Image Recovery

Convolution in Matrix-Vector Form

Convolution in matrix-vector form

- 1D convolution can be represented in a matrix-vector form:

$$y(n) = x(n) * h(n) = \sum_k x(k)h(n-k)\text{, where}\quad \begin{array}{l} x:1\times N \\ h:1\times L \\ y:1\times\left(N+L-1\right) \end{array}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \\ \vdots \\ \\ y(N+L-2) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \cdots & \cdots & 0 \\ h(1) & h(0) & 0 & \cdots & 0 \\ h(2) & h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ h(L-1) & \vdots & & & h(0) \\ 0 & h(L-1) & & & \vdots \\ \vdots & \vdots & & & h(L-1) \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \\ \vdots \\ \\ x(N-1) \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{H} \cdot \boldsymbol{x}$$

Convolution in matrix-vector form

- Circular convolution represented in a matrix-vector form:

$$\boldsymbol{y}(\boldsymbol{n}) = \boldsymbol{x}(\boldsymbol{n}) \otimes \boldsymbol{h}(\boldsymbol{n}), \text{ where } \quad \begin{matrix} \boldsymbol{x} : 1 \times \boldsymbol{N} \\ \boldsymbol{h} : 1 \times \boldsymbol{L} \\ \boldsymbol{y} : 1 \times (\boldsymbol{N} + \boldsymbol{L} - 1) \end{matrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N+L-2) \end{bmatrix} = \begin{bmatrix} h(0) & \cdots & h(L-1) & \cdots & h(2) & h(1) \\ h(1) & \ddots & & \ddots & & \vdots \\ h(2) & & \ddots & 0 & & h(L-1) \\ \vdots & & & \ddots & & \vdots \\ h(L-1) & \ddots & & & \ddots & \\ & 0 & \ddots & h(L-1) & \cdots & h(0) \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

H is a circulant matrix

$$\boldsymbol{y} : 1 \times (\boldsymbol{N} + \boldsymbol{L} - 1) \qquad \boldsymbol{H} : (\boldsymbol{N} + \boldsymbol{L} - 1) \times (\boldsymbol{N} + \boldsymbol{L} - 1) \qquad \boldsymbol{x} : 1 \times (\boldsymbol{N} + \boldsymbol{L} - 1)$$

Convolution in matrix-vector form

- The eigenvalues and eigenvectors of circulant matrices:

Let *H* be a circulant matrix:

$$H = \begin{bmatrix} h(0) & \cdots & h(M-1) \\ & \vdots & \\ & & \end{bmatrix}$$

with eigenvalues λ_n and eigenvectors w_n where $n = 1 \ldots M$:

$$Hw_n = \lambda_n w_n$$

then…

$$w_n = \begin{bmatrix} 1 & e^{j\frac{2\pi}{M}n} & e^{j\frac{2\pi}{M}2n} & \cdots & e^{j\frac{2\pi}{M}(M-1)n} \end{bmatrix}^T$$

$$\{\lambda_0, \ldots \lambda_{m-1}\} = M \cdot \mathrm{DFT}\{h(0), \ldots, h(M-1)\}$$

And the Singular Value Decomposition of H is the following:

$$H = \begin{bmatrix} w_0 & \cdots & w_{M-1} \end{bmatrix} \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{M-1} \end{bmatrix} \begin{bmatrix} w_0 & \cdots & w_{M-1} \end{bmatrix}^{-1} = WDW^{-1}$$

Convolution in matrix-vector form

- Circulant convolution in matrix-vector form

Matrix-vector form

$$y(n) = x(n) \otimes h(n) \longrightarrow y = Hx = WDW^{-1}x$$

DFT

$$W^{-1}y = DW^{-1}x$$

$$Y(k) = X(k) \cdot H(k) \qquad Y = DX$$

$$k = 0, \ldots, N-1$$

$$\begin{bmatrix} Y(0) \\ \vdots \\ Y(N-1) \end{bmatrix} = \begin{bmatrix} H(0) & & \\ & \ddots & \\ & & H(N-1) \end{bmatrix} \begin{bmatrix} X(0) \\ \vdots \\ X(N-1) \end{bmatrix}$$

- Back to images:
  - If we stack the observed image lexicographically into a vector, the degradation can be described the following way:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$$

  - If the system is LSI, then ***H*** is a ***block circulant*** matrix, which can be decomposed as a circulant matrix:

$$H = WDW^{-1}$$

Convolution in matrix-vector form

- Back to images:
  - If we stack the observed image lexicographically into a vector, the degradation can be described the following way: $\boldsymbol{y} = \boldsymbol{H}\mathbf{x} + \boldsymbol{n}$
  - If the system is LSI, then ***H*** is a *block circulant* matrix, which can be decomposed the same way as a circulant matrix:

$$H = WDW^{-1} \longrightarrow \begin{aligned} y &= Hx + n = WDW^{-1}x + n \\ W^{-1}y &= DW^{-1}x + W^{-1}n \\ Y &= DX + N \end{aligned}$$

  - Since ***D*** is diagonal, we have the following element wise equation:

$$Y(\omega_1, \omega_2) = H(\omega_1, \omega_2) \cdot X(\omega_1, \omega_2) + N(\omega_1, \omega_2), \text{ where} \begin{cases} \omega_1 = 0, \ldots, M-1 \\ \omega_2 = 0, \ldots, M-1 \end{cases}$$