly convergent. Let $f \in X^*$ be a functional. Because of the linearity $f(x_n) - f(x_0) = f(x_n - x_0)$. THis is why:

$$|f(x_n) - f(x_0)| = |f(x_n - x_0)| \leq \|f\| \cdot \|x_n - x_0\| \to 0.$$

So the sequence is also weakly convergent.

21.3 Linear functionals in Hilbert space

Let H be a Hilbert space.

Example: Let $y \in H$ be fixed. The $f_y : H \to \mathbb{R}$ functional will be defined as follows:

$$f_y(x) := \langle x, y \rangle.$$

Now because of the Cauchy-Brunyakovszki-Schwarz inequality:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|,$$

and in the case of $x = y$, we have an equality. So $\|f_y\| = \|y\|$. It is forseeable, that essentially there is no other functional.

21.4 Riesz representation theorem

Theorem: For every $f \in H^*$ functional, there exists a $y \in H$, such that

$$f(x) = \langle x, y \rangle,$$

and $\|f\| = \|y\|$.

21.5 Dual space of a Hilbert space

Statement: H and H^* are isomorphic, and every Hilbert space is reflexive.

Theme 22

22.1 Adjoint of a bounded linear operator in a Hilbert space

Definition: The adjoint of the $A \in \mathcal{B}(H)$ liner operator, is the $A^* \in \mathcal{B}(H)$ linear operator, that satisfies

$$\langle Ax, y \rangle = \langle x, A^*y, , \rangle \quad \forall x, y \in H.$$

22.1.1 Existence (P)

Statement: The adjoint operator is well defined.

Proof: Let $f(x) := \langle Ax, y \rangle$. The Riesz representation theorem sasy, that $\exists y^* \in H$, such that

$$f(x) = \langle x, y^* \rangle.$$

This means, that there is a linear operator $A^* \in \mathcal{B}(H)$, such that $A^*y = y^*$. This is the adjoint, because

$$\langle Ax, y \rangle = \langle x, y^* \rangle = \langle x, A^*y \rangle.$$

22.1.2 Examples in finiteand infinite dimension

Example: In finite dimension:
Let $H = \mathbb{R}^n$. Here a linear operator can be given as $A \in \mathbb{R}^{n \times n}$ square matrix.

$$\langle Ax, y \rangle = (Ax)^T y = x^T A^T y = \langle x, A^T y \rangle.$$

So $A^* = A^T$.

Example: In infinite dimension:
Let $H = \mathcal{L}^2[0, 1]$, where the inner product is

$$\langle u, v \rangle = \int_0^1 u(t)v(t)dt.$$

Lets have a look at the $H_0 \subset H$ subspace, which contains those infinitely differentiable $u(t)$ functions, where $u(0) = u(1) = 0$. We construct the differetial operator in H_0:

$$Au = u'.$$

In this case

$$\langle Au, v \rangle = \int_0^1 u'(t)v(t)dt = u(t)v(t)\Big|_0^1 - \int_0^1 u(t)v'(t)dt = \int_0^1 u(t)(-v'(t))dt = \langle u, -v' \rangle = \langle u, A^*v \rangle.$$

Thus the adjoint of a differential operator in H_0 is

$$A * v = -v'.$$

22.2 Self adjoint operator

Definition: The operator A is self adjoint, if $A = A^*$

Theorem: If A is self adjoint, then

1. $\|A^n\| = \|A\|^n$.
2. The spectral radius $r(A) = \|A\|$.
3. The spectrum is real: $\sigma(A) \subset \mathbb{R}$.

22.2.1 E.g.: orthogonal projection (P)

Example: Let $E \subset H$ a closed subspace. Then $\forall x \in H$ can be constructed by $x = x_E + x_0$, where $x_e \in E$ and $x_0 \perp E$. The operator of the orthogonal projection is $P : H \to E$, such that $Px = x_E$. Then

$$\langle Px, y\rangle = \langle Px, Py + y_0\rangle = \langle Px, Py\rangle + \langle Px, y_0\rangle = (**).$$

Here the second element is 0. $\langle x_0, Py\rangle$ is also 0, so

$$(**) = \langle Px, Py\rangle + \langle x_0, Py\rangle = \langle Px + x_0, Py\rangle = \langle x, Py\rangle.$$

We get, that $P = P^*$.

Theme 23

23.1 Hilbert space methods in quantum mechanics

23.1.1 Movement of a particle along a straight line

Let there be a particle, which is moving on an infinite straight line. It's state is described by a complex valued state function $f(x,t)$. The t variable represents time, x represents the state like so. The probability that the particle is in the $[a,b]$ interval at t time, is

$$\int_a^b |f(x,t)|^2 dx.$$

We obviously require, that for every t,

$$\int_{-\infty}^{\infty} |f(x,t)|^2 dx = 1.$$

Mathematic model The model above formally means, that $f \in \mathcal{L}^2(\mathbb{R})$, and $\|f\| = 1$.

Another physical property is momentum, which is given by the Fourier transform of f

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} f(x) dx.$$

Because of Parseval's equality

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(w)|^2 dw,$$

So $\widehat{f} \in \mathcal{L}^2(\mathbb{R})$, and $\left\|\widehat{f}\right\| = 1$. Let $\overline{x}, \overline{w}$ be the expected value of the state, and momentum:

$$\overline{x} = \int_{-\infty}^{\infty} x|f(x)|^2 dx, \quad \overline{w} = \int_{-\infty}^{\infty} w|\widehat{f}(w)|^2 dw,$$

and σ_x^2, σ_w^2 the variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\overline{x})^2 |f(x)|^2 dx, \quad \int_{-\infty}^{\infty} (w-\overline{w})^2 |\widehat{f}(w)|^2 dw.$$

23.1.2 Heisenberg's uncertainty principle (P)

Theorem: σ_x and σ_w can't be both small at the same time.

$$\sigma_x^2 \cdot \sigma_w^2 \geq \frac{1}{4}.$$

Proof: Lets assume, without the violation of generality, that $\overline{x} = \overline{w} = 0$. Lets define two operators in $\mathcal{L}^r(\mathbb{R})$ Hilbert space:

$$Mf(x) = x \cdot f(x),$$

$$Df(x) = f'(x).$$

It is forseeable, that these operators are defined where our functions are defined. It is true, that

$$\|Mf\|^2 = \sigma_x^2, \quad \|Df\|^2 = \sigma_w^2,$$

and also that $DM - MD = I$. Also M is self adjoined, and $D^* = -D$. We can forsee, that these perators are too defined, where our functions are defined. That means

$$\|f\|^2 = \langle f, f\rangle = \langle f, (DM-MD)f\rangle = \langle f, DMf\rangle - \langle f, MDf\rangle = \langle -Df, Mf\rangle - \langle Mf, Df\rangle = -2\langle Df, Mf\rangle = 1,$$

so

$$\frac{1}{2} = |\langle Df, Mf\rangle| \leq \|Mf\| \cdot \|Df\|,$$

and from that, we get the theorem.

Proof:

$$\|Mf\|^2 = \int_{-\infty}^{\infty} |xf(x)|^2 dx = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx = \sigma_x^2.$$

Because of the Parseval equality

$$\|Df\|^2 = \left\|\widehat{Df}\right\|^2 = \int_{-\infty}^{\infty} |\widehat{Df}(w)|^2 dw = \int_{-\infty}^{\infty} |iw\widehat{f}(w)|^2 dw = \int_{-\infty^{\infty}} w^2 |\widehat{f}(w)|^2 dw = \sigma_w^2.$$

Proof:

$$DMf = (xf(x))' = f(x) + xf'(x) = If + MDf \Rightarrow DM - MD = I.$$

Proof:

$$\langle Mf, g\rangle = \int_{-\infty}^{\infty} xf(x)g(x)dx = \int_{-\infty}^{\infty} f(x)xg(x)dx = \langle f, Mg\rangle.$$

$$\langle Df, g\rangle = \int_{-\infty}^{\infty} f'(x)g(x)dx = f(x)g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)g'(x)dx = \langle f, -Dg\rangle.$$