Image Recovery

Extension: The Wiener Filter

Wiener Filter

- Stochastic restoration approach:
  - Treat the image as a sample from a 2D random field.
  - The image is part of a class of samples (an ensemble), realizations of the same random field.
- Definitions:
  - Autocorrelation:

Expected value over many realizations of the 2D random field

$$R_{ff}(n_1, n_2, n_3, n_4) = E\left[f(n_1, n_2)\, f^*(n_3, n_4)\right]$$

    - Wide Sense Stationarity property (WSS):

$$R_{ff}(n_1, n_2, n_3, n_4) = R_{ff}(n_1 - n_3, n_2 - n_4) = R_{ff}(d_1, d_2)$$

  - Ergodicity:
    - ensemble average is equal to spatial average

$$R_{ff}(d_1, d_2) = \lim_{N \to \infty} \frac{1}{(2N+1)^2} \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} f(k_1, k_2)\, f^*(k_1 - d_1, k_2 - d_2)$$

Wiener Filter

- Stochastic restoration approach:
  - Treat the image as a sample from a 2D random field.
- Definitions:
  - Power-Spectrum:

$$P_{ff}(\omega_1, \omega_2) = \mathrm{F}\{R_{ff}(d_1, d_2)\}$$

Fourier transformation

  - Cross-Correlation:

$$R_{fg}(n_1, n_2, n_3, n_4) = E\left[f(n_1, n_2) g^*(n_3, n_4)\right]$$

$$\Rightarrow \quad P_{fg}(\omega_1, \omega_2) = \mathrm{F}\{R_{fg}(d_1, d_2)\}$$

$$P_{gf}(\omega_1, \omega_2) = \mathrm{F}\{R_{gf}(d_1, d_2)\}$$

Wiener Filter

- The degradation model:

$$y(n_1, n_2) = x(n_1, n_2) * h(n_1, n_2) + n(n_1, n_2)$$

- The objective:

$$\tilde{x}(n_1, n_2) = \underset{\tilde{x}(n_1, n_2)}{\operatorname{argmin}} \; E\left[\left| x(n_1, n_2) - \tilde{x}(n_1, n_2) \right|^2 \right]$$

- We look for the solution in the following format, assuming an LSI restoration model:

$$\tilde{x}(n_1, n_2) = r(n_1, n_2) * y(n_1, n_2)$$

  - The input image is assumed to be WSS with autocorrelation $R_{xx}(n_1, n_2)$.

Wiener Filter

- It can be shown that the autocorrelation of

$$\tilde{y}(n_1, n_2) = x(n_1, n_2) * h(n_1, n_2)$$

...is the following:

$$R_{\tilde{y}\tilde{y}}(n_1, n_2) = R_{xx}(n_1, n_2) * h(n_1, n_2) * h^*(-n_1, -n_2)$$

- In the frequency domain:

$$P_{\tilde{y}\tilde{y}}(\omega_1, \omega_2) = \left|H(\omega_1, \omega_2)\right|^2 \cdot P_{xx}(\omega_1, \omega_2)$$

similarly..

$$R_{x\tilde{y}}(n_1, n_2) = R_{xx}(n_1, n_2) * h^*(-n_1, -n_2) \quad \rightarrow \quad P_{x\tilde{y}}(\omega_1, \omega_2) = H^*(\omega_1, \omega_2) \cdot P_{xx}(\omega_1, \omega_2)$$

$$R_{\tilde{y}x}(n_1, n_2) = R_{xx}(n_1, n_2) * h(n_1, n_2) \quad \rightarrow \quad P_{\tilde{y}x}(\omega_1, \omega_2) = H(\omega_1, \omega_2) \cdot P_{xx}(\omega_1, \omega_2)$$

Wiener Filter

- Assumptions:
  - that both the input image and the noise are WSS.
  - The restoration error and the signal (the observed image) is orthogonal:

$$E\left[e(n_1,n_2)y^*(n_3,n_4)\right] = E\left[(x(n_1,n_2)-\tilde{x}(n_1,n_2))y^*(n_3,n_4)\right] = 0, \quad \forall (n_1,n_2),(n_3,n_4)$$

$$\Downarrow$$

$$E\left[x(n_1,n_2)y^*(n_3,n_4)\right] = E\left[\tilde{x}(n_1,n_2)y^*(n_3,n_4)\right]$$

$$E\left[x(n_1,n_2)y^*(n_3,n_4)\right] = E\left[(y(n_1,n_2)*r(n_1,n_2))y^*(n_3,n_4)\right]$$

$$R_{xy}(n_1,n_2) = R_{yy}(n_1,n_2)*r(n_1,n_2)$$

$$\Downarrow$$

Going to the frequency domain

$$\boxed{R(\omega_1,\omega_2) = \frac{P_{xy}(\omega_1,\omega_2)}{P_{yy}(\omega_1,\omega_2)}}$$

Wiener Filter

- Assuming zero mean, uncorrelated image and noise, we can derive the following equation for the restoration filter:

$$R(\omega_1, \omega_2) = \frac{\boxed{P_{xy}(\omega_1, \omega_2)}}{\boxed{P_{yy}(\omega_1, \omega_2)}}$$

$$\boxed{P_{xy}(\omega_1, \omega_2) = H^*(\omega_1, \omega_2) \cdot P_{xx}(\omega_1, \omega_2)}$$

$$\boxed{P_{yy}(\omega_1, \omega_2) = \left|H(\omega_1, \omega_2)\right|^2 \cdot P_{xx}(\omega_1, \omega_2) + P_{NN}(\omega_1, \omega_2)}$$

$$\Downarrow$$

$$\boxed{R(\omega_1, \omega_2) = \frac{H^*(\omega_1, \omega_2) \cdot P_{xx}(\omega_1, \omega_2)}{\left|H(\omega_1, \omega_2)\right|^2 \cdot P_{xx}(\omega_1, \omega_2) + P_{NN}(\omega_1, \omega_2)}}$$

Wiener Filter and CLS Filter

- Wiener filter:

$$R(\omega_1,\omega_2)=\frac{H^*(\omega_1,\omega_2)\cdot P_{xx}(\omega_1,\omega_2)}{\left|H(\omega_1,\omega_2)\right|^2\cdot P_{xx}(\omega_1,\omega_2)+P_{NN}(\omega_1,\omega_2)}=$$

$$=\frac{H^*(\omega_1,\omega_2)}{\left|H(\omega_1,\omega_2)\right|^2+\frac{P_{NN}(\omega_1,\omega_2)}{P_{xx}(\omega_1,\omega_2)}}=\frac{H^*(\omega_1,\omega_2)}{\left|H(\omega_1,\omega_2)\right|^2+\frac{\sigma_N^2}{P_{xx}(\omega_1,\omega_2)}}$$

Assuming white noise

- CLS filter:

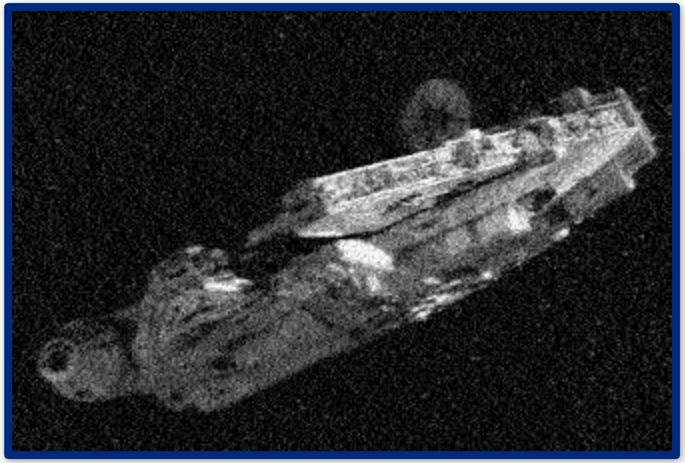
$$R(\omega_1,\omega_2)=\frac{H^*(\omega_1,\omega_2)}{\left|H(\omega_1,\omega_2)\right|^2+\alpha\left|C(\omega_1,\omega_2)\right|^2}$$

- With the right choice of C and α, CLS filter is the same as the Wiener filter.

Wiener Filter

Original Image

Motion Blurred Noisy Image



Reconstructed with Wiener filter

Reconstructed with CLS filter