# Tables of Cellular Automaton Properties

1986

## Introduction

This appendix gives tables of properties of one-dimensional cellular automata with two possible values at each site ($k = 2$), and with rules depending on nearest neighbours ($r = 1$). These cellular automata are some of the simplest that can be constructed. Yet they are already capable of a great diversity of highly complex behaviour. The tables in this appendix attempt to capture some of this behaviour, both pictorially and numerically.

There are 256 possible rules for $k = 2$, $r = 1$ cellular automata. Table 1 gives forms for these rules, together with simple equivalences among them.

Tables 2 and 3 show patterns produced by evolution according to all possible inequivalent rules, starting from "typical" disordered or random initial conditions. Several general classes of qualitative behaviour are seen (see pages 115–157 in this book):

1. A fixed, homogeneous, state is eventually reached (e.g. rules 0, 8, 136).
2. A pattern consisting of separated periodic regions is produced (e.g. rules 4, 37, 56, 73).
3. A chaotic, aperiodic, pattern is produced (e.g. rules 18, 45, 146).
4. Complex, localized structures are generated (e.g. rule 110). (This behaviour is clearly visible in the pictures of table 15.)

Much of the data in this appendix can be understood in terms of this classification.

The patterns produced with a particular rule by evolution from different disordered initial states are qualitatively similar. Nevertheless, changes in initial conditions can lead to detailed changes in the configurations produced. Table 4 shows the pattern of

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differences produced by single-site changes in initial conditions. For class 1 rules, the changes always die out. For class 2 rules, they may persist, but remain localized. Class 3 rules, however, show "instability": small changes in initial conditions can lead to an ever-expanding region of differences. "Information" on the initial state thus propagates, typically at a fixed speed, through the cellular automaton. In class 4 cellular automata, such information transmission occurs irregularly, through motion of specific localized structures.

Table 6 gives the values of some statistical quantities which characterize some of the behaviour seen in tables 2, 3 and 4. The definitions of entropies and Lyapunov exponents for cellular automata (see pages 115–157 in this book) are closely analogous to those for conventional continuous dynamical systems.

Tables 2, 3, 4 and 6 concern the generic behaviour of cellular automata with "typical" disordered initial conditions. The generation of complexity in cellular automata is however perhaps more clearly illustrated by evolution from particular, simple, initial conditions, as in table 5. With such initial conditions, some cellular automaton rules yield simple or regular patterns. But other rules yield highly complex patterns, which seem in many respects random.

Tables 2 through 6 suggest that many different $k = 2$, $r = 1$ cellular automata exhibit similar behaviour. Table 1 gives some simple equivalences between rules. Table 7 gives equivalences arising from more complex transformations. Often different regions in a cellular automaton will form "domains" which show different equivalences.

Table 8 gives further relations between rules, in the form of factorizations which express one rule as compositions of others.

An important feature of cellular automata is their capability for "self organization". Even starting from arbitrary disordered or random initial conditions, their time evolution can pick out particular "ordered" states. Tables 9 through 11 give mathematical characterizations of the sets of configurations that can occur in the evolution of $k = 2$, $r = 1$ cellular automata. Table 9 concerns blocks of site values which are filtered out by the cellular automaton evolution.

The complete set of configurations produced after any finite number of time steps can be described in terms of regular formal languages (see pages 159–202 in this book). Tables 10 and 11 give the values of quantities which characterize the certain aspects of the "complexity" of these languages.

The behaviour of class 3 and 4 cellular automata often seems to be so complex that its outcome cannot be determined except by essentially performing a direct simulation. Tables 10 and 11 may provide some quantitative basis for this supposition. Table 12 gives a more direct measure of the difficulty of computing the outcome of cellular automaton evolution in the context of a simple computational model involving Boolean functions.

The results for most of the tables here are for cellular automata on lattices with an infinite number of sites. Tables 13 and 14 give some of the more complete results that can be obtained for cellular automata on finite lattices (or with spatially periodic configurations). Table 13 shows fragments of the state transition diagrams which describe the global evolution of finite cellular automata. Table 14 plots some of their overall properties.

Many of the $k = 2$, $r = 1$ cellular automata show highly complex behaviour. Such behaviour is probably most evident in rule 110. Table 15 gives some properties of the particle-like structures which are found in this rule. One suspects that with appropriate combinations of these structures, it should be possible to perform universal computation.

The final table shows patterns produced by reversible generalizations of the standard $k = 2$, $r = 1$ cellular automata. Qualitatively similar behaviour is again seen.

It is remarkable that with such simple construction, the $k = 2$, $r = 1$ cellular automata can show such complex behaviour. The tables in this appendix give some first attempts at characterizing and quantifying this behaviour. Much, however, still remains to be done.

# Table 1: Rule Forms and Equivalences

| rule number | | | boolean expression | dep | equivalent rules | | | min |
|---|---|---|---|---|---|---|---|---|
| dec | binary | hex | | | conj | refl | c.r. | |
| 0 | 00000000 | 00 | $0$ | −−− | 255 | 0 | 255 | 0 |
| 1 | 00000001 | 01 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)$ | ●●● | 127 | 1 | 127 | 1 |
| 2 | 00000010 | 02 | $(\bar{a}_{-1}\bar{a}_0 a_1)$ | ●●● | 191 | 16 | 247 | 2 |
| 3 | 00000011 | 03 | $(\bar{a}_{-1}\bar{a}_0)$ | ●●− | 63 | 17 | 119 | 3 |
| 4 | 00000100 | 04 | $(\bar{a}_{-1}a_0\bar{a}_1)$ | ●●● | 223 | 4 | 223 | 4 |
| 5 | 00000101 | 05 | $(\bar{a}_{-1}\bar{a}_1)$ | ●−● | 95 | 5 | 95 | 5 |
| 6 | 00000110 | 06 | $(\bar{a}_{-1}a_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0 a_1)$ | ●●● | 159 | 20 | 215 | 6 |
| 7 | 00000111 | 07 | $(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)$ | ●●● | 31 | 21 | 87 | 7 |
| 8 | 00001000 | 08 | $(\bar{a}_{-1}a_0 a_1)$ | ●●● | 239 | 64 | 253 | 8 |
| 9 | 00001001 | 09 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0 a_1)$ | ●●● | 111 | 65 | 125 | 9 |
| 10 | 00001010 | 0a | $(\bar{a}_{-1}a_1)$ | ●−● | 175 | 80 | 245 | 10 |
| 11 | 00001011 | 0b | $(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_{-1}a_1)$ | ●●● | 47 | 81 | 117 | 11 |
| 12 | 00001100 | 0c | $(\bar{a}_{-1}a_0)$ | ●●− | 207 | 68 | 221 | 12 |
| 13 | 00001101 | 0d | $(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}a_0)$ | ●●● | 79 | 69 | 93 | 13 |
| 14 | 00001110 | 0e | $(\bar{a}_{-1}a_0)+(\bar{a}_{-1}a_1)$ | ●●● | 143 | 84 | 213 | 14 |
| 15 | 00001111 | 0f | $(\bar{a}_{-1})$ | ○−− | 15 | 85 | 85 | 15 |
| 16 | 00010000 | 10 | $(a_{-1}\bar{a}_0\bar{a}_1)$ | ●●● | 247 | 2 | 191 | 2 |
| 17 | 00010001 | 11 | $(\bar{a}_0\bar{a}_1)$ | −●● | 119 | 3 | 63 | 3 |
| 18 | 00010010 | 12 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0 a_1)$ | ●●● | 183 | 18 | 183 | 18 |
| 19 | 00010011 | 13 | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)$ | ●●● | 55 | 19 | 55 | 19 |
| 20 | 00010100 | 14 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0\bar{a}_1)$ | ●●● | 215 | 6 | 159 | 6 |
| 21 | 00010101 | 15 | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_1)$ | ●●● | 87 | 7 | 31 | 7 |
| 22 | 00010110 | 16 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0 a_1)$ | ●●● | 151 | 22 | 151 | 22 |
| 23 | 00010111 | 17 | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)$ | ●●● | 23 | 23 | 23 | 23 |
| 24 | 00011000 | 18 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0 a_1)$ | ●●● | 231 | 66 | 189 | 24 |
| 25 | 00011001 | 19 | $(\bar{a}_{-1}a_0 a_1)+(\bar{a}_0\bar{a}_1)$ | ●●● | 103 | 67 | 61 | 25 |
| 26 | 00011010 | 1a | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_1)$ | ●●● | 167 | 82 | 181 | 26 |
| 27 | 00011011 | 1b | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_1)$ | ●●● | 39 | 83 | 53 | 27 |
| 28 | 00011100 | 1c | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0)$ | ●●● | 199 | 70 | 157 | 28 |
| 29 | 00011101 | 1d | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0)$ | ●●● | 71 | 71 | 29 | 29 |
| 30 | 00011110 | 1e | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0)+(\bar{a}_{-1}a_1)$ | ○●● | 135 | 86 | 149 | 30 |
| 31 | 00011111 | 1f | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1})$ | ●●● | 7 | 87 | 21 | 7 |
| 32 | 00100000 | 20 | $(a_{-1}\bar{a}_0 a_1)$ | ●●● | 251 | 32 | 251 | 32 |
| 33 | 00100001 | 21 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}\bar{a}_0 a_1)$ | ●●● | 123 | 33 | 123 | 33 |
| 34 | 00100010 | 22 | $(\bar{a}_0 a_1)$ | −●● | 187 | 48 | 243 | 34 |
| 35 | 00100011 | 23 | $(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_0 a_1)$ | ●●● | 59 | 49 | 115 | 35 |
| 36 | 00100100 | 24 | $(\bar{a}_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0 a_1)$ | ●●● | 219 | 36 | 219 | 36 |
| 37 | 00100101 | 25 | $(a_{-1}\bar{a}_0 a_1)+(\bar{a}_{-1}\bar{a}_1)$ | ●●● | 91 | 37 | 91 | 37 |
| 38 | 00100110 | 26 | $(\bar{a}_{-1}a_0\bar{a}_1)+(\bar{a}_0 a_1)$ | ●●● | 155 | 52 | 211 | 38 |
| 39 | 00100111 | 27 | $(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_0 a_1)$ | ●●● | 27 | 53 | 83 | 27 |
| 40 | 00101000 | 28 | $(a_{-1}\bar{a}_0 a_1)+(\bar{a}_{-1}a_0 a_1)$ | ●●● | 235 | 96 | 249 | 40 |
| 41 | 00101001 | 29 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}\bar{a}_0 a_1)+(\bar{a}_{-1}a_0 a_1)$ | ●●● | 107 | 97 | 121 | 41 |
| 42 | 00101010 | 2a | $(\bar{a}_0 a_1)+(\bar{a}_{-1}a_1)$ | ●●● | 171 | 112 | 241 | 42 |
| 43 | 00101011 | 2b | $(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_0 a_1)+(\bar{a}_{-1}a_1)$ | ●●● | 43 | 113 | 113 | 43 |
| 44 | 00101100 | 2c | $(a_{-1}\bar{a}_0 a_1)+(\bar{a}_{-1}a_0)$ | ●●● | 203 | 100 | 217 | 44 |
| 45 | 00101101 | 2d | $(a_{-1}\bar{a}_0 a_1)+(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}a_0)$ | ○●● | 75 | 101 | 89 | 45 |
| 46 | 00101110 | 2e | $(\bar{a}_{-1}a_0)+(\bar{a}_0 a_1)$ | ●●● | 139 | 116 | 209 | 46 |

| rule number | | | boolean expression | dep | equivalent rules | | | min |
|---|---|---|---|---|---|---|---|---|
| dec | binary | hex | | | conj | refl | c.r. | |
| 47 | 00101111 | 2f | $(\bar{a}_0 a_1) + (\bar{a}_{-1})$ | ●●● | 11 | 117 | 81 | 11 |
| 48 | 00110000 | 30 | $(a_{-1}\bar{a}_0)$ | ●●− | 243 | 34 | 187 | 34 |
| 49 | 00110001 | 31 | $(\bar{a}_0\bar{a}_1) + (a_{-1}\bar{a}_0)$ | ●●● | 115 | 35 | 59 | 35 |
| 50 | 00110010 | 32 | $(a_{-1}\bar{a}_0) + (\bar{a}_0 a_1)$ | ●●● | 179 | 50 | 179 | 50 |
| 51 | 00110011 | 33 | $(\bar{a}_0)$ | −○− | 51 | 51 | 51 | 51 |
| 52 | 00110100 | 34 | $(\bar{a}_{-1}a_0\bar{a}_1) + (a_{-1}\bar{a}_0)$ | ●●● | 211 | 38 | 155 | 38 |
| 53 | 00110101 | 35 | $(\bar{a}_{-1}\bar{a}_1) + (a_{-1}\bar{a}_0)$ | ●●● | 83 | 39 | 27 | 27 |
| 54 | 00110110 | 36 | $(\bar{a}_{-1}a_0\bar{a}_1) + (a_{-1}\bar{a}_0) + (\bar{a}_0 a_1)$ | ●○● | 147 | 54 | 147 | 54 |
| 55 | 00110111 | 37 | $(\bar{a}_{-1}\bar{a}_1) + (\bar{a}_0)$ | ●●● | 19 | 55 | 19 | 19 |
| 56 | 00111000 | 38 | $(\bar{a}_{-1}a_0 a_1) + (a_{-1}\bar{a}_0)$ | ●●● | 227 | 98 | 185 | 56 |
| 57 | 00111001 | 39 | $(\bar{a}_{-1}a_0 a_1) + (\bar{a}_0\bar{a}_1) + (a_{-1}\bar{a}_0)$ | ●○● | 99 | 99 | 57 | 57 |
| 58 | 00111010 | 3a | $(a_{-1}\bar{a}_0) + (\bar{a}_{-1}a_1)$ | ●●● | 163 | 114 | 177 | 58 |
| 59 | 00111011 | 3b | $(\bar{a}_{-1}a_1) + (\bar{a}_0)$ | ●●● | 35 | 115 | 49 | 35 |
| 60 | 00111100 | 3c | $(a_{-1}\bar{a}_0) + (\bar{a}_{-1}a_0)$ | ○○− | 195 | 102 | 153 | 60 |
| 61 | 00111101 | 3d | $(\bar{a}_{-1}\bar{a}_1) + (a_{-1}\bar{a}_0) + (\bar{a}_{-1}a_0)$ | ●●● | 67 | 103 | 25 | 25 |
| 62 | 00111110 | 3e | $(\bar{a}_{-1}a_1) + (a_{-1}\bar{a}_0) + (\bar{a}_{-1}a_0)$ | ●●● | 131 | 118 | 145 | 62 |
| 63 | 00111111 | 3f | $(\bar{a}_0) + (\bar{a}_{-1})$ | ●●− | 3 | 119 | 17 | 3 |
| 64 | 01000000 | 40 | $(a_{-1}a_0\bar{a}_1)$ | ●●● | 253 | 8 | 239 | 8 |
| 65 | 01000001 | 41 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1) + (a_{-1}a_0\bar{a}_1)$ | ●●● | 125 | 9 | 111 | 9 |
| 66 | 01000010 | 42 | $(a_{-1}a_0\bar{a}_1) + (\bar{a}_{-1}\bar{a}_0 a_1)$ | ●●● | 189 | 24 | 231 | 24 |
| 67 | 01000011 | 43 | $(a_{-1}a_0\bar{a}_1) + (\bar{a}_{-1}\bar{a}_0)$ | ●●● | 61 | 25 | 103 | 25 |
| 68 | 01000100 | 44 | $(a_0\bar{a}_1)$ | −●● | 221 | 12 | 207 | 12 |
| 69 | 01000101 | 45 | $(\bar{a}_{-1}\bar{a}_1) + (a_0\bar{a}_1)$ | ●●● | 93 | 13 | 79 | 13 |
| 70 | 01000110 | 46 | $(\bar{a}_{-1}\bar{a}_0 a_1) + (a_0\bar{a}_1)$ | ●●● | 157 | 28 | 199 | 28 |
| 71 | 01000111 | 47 | $(a_0\bar{a}_1) + (\bar{a}_{-1}\bar{a}_0)$ | ●●● | 29 | 29 | 71 | 29 |
| 72 | 01001000 | 48 | $(a_{-1}a_0\bar{a}_1) + (\bar{a}_{-1}a_0 a_1)$ | ●●● | 237 | 72 | 237 | 72 |
| 73 | 01001001 | 49 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1) + (a_{-1}a_0\bar{a}_1) + (\bar{a}_{-1}a_0 a_1)$ | ●●● | 109 | 73 | 109 | 73 |
| 74 | 01001010 | 4a | $(a_{-1}a_0\bar{a}_1) + (\bar{a}_{-1}a_1)$ | ●●● | 173 | 88 | 229 | 74 |
| 75 | 01001011 | 4b | $(a_{-1}a_0\bar{a}_1) + (\bar{a}_{-1}\bar{a}_0) + (\bar{a}_{-1}a_1)$ | ○●● | 45 | 89 | 101 | 45 |
| 76 | 01001100 | 4c | $(a_0\bar{a}_1) + (\bar{a}_{-1}a_0)$ | ●●● | 205 | 76 | 205 | 76 |
| 77 | 01001101 | 4d | $(\bar{a}_{-1}\bar{a}_1) + (a_0\bar{a}_1) + (\bar{a}_{-1}a_0)$ | ●●● | 77 | 77 | 77 | 77 |
| 78 | 01001110 | 4e | $(a_0\bar{a}_1) + (\bar{a}_{-1}a_1)$ | ●●● | 141 | 92 | 197 | 78 |
| 79 | 01001111 | 4f | $(a_0\bar{a}_1) + (\bar{a}_{-1})$ | ●●● | 13 | 93 | 69 | 13 |
| 80 | 01010000 | 50 | $(a_{-1}\bar{a}_1)$ | ●−● | 245 | 10 | 175 | 10 |
| 81 | 01010001 | 51 | $(\bar{a}_0\bar{a}_1) + (a_{-1}\bar{a}_1)$ | ●●● | 117 | 11 | 47 | 11 |
| 82 | 01010010 | 52 | $(\bar{a}_{-1}\bar{a}_0 a_1) + (a_{-1}\bar{a}_1)$ | ●●● | 181 | 26 | 167 | 26 |
| 83 | 01010011 | 53 | $(a_{-1}\bar{a}_1) + (\bar{a}_{-1}\bar{a}_0)$ | ●●● | 53 | 27 | 39 | 27 |
| 84 | 01010100 | 54 | $(a_{-1}\bar{a}_1) + (a_0\bar{a}_1)$ | ●●● | 213 | 14 | 143 | 14 |
| 85 | 01010101 | 55 | $(\bar{a}_1)$ | −−○ | 85 | 15 | 15 | 15 |
| 86 | 01010110 | 56 | $(\bar{a}_{-1}\bar{a}_0 a_1) + (a_{-1}\bar{a}_1) + (a_0\bar{a}_1)$ | ●●○ | 149 | 30 | 135 | 30 |
| 87 | 01010111 | 57 | $(\bar{a}_{-1}\bar{a}_0) + (\bar{a}_1)$ | ●●● | 21 | 31 | 7 | 7 |
| 88 | 01011000 | 58 | $(\bar{a}_{-1}a_0 a_1) + (a_{-1}\bar{a}_1)$ | ●●● | 229 | 74 | 173 | 74 |
| 89 | 01011001 | 59 | $(\bar{a}_{-1}a_0 a_1) + (\bar{a}_0\bar{a}_1) + (a_{-1}\bar{a}_1)$ | ●●○ | 101 | 75 | 45 | 45 |
| 90 | 01011010 | 5a | $(a_{-1}\bar{a}_1) + (\bar{a}_{-1}a_1)$ | ○−○ | 165 | 90 | 165 | 90 |
| 91 | 01011011 | 5b | $(\bar{a}_{-1}a_0) + (a_{-1}\bar{a}_1) + (\bar{a}_{-1}a_1)$ | ●●● | 37 | 91 | 37 | 37 |
| 92 | 01011100 | 5c | $(a_{-1}\bar{a}_1) + (\bar{a}_{-1}a_0)$ | ●●● | 197 | 78 | 141 | 78 |
| 93 | 01011101 | 5d | $(\bar{a}_{-1}a_0) + (\bar{a}_1)$ | ●●● | 69 | 79 | 13 | 13 |
| 94 | 01011110 | 5e | $(\bar{a}_{-1}a_0) + (a_{-1}\bar{a}_1) + (\bar{a}_{-1}a_1)$ | ●●● | 133 | 94 | 133 | 94 |
| 95 | 01011111 | 5f | $(\bar{a}_1) + (\bar{a}_{-1})$ | ●−● | 5 | 95 | 5 | 5 |

| rule number | | | boolean expression | dep | equivalent rules | | | min |
|---|---|---|---|---|---|---|---|---|
| dec | binary | hex | | | conj | refl | c.r. | |
| 96 | 01100000 | 60 | $(a_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0a_1)$ | ●●● | 249 | 40 | 235 | 40 |
| 97 | 01100001 | 61 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0a_1)$ | ●●● | 121 | 41 | 107 | 41 |
| 98 | 01100010 | 62 | $(a_{-1}a_0\bar{a}_1)+(\bar{a}_0a_1)$ | ●●● | 185 | 56 | 227 | 56 |
| 99 | 01100011 | 63 | $(a_{-1}a_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_0a_1)$ | ●○● | 57 | 57 | 99 | 57 |
| 100 | 01100100 | 64 | $(a_{-1}\bar{a}_0a_1)+(a_0\bar{a}_1)$ | ●●● | 217 | 44 | 203 | 44 |
| 101 | 01100101 | 65 | $(a_{-1}\bar{a}_0a_1)+(\bar{a}_{-1}\bar{a}_1)+(a_0\bar{a}_1)$ | ●●○ | 89 | 45 | 75 | 45 |
| 102 | 01100110 | 66 | $(a_0\bar{a}_1)+(\bar{a}_0a_1)$ | −○○ | 153 | 60 | 195 | 60 |
| 103 | 01100111 | 67 | $(\bar{a}_{-1}\bar{a}_0)+(a_0\bar{a}_1)+(\bar{a}_0a_1)$ | ●●● | 25 | 61 | 67 | 25 |
| 104 | 01101000 | 68 | $(a_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0a_1)+(\bar{a}_{-1}a_0a_1)$ | ●●● | 233 | 104 | 233 | 104 |
| 105 | 01101001 | 69 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0a_1)+(\bar{a}_{-1}a_0a_1)$ | ○○○ | 105 | 105 | 105 | 105 |
| 106 | 01101010 | 6a | $(a_{-1}a_0\bar{a}_1)+(\bar{a}_0a_1)+(\bar{a}_{-1}a_1)$ | ●●○ | 169 | 120 | 225 | 106 |
| 107 | 01101011 | 6b | $(a_{-1}a_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_0a_1)+(\bar{a}_{-1}a_1)$ | ●●● | 41 | 121 | 97 | 41 |
| 108 | 01101100 | 6c | $(a_{-1}\bar{a}_0a_1)+(a_0\bar{a}_1)+(\bar{a}_{-1}a_0)$ | ●○● | 201 | 108 | 201 | 108 |
| 109 | 01101101 | 6d | $(a_{-1}\bar{a}_0a_1)+(\bar{a}_{-1}\bar{a}_1)+(a_0\bar{a}_1)+(\bar{a}_{-1}a_0)$ | ●●● | 73 | 109 | 73 | 73 |
| 110 | 01101110 | 6e | $(\bar{a}_{-1}a_0)+(a_0\bar{a}_1)+(\bar{a}_0a_1)$ | ●●● | 137 | 124 | 193 | 110 |
| 111 | 01101111 | 6f | $(a_0\bar{a}_1)+(\bar{a}_0a_1)+(\bar{a}_{-1})$ | ●●● | 9 | 125 | 65 | 9 |
| 112 | 01110000 | 70 | $(a_{-1}\bar{a}_1)+(a_{-1}\bar{a}_0)$ | ●●● | 241 | 42 | 171 | 42 |
| 113 | 01110001 | 71 | $(\bar{a}_0\bar{a}_1)+(a_{-1}\bar{a}_1)+(a_{-1}\bar{a}_0)$ | ●●● | 113 | 43 | 43 | 43 |
| 114 | 01110010 | 72 | $(a_{-1}\bar{a}_1)+(\bar{a}_0a_1)$ | ●●● | 177 | 58 | 163 | 58 |
| 115 | 01110011 | 73 | $(a_{-1}\bar{a}_1)+(\bar{a}_0)$ | ●●● | 49 | 59 | 35 | 35 |
| 116 | 01110100 | 74 | $(a_0\bar{a}_1)+(a_{-1}\bar{a}_0)$ | ●●● | 209 | 46 | 139 | 46 |
| 117 | 01110101 | 75 | $(a_{-1}\bar{a}_0)+(\bar{a}_1)$ | ●●● | 81 | 47 | 11 | 11 |
| 118 | 01110110 | 76 | $(a_{-1}\bar{a}_0)+(a_0\bar{a}_1)+(\bar{a}_0a_1)$ | ●●● | 145 | 62 | 131 | 62 |
| 119 | 01110111 | 77 | $(\bar{a}_1)+(\bar{a}_0)$ | −●● | 17 | 63 | 3 | 3 |
| 120 | 01111000 | 78 | $(\bar{a}_{-1}a_0a_1)+(a_{-1}\bar{a}_1)+(a_{-1}\bar{a}_0)$ | ○●● | 225 | 106 | 169 | 106 |
| 121 | 01111001 | 79 | $(\bar{a}_{-1}a_0a_1)+(\bar{a}_0\bar{a}_1)+(a_{-1}\bar{a}_1)+(a_{-1}\bar{a}_0)$ | ●●● | 97 | 107 | 41 | 41 |
| 122 | 01111010 | 7a | $(a_{-1}\bar{a}_0)+(a_{-1}\bar{a}_1)+(\bar{a}_{-1}a_1)$ | ●●● | 161 | 122 | 161 | 122 |
| 123 | 01111011 | 7b | $(a_{-1}\bar{a}_1)+(\bar{a}_{-1}a_1)+(\bar{a}_0)$ | ●●● | 33 | 123 | 33 | 33 |
| 124 | 01111100 | 7c | $(a_{-1}\bar{a}_1)+(a_{-1}\bar{a}_0)+(\bar{a}_{-1}a_0)$ | ●●● | 193 | 110 | 137 | 110 |
| 125 | 01111101 | 7d | $(a_{-1}\bar{a}_0)+(\bar{a}_{-1}a_0)+(\bar{a}_1)$ | ●●● | 65 | 111 | 9 | 9 |
| 126 | 01111110 | 7e | $(a_{-1}\bar{a}_1)+(\bar{a}_0a_1)+(\bar{a}_{-1}a_0)$ | ●●● | 129 | 126 | 129 | 126 |
| 127 | 01111111 | 7f | $(\bar{a}_1)+(\bar{a}_0)+(\bar{a}_{-1})$ | ●●● | 1 | 127 | 1 | 1 |
| 128 | 10000000 | 80 | $(a_{-1}a_0a_1)$ | ●●● | 254 | 128 | 254 | 128 |
| 129 | 10000001 | 81 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0a_1)$ | ●●● | 126 | 129 | 126 | 126 |
| 130 | 10000010 | 82 | $(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}a_0a_1)$ | ●●● | 190 | 144 | 246 | 130 |
| 131 | 10000011 | 83 | $(a_{-1}a_0a_1)+(\bar{a}_{-1}\bar{a}_0)$ | ●●● | 62 | 145 | 118 | 62 |
| 132 | 10000100 | 84 | $(\bar{a}_{-1}a_0\bar{a}_1)+(a_{-1}a_0a_1)$ | ●●● | 222 | 132 | 222 | 132 |
| 133 | 10000101 | 85 | $(a_{-1}a_0a_1)+(\bar{a}_{-1}\bar{a}_1)$ | ●●● | 94 | 133 | 94 | 94 |
| 134 | 10000110 | 86 | $(\bar{a}_{-1}a_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}a_0a_1)$ | ●●● | 158 | 148 | 214 | 134 |
| 135 | 10000111 | 87 | $(a_{-1}a_0a_1)+(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)$ | ○●● | 30 | 149 | 86 | 30 |
| 136 | 10001000 | 88 | $(a_0a_1)$ | −●● | 238 | 192 | 252 | 136 |
| 137 | 10001001 | 89 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_0a_1)$ | ●●● | 110 | 193 | 124 | 110 |
| 138 | 10001010 | 8a | $(\bar{a}_{-1}a_1)+(a_0a_1)$ | ●●● | 174 | 208 | 244 | 138 |
| 139 | 10001011 | 8b | $(\bar{a}_{-1}\bar{a}_0)+(a_0a_1)$ | ●●● | 46 | 209 | 116 | 46 |
| 140 | 10001100 | 8c | $(\bar{a}_{-1}a_0)+(a_0a_1)$ | ●●● | 206 | 196 | 220 | 140 |
| 141 | 10001101 | 8d | $(\bar{a}_{-1}\bar{a}_1)+(a_0a_1)$ | ●●● | 78 | 197 | 92 | 78 |
| 142 | 10001110 | 8e | $(\bar{a}_{-1}a_0)+(\bar{a}_{-1}a_1)+(a_0a_1)$ | ●●● | 142 | 212 | 212 | 142 |
| 143 | 10001111 | 8f | $(a_0a_1)+(\bar{a}_{-1})$ | ●●● | 14 | 213 | 84 | 14 |
| 144 | 10010000 | 90 | $(a_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0a_1)$ | ●●● | 246 | 130 | 190 | 130 |
| 145 | 10010001 | 91 | $(a_{-1}a_0a_1)+(\bar{a}_0\bar{a}_1)$ | ●●● | 118 | 131 | 62 | 62 |

| rule number | | | boolean expression | dep | equivalent rules | | | min |
|---|---|---|---|---|---|---|---|---|
| dec | binary | hex | | | conj | refl | c.r. | |
| 146 | 10010010 | 92 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}a_0a_1)$ | ●●● | 182 | 146 | 182 | 146 |
| 147 | 10010011 | 93 | $(a_{-1}a_0a_1)+(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)$ | ●○● | 54 | 147 | 54 | 54 |
| 148 | 10010100 | 94 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0\bar{a}_1)+(a_{-1}a_0a_1)$ | ●●● | 214 | 134 | 158 | 134 |
| 149 | 10010101 | 95 | $(a_{-1}a_0a_1)+(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_1)$ | ●●○ | 86 | 135 | 30 | 30 |
| 150 | 10010110 | 96 | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}a_0a_1)$ | ○○○ | 150 | 150 | 150 | 150 |
| 151 | 10010111 | 97 | $(a_{-1}a_0a_1)+(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)$ | ●●● | 22 | 151 | 22 | 22 |
| 152 | 10011000 | 98 | $(a_{-1}\bar{a}_0\bar{a}_1)+(a_0a_1)$ | ●●● | 230 | 194 | 188 | 152 |
| 153 | 10011001 | 99 | $(\bar{a}_0\bar{a}_1)+(a_0a_1)$ | −○○ | 102 | 195 | 60 | 60 |
| 154 | 10011010 | 9a | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_1)+(a_0a_1)$ | ●●○ | 166 | 210 | 180 | 154 |
| 155 | 10011011 | 9b | $(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_0\bar{a}_1)+(a_0a_1)$ | ●●● | 38 | 211 | 52 | 38 |
| 156 | 10011100 | 9c | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0)+(a_0a_1)$ | ●○● | 198 | 198 | 156 | 156 |
| 157 | 10011101 | 9d | $(\bar{a}_{-1}a_0)+(\bar{a}_0\bar{a}_1)+(a_0a_1)$ | ●●● | 70 | 199 | 28 | 28 |
| 158 | 10011110 | 9e | $(a_{-1}\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_0)+(\bar{a}_{-1}a_1)+(a_0a_1)$ | ●●● | 134 | 214 | 148 | 134 |
| 159 | 10011111 | 9f | $(\bar{a}_0\bar{a}_1)+(a_0a_1)+(\bar{a}_{-1})$ | ●●● | 6 | 215 | 20 | 6 |
| 160 | 10100000 | a0 | $(a_{-1}a_1)$ | ●−● | 250 | 160 | 250 | 160 |
| 161 | 10100001 | a1 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 122 | 161 | 122 | 122 |
| 162 | 10100010 | a2 | $(\bar{a}_0a_1)+(a_{-1}a_1)$ | ●●● | 186 | 176 | 242 | 162 |
| 163 | 10100011 | a3 | $(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_1)$ | ●●● | 58 | 177 | 114 | 58 |
| 164 | 10100100 | a4 | $(\bar{a}_{-1}a_0\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 218 | 164 | 218 | 164 |
| 165 | 10100101 | a5 | $(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)$ | ○−○ | 90 | 165 | 90 | 90 |
| 166 | 10100110 | a6 | $(\bar{a}_{-1}a_0\bar{a}_1)+(\bar{a}_0a_1)+(a_{-1}a_1)$ | ●●○ | 154 | 180 | 210 | 154 |
| 167 | 10100111 | a7 | $(\bar{a}_{-1}\bar{a}_0)+(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 26 | 181 | 82 | 26 |
| 168 | 10101000 | a8 | $(a_{-1}a_1)+(a_0a_1)$ | ●●● | 234 | 224 | 248 | 168 |
| 169 | 10101001 | a9 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_1)+(a_0a_1)$ | ●●○ | 106 | 225 | 120 | 106 |
| 170 | 10101010 | aa | $(a_1)$ | −−○ | 170 | 240 | 240 | 170 |
| 171 | 10101011 | ab | $(\bar{a}_{-1}\bar{a}_0)+(a_1)$ | ●●● | 42 | 241 | 112 | 42 |
| 172 | 10101100 | ac | $(\bar{a}_{-1}a_0)+(a_{-1}a_1)$ | ●●● | 202 | 228 | 216 | 172 |
| 173 | 10101101 | ad | $(\bar{a}_{-1}a_0)+(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 74 | 229 | 88 | 74 |
| 174 | 10101110 | ae | $(\bar{a}_{-1}a_0)+(a_1)$ | ●●● | 138 | 244 | 208 | 138 |
| 175 | 10101111 | af | $(\bar{a}_{-1})+(a_1)$ | ●−● | 10 | 245 | 80 | 10 |
| 176 | 10110000 | b0 | $(a_{-1}\bar{a}_0)+(a_{-1}a_1)$ | ●●● | 242 | 162 | 186 | 162 |
| 177 | 10110001 | b1 | $(\bar{a}_0\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 114 | 163 | 58 | 58 |
| 178 | 10110010 | b2 | $(a_{-1}\bar{a}_0)+(\bar{a}_0a_1)+(a_{-1}a_1)$ | ●●● | 178 | 178 | 178 | 178 |
| 179 | 10110011 | b3 | $(a_{-1}a_1)+(\bar{a}_0)$ | ●●● | 50 | 179 | 50 | 50 |
| 180 | 10110100 | b4 | $(\bar{a}_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0)+(a_{-1}a_1)$ | ○●● | 210 | 166 | 154 | 154 |
| 181 | 10110101 | b5 | $(a_{-1}\bar{a}_0)+(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 82 | 167 | 26 | 26 |
| 182 | 10110110 | b6 | $(\bar{a}_{-1}a_0\bar{a}_1)+(a_{-1}\bar{a}_0)+(\bar{a}_0a_1)+(a_{-1}a_1)$ | ●●● | 146 | 182 | 146 | 146 |
| 183 | 10110111 | b7 | $(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)+(\bar{a}_0)$ | ●●● | 18 | 183 | 18 | 18 |
| 184 | 10111000 | b8 | $(a_{-1}\bar{a}_0)+(a_0a_1)$ | ●●● | 226 | 226 | 184 | 184 |
| 185 | 10111001 | b9 | $(a_{-1}\bar{a}_0)+(\bar{a}_0\bar{a}_1)+(a_0a_1)$ | ●●● | 98 | 227 | 56 | 56 |
| 186 | 10111010 | ba | $(a_{-1}\bar{a}_0)+(a_1)$ | ●●● | 162 | 242 | 176 | 162 |
| 187 | 10111011 | bb | $(\bar{a}_0)+(a_1)$ | −●● | 34 | 243 | 48 | 34 |
| 188 | 10111100 | bc | $(a_{-1}a_1)+(a_{-1}\bar{a}_0)+(\bar{a}_{-1}a_0)$ | ●●● | 194 | 230 | 152 | 152 |
| 189 | 10111101 | bd | $(\bar{a}_0\bar{a}_1)+(a_{-1}a_1)+(\bar{a}_{-1}a_0)$ | ●●● | 66 | 231 | 24 | 24 |
| 190 | 10111110 | be | $(a_{-1}\bar{a}_0)+(\bar{a}_{-1}a_0)+(a_1)$ | ●●● | 130 | 246 | 144 | 130 |
| 191 | 10111111 | bf | $(\bar{a}_0)+(\bar{a}_{-1})+(a_1)$ | ●●● | 2 | 247 | 16 | 2 |
| 192 | 11000000 | c0 | $(a_{-1}a_0)$ | ●●− | 252 | 136 | 238 | 136 |
| 193 | 11000001 | c1 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 124 | 137 | 110 | 110 |
| 194 | 11000010 | c2 | $(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}a_0)$ | ●●● | 188 | 152 | 230 | 152 |
| 195 | 11000011 | c3 | $(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)$ | ○○− | 60 | 153 | 102 | 60 |

| rule number | | | boolean expression | dep | equivalent rules | | | min |
|---|---|---|---|---|---|---|---|---|
| dec | binary | hex | | | conj | refl | c.r. | |
| 196 | 11000100 | c4 | $(a_0\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 220 | 140 | 206 | 140 |
| 197 | 11000101 | c5 | $(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 92 | 141 | 78 | 78 |
| 198 | 11000110 | c6 | $(\bar{a}_{-1}\bar{a}_0a_1)+(a_0\bar{a}_1)+(a_{-1}a_0)$ | ●○● | 156 | 156 | 198 | 156 |
| 199 | 11000111 | c7 | $(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)$ | ●●● | 28 | 157 | 70 | 28 |
| 200 | 11001000 | c8 | $(a_{-1}a_0)+(a_0a_1)$ | ●●● | 236 | 200 | 236 | 200 |
| 201 | 11001001 | c9 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0)+(a_0a_1)$ | ●○● | 108 | 201 | 108 | 108 |
| 202 | 11001010 | ca | $(a_{-1}a_0)+(\bar{a}_{-1}a_1)$ | ●●● | 172 | 216 | 228 | 172 |
| 203 | 11001011 | cb | $(\bar{a}_{-1}a_1)+(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)$ | ●●● | 44 | 217 | 100 | 44 |
| 204 | 11001100 | cc | $(a_0)$ | −○− | 204 | 204 | 204 | 204 |
| 205 | 11001101 | cd | $(\bar{a}_{-1}\bar{a}_1)+(a_0)$ | ●●● | 76 | 205 | 76 | 76 |
| 206 | 11001110 | ce | $(\bar{a}_{-1}a_1)+(a_0)$ | ●●● | 140 | 220 | 196 | 140 |
| 207 | 11001111 | cf | $(\bar{a}_{-1})+(a_0)$ | ●●− | 12 | 221 | 68 | 12 |
| 208 | 11010000 | d0 | $(a_{-1}\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 244 | 138 | 174 | 138 |
| 209 | 11010001 | d1 | $(\bar{a}_0\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 116 | 139 | 46 | 46 |
| 210 | 11010010 | d2 | $(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}\bar{a}_1)+(a_{-1}a_0)$ | ○●● | 180 | 154 | 166 | 154 |
| 211 | 11010011 | d3 | $(a_{-1}\bar{a}_1)+(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)$ | ●●● | 52 | 155 | 38 | 38 |
| 212 | 11010100 | d4 | $(a_{-1}\bar{a}_1)+(a_0\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 212 | 142 | 142 | 142 |
| 213 | 11010101 | d5 | $(a_{-1}a_0)+(\bar{a}_1)$ | ●●● | 84 | 143 | 14 | 14 |
| 214 | 11010110 | d6 | $(\bar{a}_{-1}\bar{a}_0a_1)+(a_{-1}\bar{a}_1)+(a_0\bar{a}_1)+(a_{-1}a_0)$ | ●●● | 148 | 158 | 134 | 134 |
| 215 | 11010111 | d7 | $(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)+(\bar{a}_1)$ | ●●● | 20 | 159 | 6 | 6 |
| 216 | 11011000 | d8 | $(a_{-1}\bar{a}_1)+(a_0a_1)$ | ●●● | 228 | 202 | 172 | 172 |
| 217 | 11011001 | d9 | $(a_{-1}a_0)+(\bar{a}_0\bar{a}_1)+(a_0a_1)$ | ●●● | 100 | 203 | 44 | 44 |
| 218 | 11011010 | da | $(a_{-1}a_0)+(a_{-1}\bar{a}_1)+(\bar{a}_{-1}a_1)$ | ●●● | 164 | 218 | 164 | 164 |
| 219 | 11011011 | db | $(\bar{a}_0\bar{a}_1)+(\bar{a}_{-1}a_1)+(a_{-1}a_0)$ | ●●● | 36 | 219 | 36 | 36 |
| 220 | 11011100 | dc | $(a_{-1}\bar{a}_1)+(a_0)$ | ●●● | 196 | 206 | 140 | 140 |
| 221 | 11011101 | dd | $(\bar{a}_1)+(a_0)$ | −●● | 68 | 207 | 12 | 12 |
| 222 | 11011110 | de | $(a_{-1}\bar{a}_1)+(\bar{a}_{-1}a_1)+(a_0)$ | ●●● | 132 | 222 | 132 | 132 |
| 223 | 11011111 | df | $(\bar{a}_1)+(\bar{a}_{-1})+(a_0)$ | ●●● | 4 | 223 | 4 | 4 |
| 224 | 11100000 | e0 | $(a_{-1}a_0)+(a_{-1}a_1)$ | ●●● | 248 | 168 | 234 | 168 |
| 225 | 11100001 | e1 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0)+(a_{-1}a_1)$ | ○●● | 120 | 169 | 106 | 106 |
| 226 | 11100010 | e2 | $(a_{-1}a_0)+(\bar{a}_0a_1)$ | ●●● | 184 | 184 | 226 | 184 |
| 227 | 11100011 | e3 | $(a_{-1}a_1)+(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)$ | ●●● | 56 | 185 | 98 | 56 |
| 228 | 11100100 | e4 | $(a_0\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 216 | 172 | 202 | 172 |
| 229 | 11100101 | e5 | $(a_{-1}a_0)+(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)$ | ●●● | 88 | 173 | 74 | 74 |
| 230 | 11100110 | e6 | $(a_{-1}a_0)+(a_0\bar{a}_1)+(\bar{a}_0a_1)$ | ●●● | 152 | 188 | 194 | 152 |
| 231 | 11100111 | e7 | $(\bar{a}_{-1}\bar{a}_1)+(\bar{a}_0a_1)+(a_{-1}a_0)$ | ●●● | 24 | 189 | 66 | 24 |
| 232 | 11101000 | e8 | $(a_{-1}a_0)+(a_{-1}a_1)+(a_0a_1)$ | ●●● | 232 | 232 | 232 | 232 |
| 233 | 11101001 | e9 | $(\bar{a}_{-1}\bar{a}_0\bar{a}_1)+(a_{-1}a_0)+(a_{-1}a_1)+(a_0a_1)$ | ●●● | 104 | 233 | 104 | 104 |
| 234 | 11101010 | ea | $(a_{-1}a_0)+(a_1)$ | ●●● | 168 | 248 | 224 | 168 |
| 235 | 11101011 | eb | $(\bar{a}_{-1}\bar{a}_0)+(a_{-1}a_0)+(a_1)$ | ●●● | 40 | 249 | 96 | 40 |
| 236 | 11101100 | ec | $(a_{-1}a_1)+(a_0)$ | ●●● | 200 | 236 | 200 | 200 |
| 237 | 11101101 | ed | $(\bar{a}_{-1}\bar{a}_1)+(a_{-1}a_1)+(a_0)$ | ●●● | 72 | 237 | 72 | 72 |
| 238 | 11101110 | ee | $(a_0)+(a_1)$ | −●● | 136 | 252 | 192 | 136 |
| 239 | 11101111 | ef | $(\bar{a}_{-1})+(a_0)+(a_1)$ | ●●● | 8 | 253 | 64 | 8 |
| 240 | 11110000 | f0 | $(a_{-1})$ | ○−− | 240 | 170 | 170 | 170 |
| 241 | 11110001 | f1 | $(\bar{a}_0\bar{a}_1)+(a_{-1})$ | ●●● | 112 | 171 | 42 | 42 |
| 242 | 11110010 | f2 | $(\bar{a}_0a_1)+(a_{-1})$ | ●●● | 176 | 186 | 162 | 162 |
| 243 | 11110011 | f3 | $(\bar{a}_0)+(a_{-1})$ | ●●− | 48 | 187 | 34 | 34 |
| 244 | 11110100 | f4 | $(a_0\bar{a}_1)+(a_{-1})$ | ●●● | 208 | 174 | 138 | 138 |

| rule number | | | boolean expression | dep | equivalent rules | | | min |
|---|---|---|---|---|---|---|---|---|
| dec | binary | hex | | | conj | refl | c.r. | |
| 245 | 11110101 | f5 | $(\bar{a}_1)+(a_{-1})$ | •−• | 80 | 175 | 10 | 10 |
| 246 | 11110110 | f6 | $(a_0\bar{a}_1)+(\bar{a}_0a_1)+(a_{-1})$ | ••• | 144 | 190 | 130 | 130 |
| 247 | 11110111 | f7 | $(\bar{a}_1)+(\bar{a}_0)+(a_{-1})$ | ••• | 16 | 191 | 2 | 2 |
| 248 | 11111000 | f8 | $(a_0a_1)+(a_{-1})$ | ••• | 224 | 234 | 168 | 168 |
| 249 | 11111001 | f9 | $(\bar{a}_0\bar{a}_1)+(a_0a_1)+(a_{-1})$ | ••• | 96 | 235 | 40 | 40 |
| 250 | 11111010 | fa | $(a_{-1})+(a_1)$ | •−• | 160 | 250 | 160 | 160 |
| 251 | 11111011 | fb | $(\bar{a}_0)+(a_{-1})+(a_1)$ | ••• | 32 | 251 | 32 | 32 |
| 252 | 11111100 | fc | $(a_{-1})+(a_0)$ | ••− | 192 | 238 | 136 | 136 |
| 253 | 11111101 | fd | $(\bar{a}_1)+(a_{-1})+(a_0)$ | ••• | 64 | 239 | 8 | 8 |
| 254 | 11111110 | fe | $(a_{-1})+(a_0)+(a_1)$ | ••• | 128 | 254 | 128 | 128 |
| 255 | 11111111 | ff | 1 | −−− | 0 | 255 | 0 | 0 |

**Forms of rules and equivalences between rules.**

The table lists all 256 possible rules for $k = 2$, $r = 1$ one-dimensional cellular automata. Such cellular automata consist of a line of sites, each with value 0 or 1. At each time step, the value $a_i$ of a site at position $i$ is updated according to the rule

$$a_i' = \phi(a_{i-1}, a_i, a_{i+1}).$$

This table lists the $2^{2^3} = 256$ possible choices of $\phi$.

Each digit in the binary representation of the rule number gives the value of $\phi$ for a particular set of $(a_{i-1}, a_i, a_{i+1})$. The digit corresponding to the coefficient of $2^n$ in the rule number gives the value of $\phi(n_2, n_1, n_0)$, where $n = 4n_2 + 2n_1 + n_0$. Thus the leftmost digit in the binary representation of the rule number gives $\phi(1, 1, 1)$, the next gives $\phi(1, 1, 0)$, and so on, down to $\phi(0, 0, 0)$.

The table also gives the decimal and hexadecimal representations of the rule numbers.

Each $\phi$ can be considered a Boolean function of three variables, say $a_{-1}$, $a_0$ and $a_1$. The table gives the minimal disjunctive normal form representations for these Boolean functions. Boolean multiplication and addition are used (corresponding to AND and OR operations). Bar denotes complementation. In each case, the expression with the minimal number of components, using only these operations, is given.

The column labelled "dep" gives the dependence of $\phi(a_{-1}, a_0, a_1)$ on each of the $a_{-1}$, $a_0$ and $a_1$. The symbol – indicates no change in $\phi$ when the corresponding $a_j$ is changed. The symbol ∘ denotes linear dependence of $\phi$ on the corresponding $a_j$: whenever $a_j$ changes, $\phi$ also changes. The symbol • denotes arbitrary dependence of $\phi$. Rules such as 90 in which only ∘ and – dependence occurs, are called additive, and can be represented as linear functions modulo two.

For each rule, the table gives rules equivalent under simple transformations. "conj" denotes conjugation: interchange of the roles of 0 and 1. "refl" denotes reflection. Rules invariant under reflection are symmetric. "c.r." denotes the combined operation of conjugation and reflection.

Many of the properties considered in this Appendix are unaffected by these transformations. The rules form equivalence classes under these transformations, and it is usually convenient to consider only the minimal (lowest-numbered) representatives of each class, as given by the last column in the table.

In some cases, further equivalences between rules can be used. Table 7 gives one important set of such further equivalences.

Some special rules are:

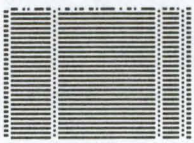
| | |
|---|---|
| 51 | complement |
| 170 | left shift |
| 204 | identity |
| 240 | right shift |

Table by Lyman P. Hurd (*Mathematics Department, Princeton University*). (Boolean expressions by S. Wolfram.)

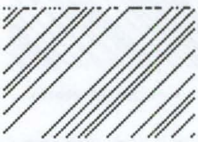
# Table 2: Patterns from Disordered States



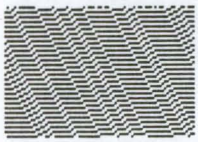
rule 0 (00000000)



rule 1 (00000001)

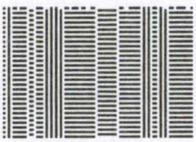


rule 2 (00000010)



rule 3 (00000011)

rule 4 (00000100)



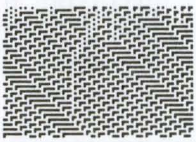
rule 5 (00000101)

rule 6 (00000110)

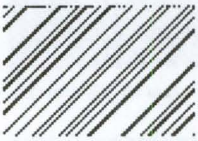
rule 7 (00000111)



rule 8 (00001000)



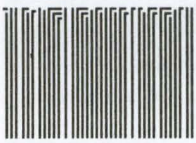
rule 9 (00001001)



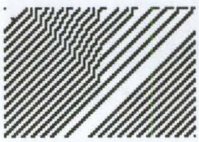
rule 10 (00001010)

rule 11 (00001011)

rule 12 (00001100)

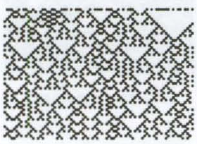


rule 13 (00001101)

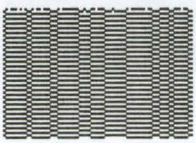


rule 14 (00001110)

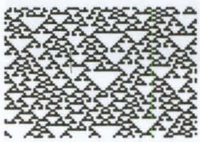
rule 15 (00001111)



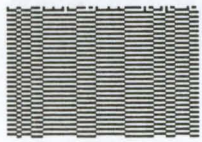
rule 18 (00010010)



rule 19 (00010011)



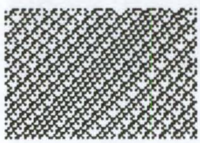
rule 22 (00010110)



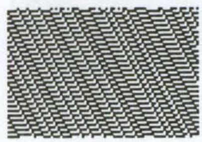
rule 23 (00010111)

rule 24 (00011000)

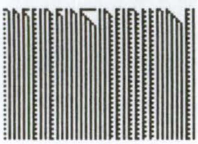
rule 25 (00011001)



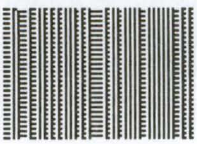
rule 26 (00011010)



rule 27 (00011011)



rule 28 (00011100)



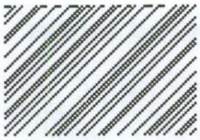
rule 29 (00011101)

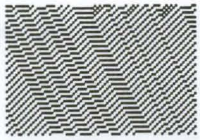
rule 30 (00011110)

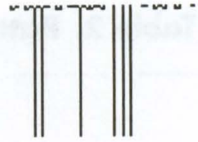


rule 32 (00100000)

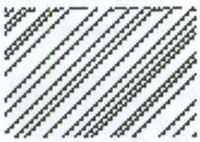

rule 33 (00100001)


rule 34 (00100010)

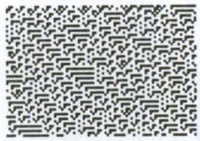

rule 35 (00100011)


rule 36 (00100100)


rule 37 (00100101)


rule 38 (00100110)

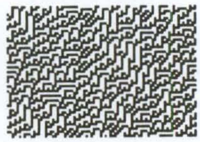

rule 40 (00101000)


rule 41 (00101001)

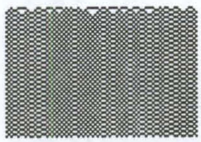

rule 42 (00101010)

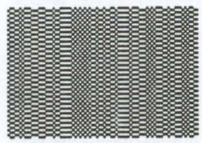

rule 43 (00101011)

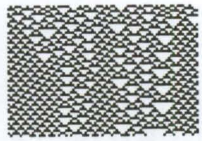

rule 44 (00101100)

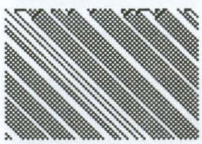

rule 45 (00101101)

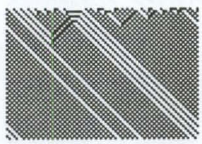

rule 46 (00101110)

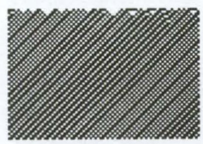

rule 50 (00110010)

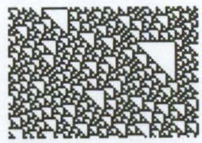

rule 51 (00110011)

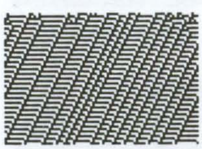

rule 54 (00110110)

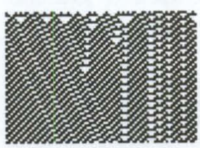

rule 56 (00111000)


rule 57 (00111001)

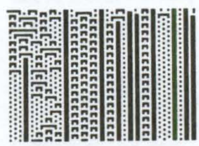

rule 58 (00111010)

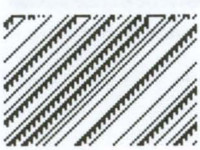

rule 60 (00111100)

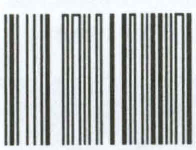

rule 61 (00111101)


rule 62 (00111110)

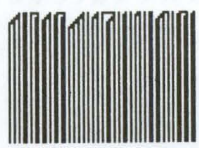

rule 72 (01001000)

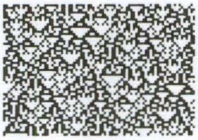

rule 73 (01001001)

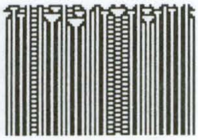

rule 74 (01001010)

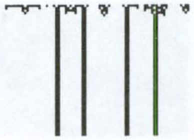

rule 76 (01001100)

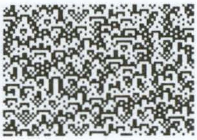

rule 77 (01001101)

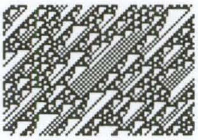

rule 78 (01001110)

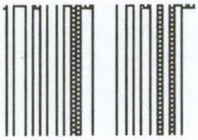

rule 90 (01011010)

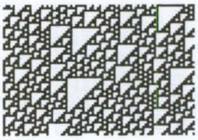

rule 94 (01011110)

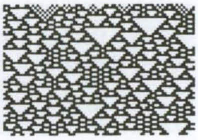

rule 104 (01101000)

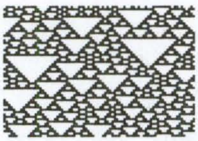

rule 105 (01101001)


rule 106 (01101010)

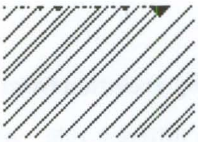

rule 108 (01101100)

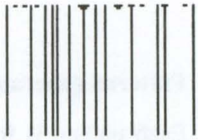

rule 110 (01101110)

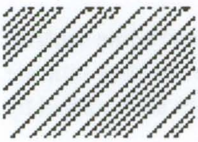

rule 122 (01111010)

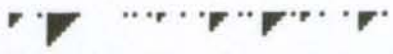

rule 126 (01111110)


rule 128 (10000000)


rule 130 (10000010)


rule 132 (10000100)

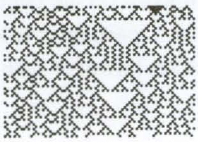

rule 134 (10000110)

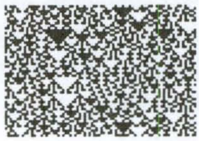

rule 136 (10001000)


rule 138 (10001010)

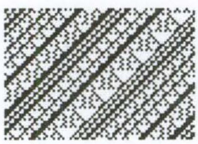

rule 140 (10001100)

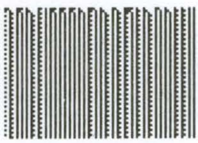

rule 142 (10001110)


rule 146 (10010010)

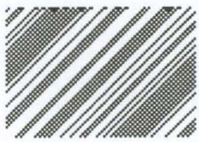

rule 150 (10010110)

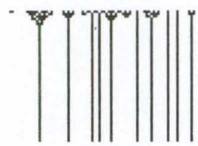

rule 152 (10011000)


rule 154 (10011010)


rule 156 (10011100)

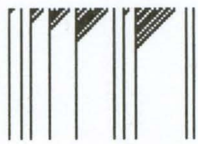

rule 160 (10100000)

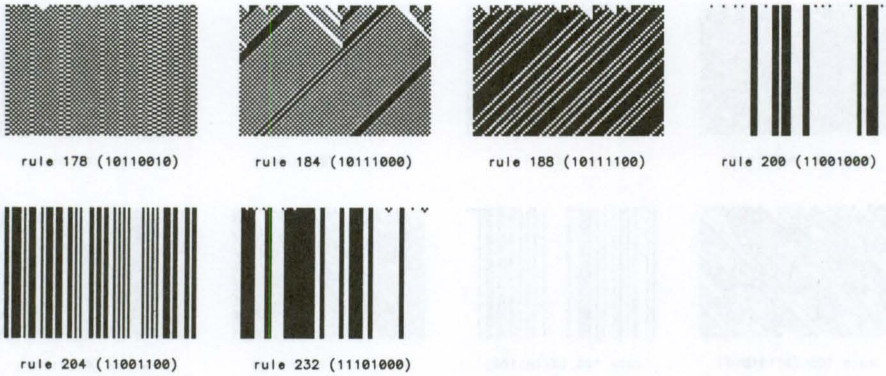

rule 162 (10100010)


rule 164 (10100100)


rule 168 (10101000)


rule 170 (10101010)


rule 172 (10101100)




**Patterns generated by evolution from disordered initial states.**

Each picture is for a different rule. All the "minimal representative" rules of table 1 are included. (Other rules have patterns equivalent to those of their minimal representatives.)

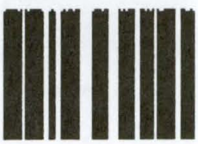
Sites with values 1 and 0 are represented respectively by black and white squares. The initial configuration is at the top of each picture. The values of sites in it are chosen randomly to be 0 or 1 with probability 1/2. Successive lines are obtained by applications of the cellular automaton rule.

These pictures show the evolution of cellular automata with 80 sites for 60 time steps. Periodic boundary conditions were imposed on the edges.

Different specific initial configurations for a particular rule almost always yield qualitatively similar patterns. Different rules are however seen to give a wide variety of different kinds of patterns.

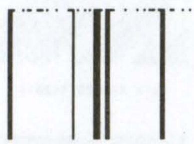
## Table 3: Blocked Patterns from Disordered States

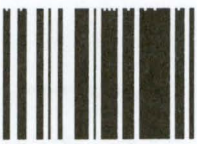

rule 0 (00000000)


rule 1 (00000001)


rule 2 (00000010)


rule 3 (00000011)


rule 4 (00000100)


rule 5 (00000101)

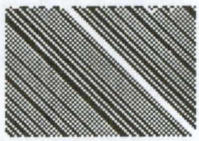

rule 6 (00000110)


rule 7 (00000111)

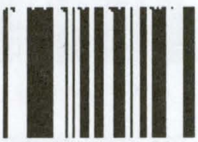

rule 8 (00001000)

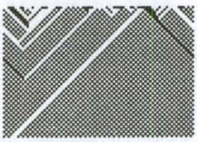

rule 9 (00001001)


rule 10 (00001010)


rule 11 (00001011)


rule 12 (00001100)


rule 13 (00001101)


rule 14 (00001110)

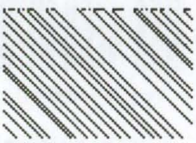

rule 15 (00001111)


rule 18 (00010010)


rule 19 (00010011)

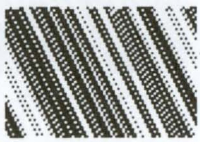

rule 22 (00010110)


rule 23 (00010111)

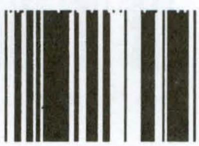

rule 24 (00011000)


rule 25 (00011001)


rule 26 (00011010)

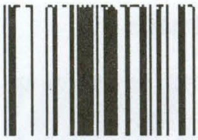

rule 27 (00011011)

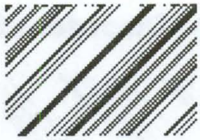

rule 28 (00011100)


rule 29 (00011101)

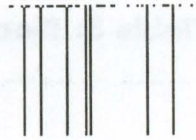

rule 30 (00011110)

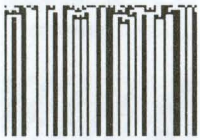

rule 32 (00100000)

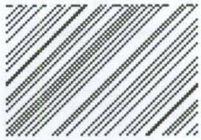

rule 33 (00100001)


rule 34 (00100010)

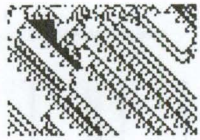

rule 35 (00100011)


rule 36 (00100100)


rule 37 (00100101)

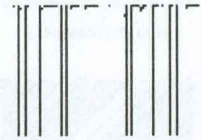

rule 38 (00100110)

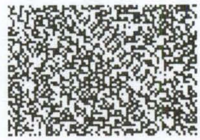

rule 40 (00101000)


rule 41 (00101001)


rule 42 (00101010)


rule 43 (00101011)

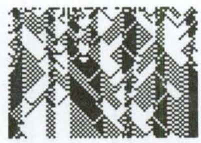

rule 44 (00101100)


rule 45 (00101101)

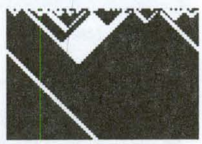

rule 46 (00101110)


rule 50 (00110010)

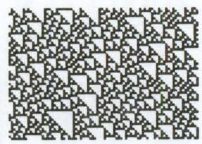

rule 51 (00110011)

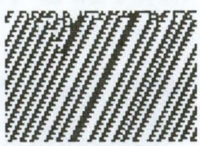

rule 54 (00110110)

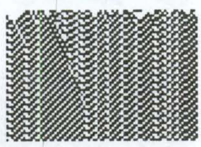

rule 56 (00111000)

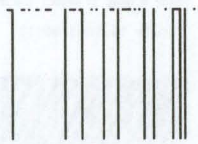

rule 57 (00111001)

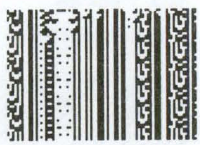

rule 58 (00111010)

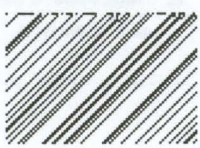

rule 60 (00111100)


rule 61 (00111101)

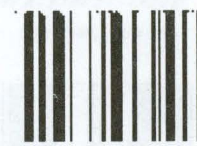

rule 62 (00111110)

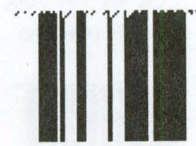

rule 72 (01001000)

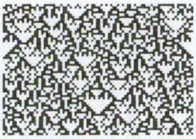

rule 73 (01001001)

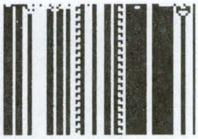

rule 74 (01001010)

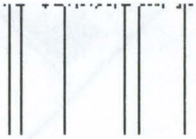

rule 76 (01001100)

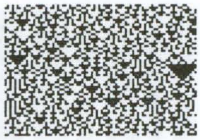

rule 77 (01001101)


rule 78 (01001110)

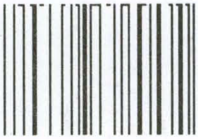

rule 90 (01011010)

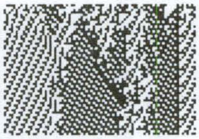

rule 94 (01011110)

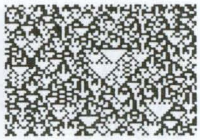

rule 104 (01101000)


rule 105 (01101001)


rule 106 (01101010)

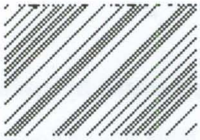

rule 108 (01101100)

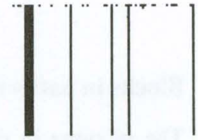

rule 110 (01101110)

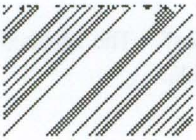

rule 122 (01111010)


rule 126 (01111110)

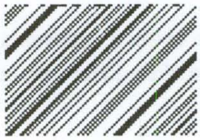

rule 128 (10000000)

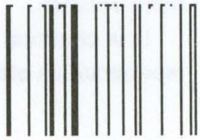

rule 130 (10000010)

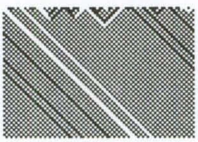

rule 132 (10000100)


rule 134 (10000110)

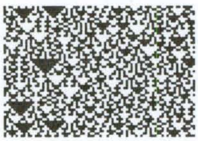

rule 136 (10001000)


rule 138 (10001010)

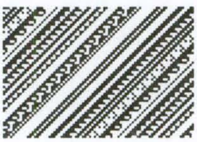

rule 140 (10001100)


rule 142 (10001110)


rule 146 (10010010)

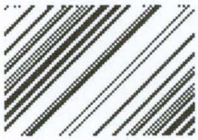

rule 150 (10010110)


rule 152 (10011000)


rule 154 (10011010)


rule 156 (10011100)

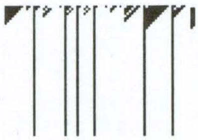

rule 160 (10100000)

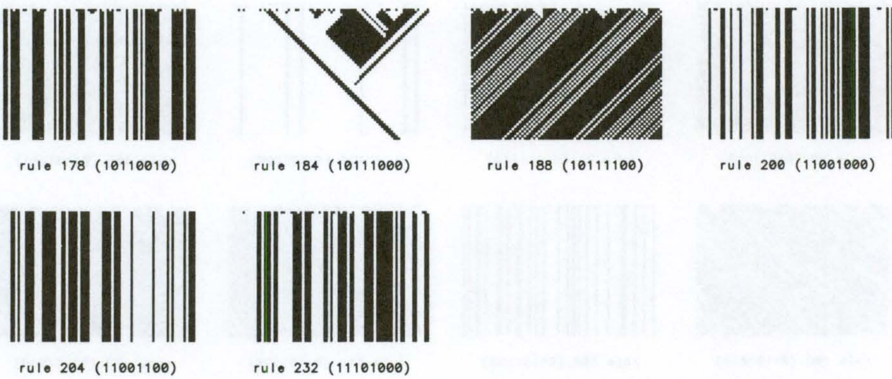

rule 162 (10100010)


rule 164 (10100100)


rule 168 (10101000)


rule 170 (10101010)


rule 172 (10101100)

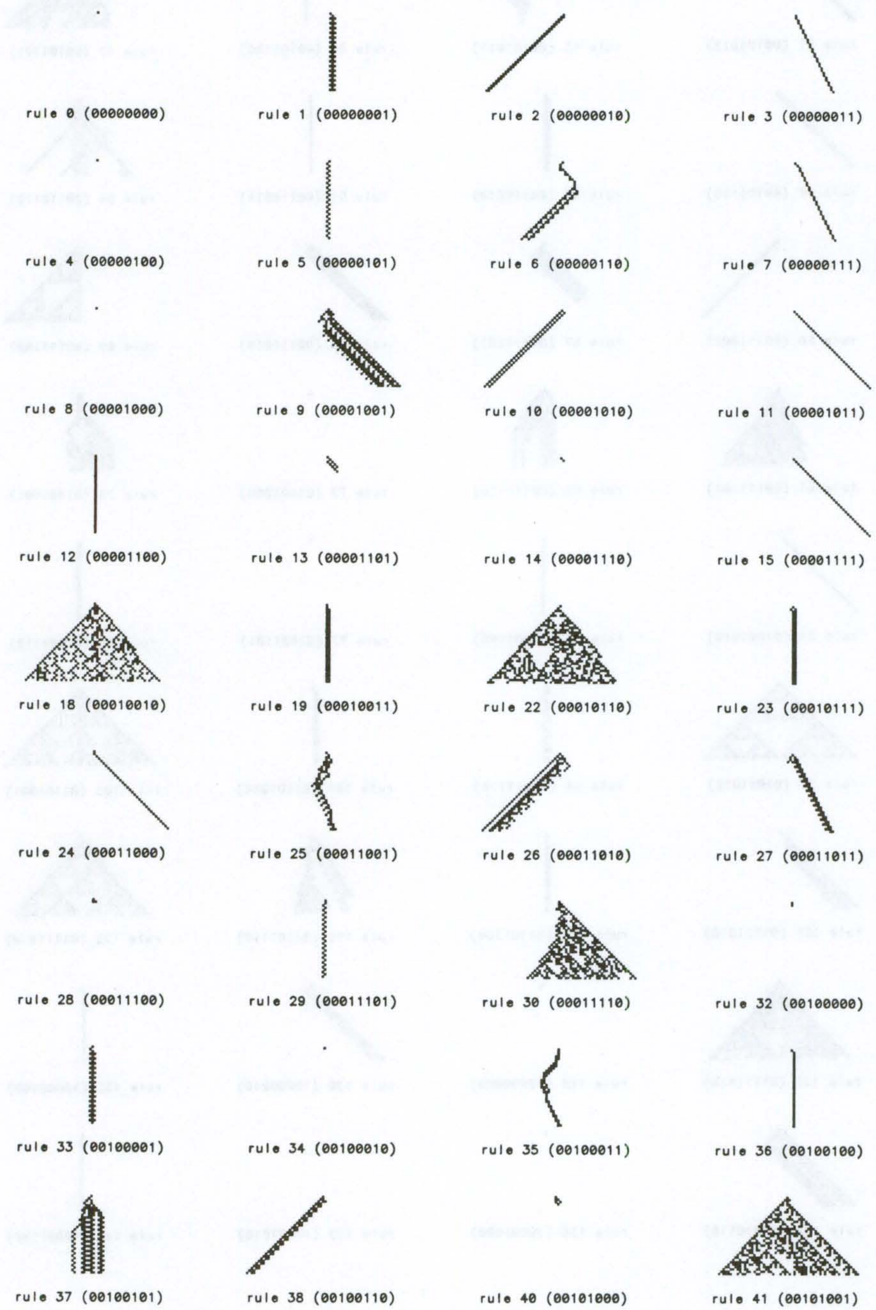


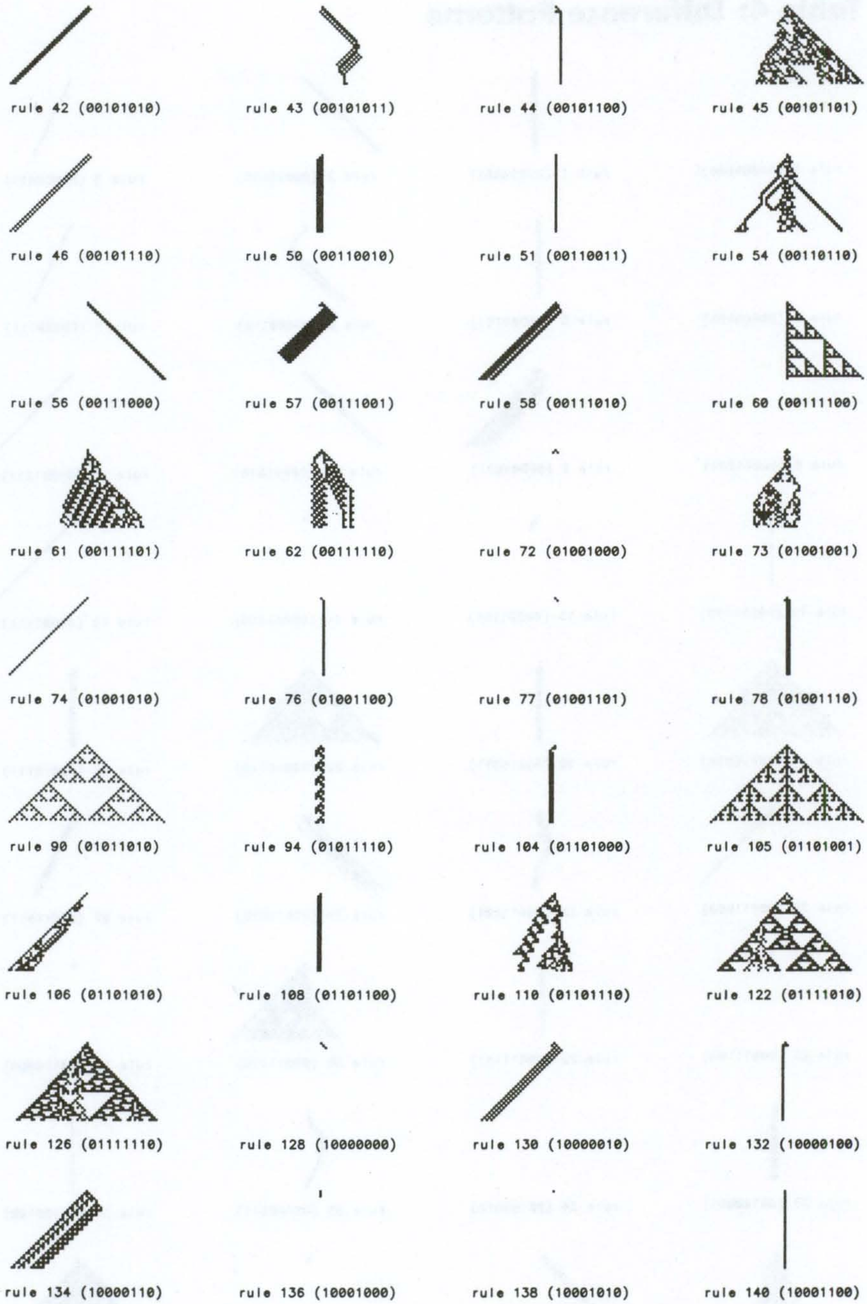

**Blocks in patterns generated by evolution from disordered initial states.**

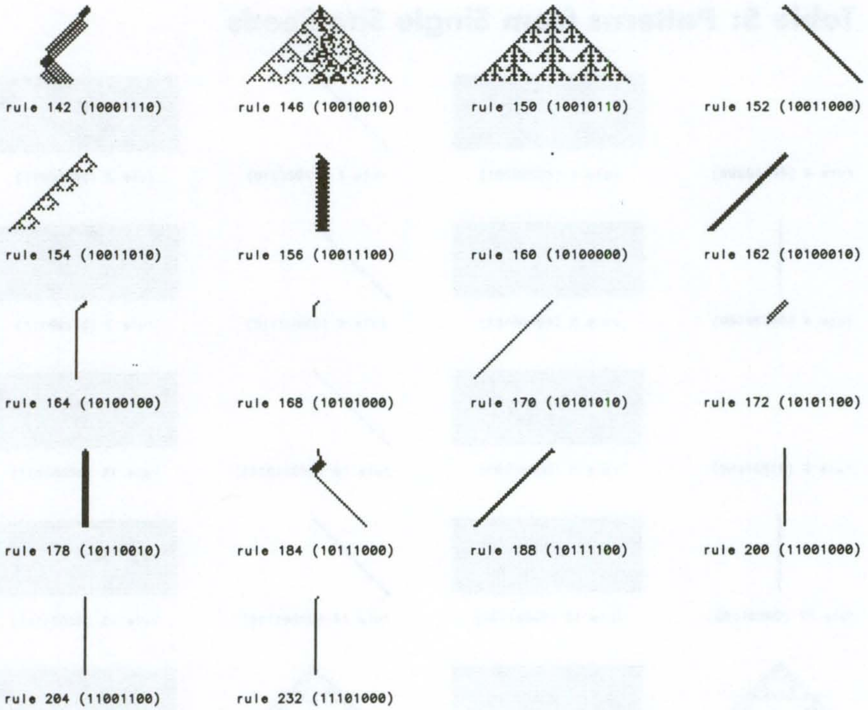
The pictures in this table are analogous to those in table 2, but show only every other site in both space and time. Certain features become clearer in this "blocked" representation.

It is common for cellular automata to exhibit several "phases". The blocked representation often makes differences between these phases visible.

## Table 4: Difference Patterns




rule 42 (00101010)
rule 43 (00101011)
rule 44 (00101100)
rule 45 (00101101)
rule 46 (00101110)
rule 50 (00110010)
rule 51 (00110011)
rule 54 (00110110)
rule 56 (00111000)
rule 57 (00111001)
rule 58 (00111010)
rule 60 (00111100)
rule 61 (00111101)
rule 62 (00111110)
rule 72 (01001000)
rule 73 (01001001)
rule 74 (01001010)
rule 76 (01001100)
rule 77 (01001101)
rule 78 (01001110)
rule 90 (01011010)
rule 94 (01011110)
rule 104 (01101000)
rule 105 (01101001)
rule 106 (01101010)
rule 108 (01101100)
rule 110 (01101110)
rule 122 (01111010)
rule 126 (01111110)
rule 128 (10000000)
rule 130 (10000010)
rule 132 (10000100)
rule 134 (10000110)
rule 136 (10001000)
rule 138 (10001010)
rule 140 (10001100)



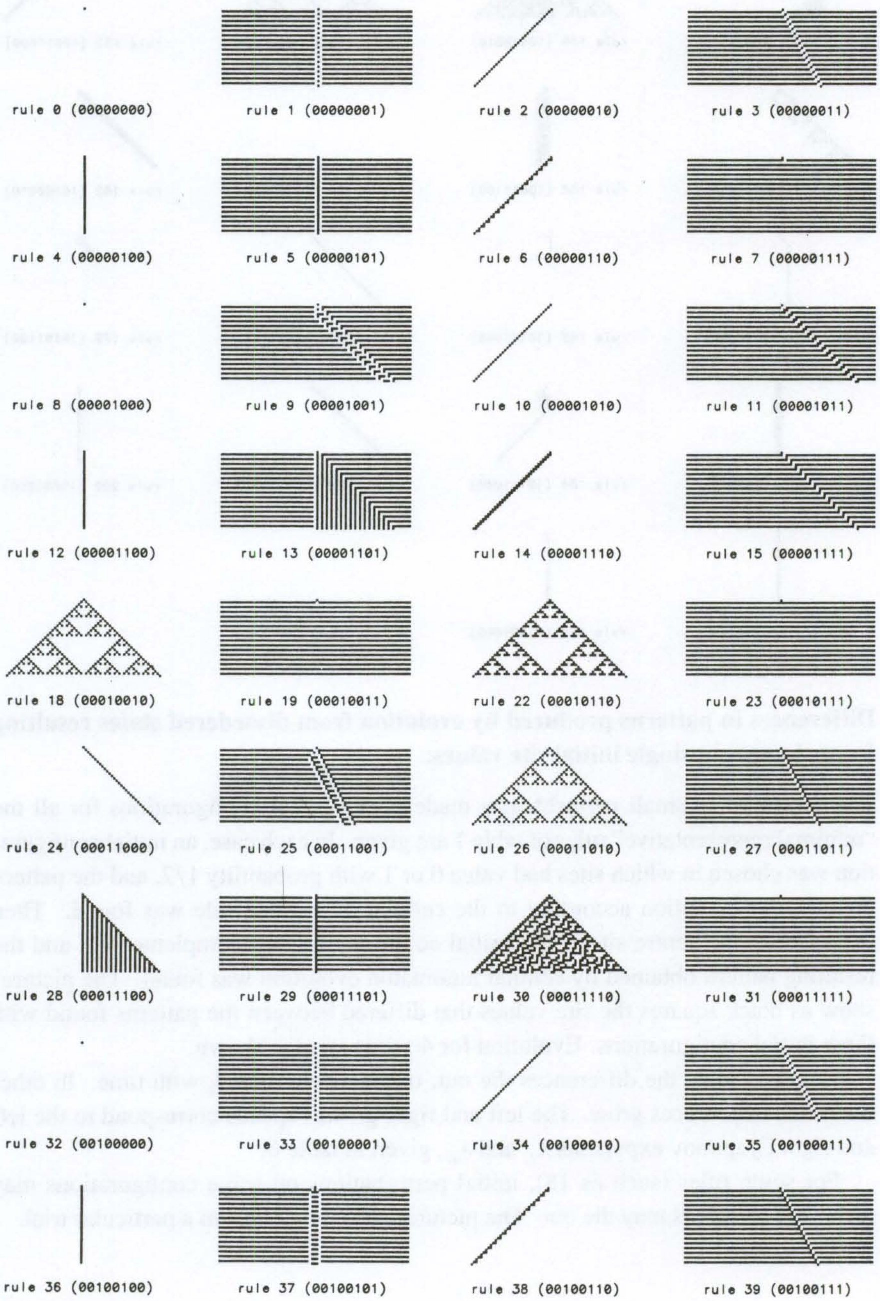

**Differences in patterns produced by evolution from disordered states resulting from changes in single initial site values.**

The evolution of small perturbations made in the initial configurations for all the "minimal representative" rules of table 1 are given. In each case, an initial configuration was chosen in which sites had value 0 or 1 with probability $1/2$, and the pattern obtained by evolution according to the cellular automaton rule was found. Then the value of the centre site in the initial configuration was complemented, and the resulting pattern obtained by cellular automaton evolution was found. The pictures show as black squares the site values that differed between the patterns found with these initial configurations. Evolution for 40 time steps is shown.

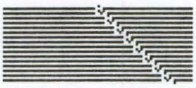
In some cases, the differences die out, or remain localized, with time. In other cases, the differences grow. The left and right growth speeds correspond to the left and right Lyapunov exponents $\lambda_L$ and $\lambda_R$, given in table 6.

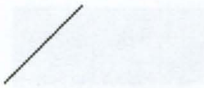
For some rules (such as 18), initial perturbations on some configurations may grow, but on others may die out. The pictures show results from a particular trial.

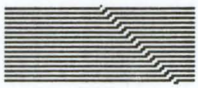
# Table 5: Patterns from Single Site Seeds



rule 40 (00101000)

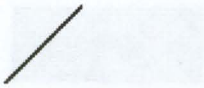

rule 41 (00101001)

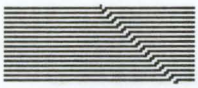

rule 42 (00101010)


rule 43 (00101011)

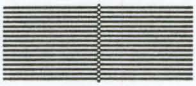

rule 44 (00101100)


rule 45 (00101101)

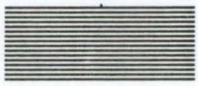

rule 46 (00101110)


rule 47 (00101111)


rule 50 (00110010)


rule 51 (00110011)

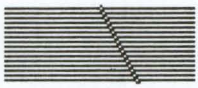

rule 54 (00110110)


rule 55 (00110111)

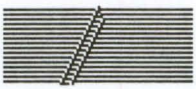

rule 56 (00111000)

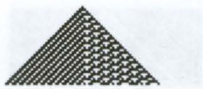

rule 57 (00111001)

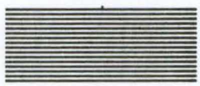

rule 58 (00111010)


rule 59 (00111011)


rule 60 (00111100)

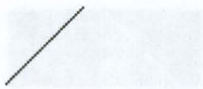

rule 61 (00111101)


rule 62 (00111110)


rule 63 (00111111)

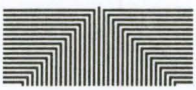

rule 72 (01001000)


rule 73 (01001001)

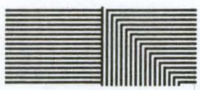

rule 74 (01001010)

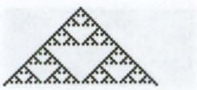

rule 75 (01001011)

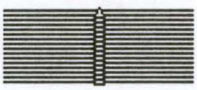

rule 76 (01001100)

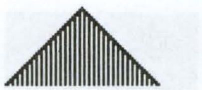

rule 77 (01001101)

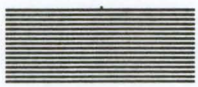

rule 78 (01001110)


rule 79 (01001111)


rule 90 (01011010)


rule 91 (01011011)

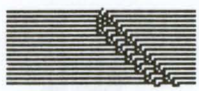

rule 94 (01011110)


rule 95 (01011111)

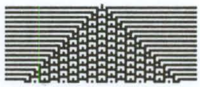

rule 104 (01101000)


rule 105 (01101001)

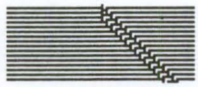

rule 106 (01101010)


rule 107 (01101011)

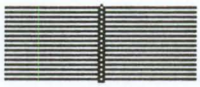

rule 108 (01101100)

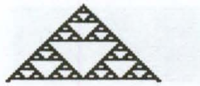

rule 109 (01101101)

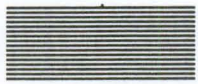

rule 110 (01101110)


rule 111 (01101111)


rule 122 (01111010)

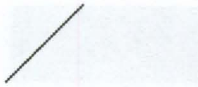

rule 123 (01111011)

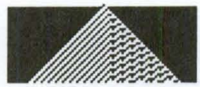

rule 126 (01111110)


rule 127 (01111111)


rule 128 (10000000)


rule 129 (10000001)


rule 130 (10000010)


rule 131 (10000011)


rule 132 (10000100)

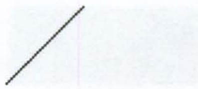

rule 133 (10000101)


rule 134 (10000110)


rule 135 (10000111)

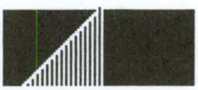

rule 136 (10001000)


rule 137 (10001001)


rule 138 (10001010)

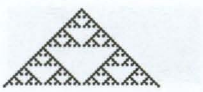

rule 139 (10001011)


rule 140 (10001100)

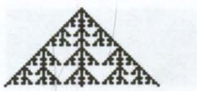

rule 141 (10001101)


rule 142 (10001110)


rule 143 (10001111)


rule 146 (10010010)

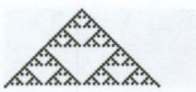

rule 147 (10010011)


rule 150 (10010110)

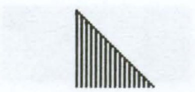

rule 151 (10010111)


rule 152 (10011000)


rule 153 (10011001)


rule 154 (10011010)

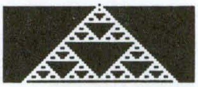
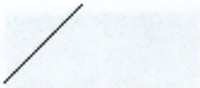

rule 155 (10011011)


rule 156 (10011100)


rule 157 (10011101)

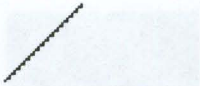

rule 158 (10011110)


rule 159 (10011111)

rule 160 (10100000)

rule 161 (10100001)

rule 162 (10100010)

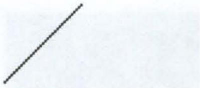
rule 163 (10100011)


rule 164 (10100100)

rule 165 (10100101)

rule 166 (10100110)

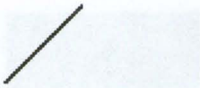
rule 167 (10100111)


rule 168 (10101000)

rule 169 (10101001)

rule 170 (10101010)

rule 171 (10101011)


rule 172 (10101100)

rule 173 (10101101)

rule 174 (10101110)

rule 175 (10101111)


rule 178 (10110010)

rule 179 (10110011)

rule 182 (10110110)

rule 183 (10110111)


rule 184 (10111000)

rule 185 (10111001)

rule 186 (10111010)

rule 187 (10111011)

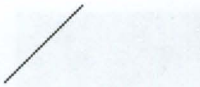

rule 188 (10111100)

rule 189 (10111101)

rule 190 (10111110)

rule 191 (10111111)

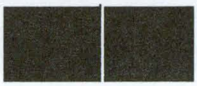

rule 200 (11001000)

rule 201 (11001001)

rule 202 (11001010)

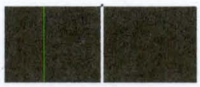
rule 203 (11001011)


rule 204 (11001100)

rule 205 (11001101)

rule 206 (11001110)

rule 207 (11001111)


rule 218 (11011010)


rule 219 (11011011)


rule 222 (11011110)


rule 223 (11011111)


rule 232 (11101000)


rule 233 (11101001)


rule 234 (11101010)


rule 235 (11101011)


rule 236 (11101100)


rule 237 (11101101)


rule 238 (11101110)

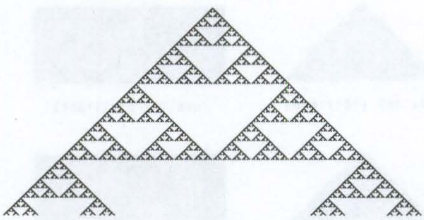

rule 239 (11101111)

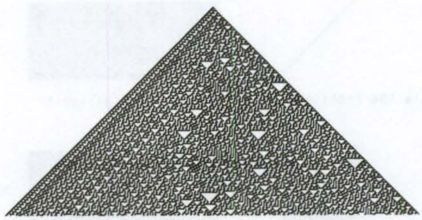

rule 250 (11111010)

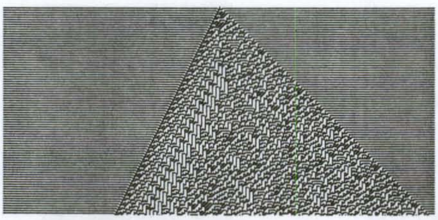

rule 251 (11111011)

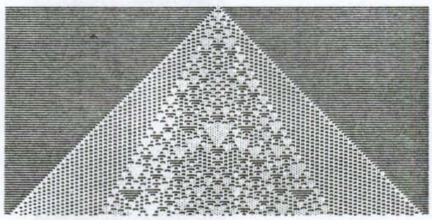

rule 254 (11111110)

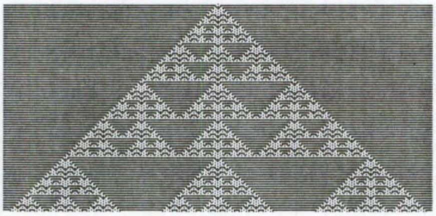

rule 255 (11111111)


rule 18 (00010010)

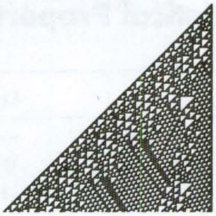

rule 30 (00011110)


rule 45 (00101101)

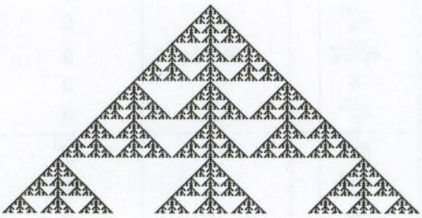

rule 73 (01001001)



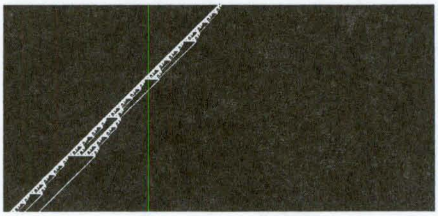
rule 105 (01101001)



rule 110 (01101110)



rule 150 (10010110)



rule 169 (10101001)

**Patterns generated by evolution from configurations containing a single nonzero site.**

The first part of the table shows pictures for all distinct rules. Since the initial configuration is not invariant under complementation, rules which differ by complementation can produce different patterns, and are shown separately. Only the minimal representative is shown for rules related by reflection. In all cases, the patterns correspond to evolution for 38 time steps.

Many rules are seen to yield equivalent patterns. The results of table 7 can often be used to deduce these equivalences.

Some rules (such as 122) yield asymptotically homogeneous patterns. Others (such as 90 and 150) yield asymptotically self similar or fractal patterns. (The fractal dimensions of the patterns obtained from rules 90 and 150 are respectively $\log_2 3 \simeq 1.59$ and $\log_2(1 + \sqrt{5}) \simeq 1.69$.) But some rules (such as 30 and 73) yield irregular patterns which show no periodic or almost periodic behaviour. The second part of the table gives some of the distinct patterns obtained by evolution for 360 time steps. Note that the structure on the right of the pattern generated by rule 110 eventually dies out, leaving an essentially periodic structure.

# Table 6: Statistical Properties

| | density | $h_\mu^{(x)}$ | $\lambda_L$ | $\lambda_R$ | $h_\mu^{(t)}$ | $\mathbf{h}_\mu$ | $\mathbf{h}_\mu^{[min]}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | — | — | 0 | 0 | 0 |
| 1 | 1/8 | .43536 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1/8 | .48752 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 3 | 1/4 | .70121 | −1/2 | 1/2 | $h_\mu^{(x)}/2$ | $h_\mu^{(x)}/2$ | 0 |
| 4 | 1/8 | .51771 | 0 | 0 | 0 | 0 | 0 |
| 5 | 7/16 | .702 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 6 | .241 ± .001 | <.573 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 7 | .469 ± .001 | <.502 ± .001 | −1/2 | 1/2 | $h_\mu^{(x)}/2$ | $h_\mu^{(x)}/2$ | 0 |
| 8 | 0 | 0 | — | — | 0 | 0 | 0 |
| 9 | .410 ± .001 | <.264 ± .002 | −1 | 1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 10 | 1/4 | .68872 | 1 | −1 | $h_\mu^{(x)}$ | 0 | $h_\mu^{(x)}$ |
| 11 | 1/2 | <.567 ± .001 | −1 | 1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 12 | 1/4 | .68872 | 0 | 0 | 0 | 0 | 0 |
| 13 | .437 ± .001 | .378 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 14 | 1/2 | 0 | (−1, 1) | (1, −1) | 0 | 0 | 0 |
| 15 | 1/2 | 1 | −1 | 1 | 1.0 | 1.0 | 0 |
| 18 | 1/4 | 1/2 | 1 | 1 | 0.5 | 1.0 | 1.0 |
| 19 | 1/2 | .62351 | 0 | 0 | 0 | 0 | 0 |
| 22 | .35095 ± .00002 | <.795 ± .001 | .7660 ± .0002 | .7660 ± .0002 | .744 ± .003 | <.9146 ± .0007 | <.9146 ± .0007 |
| 23 | 1/2 | .599 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 24 | 3/16 | .55081 | −1 | 1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 25 | .447 ± .001 | <.180 ± .001 | −1/2 | 1/2 | $h_\mu^{(x)}/2$ | $h_\mu^{(x)}/2$ | 0 |
| 26 | .386 ± .001 | <.790 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 27 | .531 ± .001 | <.800 ± .001 | −1/2 | 1/2 | $h_\mu^{(x)}/2$ | $h_\mu^{(x)}/2$ | 0 |
| 28 | 1/2 | .500 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 29 | 1/2 | .86742 | 0 | 0 | 0 | 0 | 0 |
| 30 | 1/2 | 1 | .2428 ± .0002 | 1 | 1 | <1.15436 | <.763141 |
| 32 | 0 | 0 | — | — | 0 | 0 | 0 |
| 33 | .396 ± .001 | <.637 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 34 | 1/4 | .68872 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 35 | .375 ± .001 | <.645 ± .001 | −1/2 | 1/2 | $h_\mu^{(x)}/2$ | $h_\mu^{(x)}/2$ | 0 |
| 36 | 1/16 | .32483 | 0 | 0 | 0 | 0 | 0 |
| 37 | .384 ± .001 | .506 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 38 | 9/32 | .73733 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 40 | 0 | 0 | — | — | 0 | 0 | 0 |
| 41 | .372 ± .001 | <.360 ± .001 | −1 | 1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 42 | 3/8 | .85684 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 43 | 1/2 | 0 | (−1, 1) | (1, −1) | 0 | 0 | 0 |
| 44 | .167 ± .001 | .528 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 45 | 1/2 | 1 | .1724 ± .0003 | 1 | 1 | <1.13036 | <.673893 |
| 46 | 3/8 | .55081 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 50 | 1/2 | .601 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 51 | 1/2 | 1 | 0 | 0 | 0 | 0 | 0 |

| | density | $h_\mu^{(x)}$ | $\lambda_L$ | $\lambda_R$ | $h_\mu^{(t)}$ | $\mathbf{h}_\mu$ | $\mathbf{h}_\mu^{[min]}$ |
|---|---|---|---|---|---|---|---|
| 54 | .49 ± .01 | <.2720 ± .0005 | .553 ± .002 | .553 ± .002 | <.250 ± .002 | <.250 ± .002 | <.250 ± .002 |
| 56 | .376 ± .001 | <.589 ± .001 | −1 | 1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 57 | 1/2 | 0 | (−1, 1) | (1, −1) | 0 | 0 | 0 |
| 58 | .625 ± .001 | <.332 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 60 | 1/2 | 1 | 0 | 1 | 1 | 2 | 2 |
| 62 | .644 ± .002 | <.262 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 72 | 1/8 | .32483 | 0 | 0 | 0 | 0 | 0 |
| 73 | .463 ± .001 | <.714 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 74 | .318 ± .001 | <.629 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 76 | 3/8 | .85060 | 0 | 0 | 0 | 0 | 0 |
| 77 | 1/2 | .599 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 78 | .562 ± .001 | .377 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 90 | 1/2 | 1 | 1 | 1 | 1 | 2 | 2 |
| 94 | .584 ± .001 | <.562 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 104 | .068 ± .001 | .208 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 105 | 1/2 | 1 | 1 | 1 | 1 | 2 | 2 |
| 106 | 1/2 | 1 | 1 | −.1335 ± .0006 | 1 | <1.06985 | <.461366 |
| 108 | 5/16 | .78025 | 0 | 0 | 0 | 0 | 0 |
| 110 | 4/7 | 0 | (.26 — 5) | (−.27 — 0.) | 0 | 0 | 0 |
| 122 | 1/2 | 1/2 | 1 | 1 | 0.5 | 1.0 | 1.0 |
| 126 | 1/2 | 1/2 | 1 | 1 | 0.5 | 1.0 | 1.0 |
| 128 | 0 | 0 | — | — | 0 | 0 | 0 |
| 130 | .167 ± .001 | .525 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 132 | 1/8 | .599 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 134 | .292 ± .001 | <.533 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 136 | 0 | 0 | — | — | 0 | 0 | 0 |
| 138 | 3/8 | .806 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 140 | 1/4 | <.678 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 142 | 1/2 | 0 | (−1, 1) | (1, −1) | 0 | 0 | 0 |
| 146 | 1/4 | 1/2 | 1 | 1 | 0.5 | 1.0 | 1.0 |
| 150 | 1/2 | 1 | 1 | 1 | 1 | 2 | 2 |
| 152 | .185 ± .001 | .515 ± .001 | −1 | 1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 154 | 1/2 | 1 | 1 | −1 | 1 | 1 | 0 |
| 156 | 1/2 | .502 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 160 | 0 | 0 | — | — | 0 | 0 | 0 |
| 162 | .333 ± .001 | .667 ± .001 | 1 | −1 | $h_\mu^{(x)}$ | $h_\mu^{(x)}$ | 0 |
| 164 | .083 ± .001 | .389 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 168 | 0 | 0 | — | — | 0 | 0 | 0 |
| 170 | 1/2 | −1 | 1 | 1 | 1.0 | 1.0 | 0 |
| 172 | 1/8 | .485 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 178 | 1/2 | .599 ± .001 | 0 | 0 | 0 | 0 | 0 |
| 184 | 1/2 | 0 | (−1, 1) | (1, −1) | 0 | 0 | 0 |
| 200 | 3/8 | .70121 | 0 | 0 | 0 | 0 | 0 |
| 204 | 1/2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 232 | 1/2 | .599 ± .001 | 0 | 0 | 0 | 0 | 0 |

**Statistical properties of evolution from disordered states.**

Results are given for all the "minimal representative" rules of table 1. In all cases, initial configurations were used in which each site has value 0 or 1 with probability 1/2. Some properties of some rules remain unchanged with different kinds of initial configurations.

Rational numbers, or numbers without errors, are quoted whenever analytical arguments yield exact results. In a few cases, the rigour of these arguments may be subject to question.

The column labelled "density" gives the asymptotic density of nonzero sites. For some, but not all, rules this depends on the initial density, here taken to be 1/2. For most rules, the relaxation to the final density appears to be approximately exponential. For some rules (such as 18), in which particle-like excitations undergo random annihilation, the relaxation may be like $t^{-1/2}$, or slower. Rule 110 shows particularly slow relaxation.

The column labelled $h_\mu^{(x)}$ gives estimates for the asymptotic spatial measure entropy, as defined in pages 115–157 in this book. This quantity gives a measure of the "information content" of cellular automaton configurations. It is computed by breaking the configuration into blocks of sites, say of length $X$, then evaluating the quantity $-\frac{1}{X}\sum p_i \log_2 p_i$, where the sum runs over all $2^X$ possible blocks, which are taken to occur with probabilities $p_i$. $h_\mu^{(x)}$ is the limit of this quantity as $X \to \infty$. The values decrease monotonically with $X$, allowing upper bounds on the $X \to \infty$ limit to be derived from finite $X$ results. Where errors are quoted, the values or bounds on $h_\mu^{(x)}$ given in the table were obtained after 400 time steps, with blocks up to length $X = 11$ considered. (More accurate results were obtained for rules 22 and 54.) Fits to values obtained as a function of $X$ suggest that the exact $h_\mu^{(x)}$ for rules 22 and 54 may in fact be zero.

The definition of $h_\mu^{(x)}$ implies that it achieves its maximal value of 1 only when all possible sequences of site values occur with equal probability, so that each site has value 0 or 1 with independent probability 1/2. $h_\mu^{(x)} = 0$ if only a finite number of complete cellular automaton configurations can occur.

Results for $h_\mu^{(x)}$ given without errors in the table were obtained by explicit construction of probabilistic regular languages which represent the sets of configurations produced by cellular automaton evolution, as in table 11.

The quantities $\lambda_L$ and $\lambda_R$ are left and right Lyapunov exponents, which measure the rate of information transmission. They give the slopes of the left and right boundaries of the difference patterns illustrated in table 4. Thus they measure the rate at which perturbations in cellular automaton configurations spread to the left and right.

The notation — indicates that almost all changes in initial configurations die out, so that the $\lambda_{L,R}$ are not defined.

The notation $(-1, 1)$ indicates that the information propagation direction can alternate, typically as progressively more distant particle-like structures from the initial configuration are encountered. There is probably no definite infinite size limit for the $\lambda_{L,R}$ in such cases.

Rule 110 shows highly complex information transmission properties, associated with the particle-like structures of table 15. The values of $\lambda_{L,R}$ given in the table for this case are possible bounds associated with the fastest and slowest-moving particle-like structures.

The quantity $h_\mu^{(t)}$ is the temporal measure entropy, which measures the information content of time sequences of values of individual sites. It is evaluated by applying the same procedure as for $h_\mu^{(x)}$ but to sequences of values of a single site attained on many successive time steps. It can be shown (see pages 115–157 in this book) that $h_\mu^{(t)} \leq (\lambda_L + \lambda_R) h_\mu^{(x)}$.

The quantities $h_\mu^{(x)}$ and $h_\mu^{(t)}$ measure respectively the information content of spatial and temporal sequences that are one site wide. The quantity $\mathbf{h}_\mu$ gives the entropy associated with spacetime patches of sites of arbitrary width. (Nevertheless, for many rules, the exact value of $\mathbf{h}_\mu$ is in fact obtained from patches of width 1 or 2.) In general, $\mathbf{h}_\mu \leq 2h_\mu^{(t)}$, and $h_\mu^{(t)} \leq \mathbf{h}_\mu \leq (\lambda_L + \lambda_R) h_\mu^{(x)}$.

The quantity $\mathbf{h}_\mu$ is evaluated by considering spacetime patches of sites that extend in the time direction. The last column of the table uses a generalization in which the patches can extend in any spacetime direction. It gives the minimum value $\mathbf{h}_\mu$ obtained as a function of direction. (The actual bounds given in the table were obtained from vertical or diagonal patches; other directions may yield stricter bounds.)

Table by Peter Grassberger (*Physics Department, University of Wuppertal*).

# Table 7: Blocking Transformation Equivalences

| | | |
|---|---|---|
| 0 | 0: | 00 10 (1) |
| 1 | 0: | 11 10 (2) |
| | 200: | 00 11 (2) |
| | 204: | 000 111 (2) |
| 2 | 34: | 00 10 (2) |
| | 170: | 000 100 (3) |
| | 0: | 1000 1100 (4) |
| 3 | 0: | 11 10 (2) |
| | 240: | 00 11 (4) |
| 4 | 204: | 00 10 (1) |
| | 0: | 00 11 (1) |
| 5 | 200: | 00 10 (2) |
| | 204: | 000 100 (2) |
| | 0: | 111 110 (2) |
| | 51: | 00010 11010 (1) |
| 6 | 184: | 00 10 (2) |
| | 34: | 00 11 (2) |
| | 170: | 0000 1000 (4) |
| | 128: | 0100 1100 (4) |
| | 240: | 1000 1010 (4) |
| | 85: | 10000 11000 (5) |
| | 0: | 11000 11100 (10) |
| 7 | 192: | 00 10 (2) |
| | 0: | 000 100 (2) |
| | 240: | 000 111 (6) |
| 8 | 0: | 00 10 (1) |
| 9 | 0: | 0010 1110 (2) |
| | 170: | 1000 0010 (6) |
| | 34: | 1000 0011 (6) |
| | 204: | 0100000 1100000 (5) |
| | 240: | 00000000 11010000 (8) |
| 10 | 34: | 00 10 (2) |
| | 170: | 000 100 (3) |
| | 0: | 010 110 (3) |
| 11 | 240: | 00 11 (2) |
| | 0: | 010 110 (3) |
| | 15: | 000 111 (3) |
| | 128: | 1100 1001 (4) |
| | 170: | 1100100 1001100 (7) |
| 12 | 204: | 00 10 (1) |
| | 0: | 100 110 (1) |
| 13 | 192: | 00 10 (2) |
| | 0: | 100 110 (1) |
| | 204: | 10100 10010 (1) |
| 14 | 240: | 10 01 (2) |
| | 34: | 00 11 (2) |
| | 15: | 010 101 (3) |
| | 0: | 1100 1000 (4) |
| | 170: | 0000 1100 (4) |
| | 128: | 1100 0110 (4) |
| 15 | 240: | 00 10 (2) |
| | 15: | 110 001 (3) |
| 18 | 90: | 00 10 (2) |
| | 204: | 11000 10100 (2) |
| | 0: | 10100 11100 (2) |
| 19 | 51: | 00 11 (1) |
| | 0: | 11 10 (2) |
| | 204: | 00 11 (2) |
| 22 | 146: | 00 10 (2) |
| | 90: | 0000 1000 (4) |
| | 0: | 11011000 1111000 (4) |
| 23 | 51: | 00 11 (1) |
| | 128: | 00 10 (2) |
| | 204: | 00 11 (2) |
| | 0: | 000 100 (2) |
| 24 | 48: | 00 10 (2) |
| | 240: | 000 100 (3) |
| | 0: | 100 011 (3) |
| 25 | 0: | 1101000 1111000 (7) |
| | 240: | 0000000 1101000 (14) |
| 26 | 90: | 00 10 (2) |
| | 85: | 010 110 (3) |
| | 170: | 100100 101100 (6) |
| | 0: | 10110100 10111100 (8) |
| | 240: | 11100100 10011100 (16) |
| 27 | 48: | 11 10 (4) |
| | 85: | 010 110 (3) |
| | 240: | 000 100 (6) |
| | 0: | 0100 1100 (8) |
| | 170: | 100100 101100 (6) |
| 28 | 192: | 00 10 (2) |
| | 200: | 10 01 (2) |
| | 51: | 100 110 (1) |
| | 204: | 100 110 (2) |
| | 0: | 1010 1100 (1) |
| 29 | 204: | 00 10 (2) |
| | 200: | 10 01 (2) |
| | 51: | 100 110 (1) |
| | 0 : | 1010 1100 (1) |
| 30 | | |
| 32 | 0: | 00 11 (1) |
| | 128: | 00 10 (2) |
| 33 | 132: | 00 10 (2) |
| | 200: | 00 11 (2) |
| | 0: | 111 100 (2) |
| | 204: | 000 111 (2) |
| | 128: | 0000 1010 (4) |
| 34 | 170: | 00 10 (2) |
| | 0: | 100 110 (3) |

| | | |
|---|---|---|
| 35 | 240: | 00 11 (4) |
| | 0: | 100 110 (3) |
| | 170: | 10100 10010 (5) |
| 36 | 0: | 00 11 (1) |
| | 4: | 00 10 (2) |
| | 204: | 000 100 (1) |
| 37 | 200: | 00 11 (2) |
| | 0: | 1111 1100 (2) |
| | 204: | 0000 1111 (2) |
| | 170: | 100000 110000 (6) |
| | 240: | 010000 110000 (6) |
| | 128: | 010000 111000 (6) |
| 38 | 34: | 00 10 (2) |
| | 85: | 100 110 (3) |
| | 170: | 0000 1000 (4) |
| | 0: | 1100 1110 (4) |
| 40 | 128: | 00 10 (2) |
| | 0: | 00 11 (2) |
| | 170: | 11010 10110 (5) |
| 41 | 148: | 00 10 (2) |
| | 184: | 0000 1000 (4) |
| | 176: | 0000 1010 (4) |
| | 170: | 11010 10110 (5) |
| | 240: | 00000000 10000000 (8) |
| | 128: | 10000000 10100000 (8) |
| | 0: | 11111000 11001000 (8) |
| | 136: | 01010000 10101101 (8) |
| 42 | 170: | 00 10 (2) |
| | 34: | 00 11 (2) |
| | 0: | 1100 1110 (4) |
| 43 | 170: | 10 01 (2) |
| | 240: | 00 11 (2) |
| | 15: | 000 111 (3) |
| | 0: | 1001 1000 (4) |
| | 128: | 1100 0110 (4) |
| 44 | 12: | 00 10 (2) |
| | 204: | 000 100 (1) |
| | 0: | 1000 1100 (1) |
| 45 | | |
| 46 | 34: | 00 11 (2) |
| | 0: | 110 100 (3) |
| | 170: | 000 110 (3) |
| 50 | 51: | 10 01 (1) |
| | 128: | 00 10 (2) |
| | 204: | 10 01 (2) |
| | 0: | 1010 1000 (2) |
| 51 | 51: | 10 01 (1) |
| | 204: | 00 10 (2) |

| | | |
|---|---|---|
| 54 | 50: | 00 10 (2) |
| | 51: | 1000 0010 (2) |
| | 128: | 0000 1000 (4) |
| | 204: | 1000 0010 (4) |
| | 170: | 1000 1110 (4) |
| | 240: | 0010 1110 (4) |
| | 0: | 000010 111010 (4) |
| 56 | 240: | 00 10 (2) |
| | 128: | 10 01 (2) |
| | 184: | 010 101 (3) |
| | 0: | 1010 0110 (2) |
| | 34: | 1010 1101 (4) |
| | 170: | 11010 10110 (5) |
| 57 | 128: | 10 01 (2) |
| | 184: | 010 101 (3) |
| | 0: | 1010 0110 (2) |
| | 48: | 1010 0100 (4) |
| | 34: | 1010 1101 (4) |
| | 240: | 10100 10010 (5) |
| | 170: | 11010 10110 (5) |
| 58 | 128: | 00 10 (2) |
| | 0: | 101 100 (3) |
| | 240: | 1100 1110 (8) |
| | 170: | 11010 10110 (5) |
| 60 | 60: | 00 10 (2) |
| 62 | 240: | 1100 1110 (8) |
| | 204: | 11000 11010 (3) |
| | 0: | 111110 100000 (3) |
| 72 | 0: | 00 10 (1) |
| | 4: | 00 11 (2) |
| | 204: | 000 110 (1) |
| 73 | 204: | 1100 0110 (2) |
| | 51: | 11000 11010 (1) |
| | 0: | 10110 10000 (2) |
| 74 | 34: | 00 10 (2) |
| | 170: | 000 100 (3) |
| | 0: | 10000 10100 (5) |
| | 85: | 1110000 1101000 (7) |
| 76 | 204: | 00 10 (1) |
| | 0: | 1010 1110 (1) |
| 77 | 204: | 10 01 (1) |
| | 128: | 00 10 (2) |
| | 0: | 1010 1000 (1) |
| 78 | 0: | 101 100 (1) |
| | 204: | 11010 10110 (1) |
| 90 | 90: | 00 10 (2) |
| 94 | 90: | 00 11 (2) |
| | 0: | 1010 1110 (1) |
| | 204: | 1010 0101 (2) |
| | 51: | 10010 11110 (1) |
| | 136: | 111100 110110 (6) |
| | 192: | 110110 011110 (6) |

| | | |
|---|---|---|
| 104 | 128: | 00 10 (2) |
| | 4: | 00 11 (2) |
| | 0: | 000 100 (1) |
| | 204: | 0000 1100 (1) |
| 105 | 150: | 00 10 (2) |
| 106 | 170: | 00 10 (2) |
| 108 | 76: | 00 10 (2) |
| | 204: | 000 100 (1) |
| | 51: | 10100 11100 (1) |
| | 0: | 10010 11110 (2) |
| 110 | 0: | 110100 101100 (9) |
| | 240: | 11000 100110 (9) |
| | 170: | 10011000 11111000 (16) |
| 122 | 128: | 00 10 (2) |
| | 90: | 00 11 (2) |
| | 0: | 1010 1000 (2) |
| | 204: | 11100 11110 (2) |
| 126 | 90: | 00 11 (2) |
| | 204: | 11100 11110 (2) |
| | 0: | 01110 10001 (2) |
| 128 | 0: | 00 10 (1) |
| | 128: | 00 11 (2) |
| 130 | 34: | 00 10 (2) |
| | 170: | 000 100 (3) |
| | 0: | 1000 1100 (4) |
| | 128: | 1000 1111 (4) |
| 132 | 204: | 00 10 (1) |
| | 128: | 00 11 (2) |
| | 0: | 000 110 (1) |
| 134 | 184: | 00 10 (2) |
| | 162: | 00 11 (2) |
| | 170: | 0000 1000 (4) |
| | 128: | 0100 1100 (4) |
| | 240: | 1000 1010 (4) |
| | 85: | 10000 11000 (5) |
| | 0: | 100000 101100 (6) |
| 136 | 0: | 00 10 (1) |
| | 136: | 00 11 (2) |
| 138 | 34: | 00 10 (2) |
| | 170: | 00 11 (2) |
| | 0: | 010 110 (3) |
| 140 | 204: | 00 10 (1) |
| | 0: | 100 110 (1) |
| 142 | 240: | 10 01 (2) |
| | 170: | 00 11 (2) |
| | 15: | 010 101 (3) |
| | 0: | 1100 1000 (4) |
| | 128: | 1100 0110 (4) |

| | | |
|---|---|---|
| 146 | 90: | 00 10 (2) |
| | 204: | 11000 10100 (2) |
| | 0: | 10010 11110 (2) |
| 150 | 150: | 00 10 (2) |
| 152 | 48: | 00 10 (2) |
| | 240: | 000 100 (3) |
| | 136: | 1000 1111 (4) |
| | 0: | 01000 11000 (5) |
| 154 | 90: | 00 10 (2) |
| | 85: | 010 110 (3) |
| | 170: | 1100 0110 (4) |
| 156 | 192: | 00 10 (2) |
| | 200: | 10 01 (2) |
| | 136: | 10 11 (2) |
| | 51: | 100 110 (1) |
| | 204: | 100 110 (2) |
| | 0: | 1010 1100 (1) |
| 160 | 128: | 00 10 (2) |
| | 0: | 000 100 (1) |
| 162 | 170: | 00 10 (2) |
| | 128: | 10 11 (2) |
| | 0: | 100 110 (3) |
| 164 | 90: | 11 10 (2) |
| | 128: | 00 11 (2) |
| | 204: | 000 100 (1) |
| | 0: | 0000 1100 (1) |
| 168 | 128: | 00 10 (2) |
| | 136: | 00 11 (2) |
| | 170: | 10 11 (2) |
| | 0: | 000 100 (1) |
| 170 | 170: | 00 10 (2) |
| 172 | 34: | 11 10 (2) |
| | 204: | 000 100 (1) |
| | 170: | 110 111 (3) |
| | 0: | 1000 1100 (1) |
| 178 | 51: | 10 01 (1) |
| | 128: | 00 10 (2) |
| | 204: | 10 01 (2) |
| | 0: | 1010 1000 (2) |
| 184 | 240: | 00 10 (2) |
| | 128: | 10 01 (2) |
| | 170: | 10 11 (2) |
| | 184: | 010 101 (3) |
| | 0: | 1010 0110 (2) |
| 200 | 0: | 00 10 (1) |
| | 204: | 00 11 (1) |
| 204 | 204: | 00 10 (1) |
| 232 | 204: | 00 11 (1) |
| | 128: | 00 10 (2) |
| | 0: | 000 100 (1) |

**Equivalences between rules under blocking transformations.**

When only particular blocks of site values occur, the evolution of one cellular automaton rule (say $R$) may be equivalent to that of another (say $R'$). Thus for example, the evolution under rule 1 of configurations consisting of the blocks 000 and 111 is equivalent to evolution under rule 204 in which 000 is replaced by 0, and 111 is replaced by 1. (Two time steps in evolution according to rule 1 are necessary to reproduce one time step of evolution according to rule 204.) Since rule 204 is the identity, this implies that configurations consisting only of the blocks 000 and 111 must be periodic under rule 1 (with period 2).

In general, one may consider replacing site values 0 and 1 in evolution according to rule $R$ by blocks $B_0$ and $B_1$. In some cases, the resulting evolution may correspond to $T$ time steps of another rule $R'$. Evolution according to rule $R'$ can thus be "simulated" by evolution according to rule $R$, under the blocking transformation $0 \to B_0$, $1 \to B_1$. Such blocking transformations can be considered analogous to block spin transformations in the renormalization group approach.

The table gives possible simulations for all the "minimal representative" rules of table 1. The notation $R'$: $B_0$ $B_1$ $(T)$ indicates simulation of rule $R'$ by replacing 0 with the block $B_0$, and 1 with $B_1$; $T$ steps of rule $R$ are needed to reproduce one step of rule $R'$ evolution.

The table includes all simulations for block lengths up to 8. The blocks $B_0$ and $B_1$ are always assumed distinct. Only one representative set of blocks is given for each simulation. (Thus for example, only the blocks 00 and 10 are given for the simulation of rule 90 by rule 18; the blocks 00 and 01 would also suffice.) Simulations with block length 1 are not included; these correspond to transformations given in table 1. No simulations are found for rules 30 and 45 up to block length 8.

Many rules are seen to be equivalent under blocking transformations to simple rules, such as 204 (the identity), 170 (left shift), 240 (right shift), 51 (complementation) and 0. Equivalence is also often found to the additive rules 90 and 150. An important property of all these simple rules is that they simulate themselves under blocking transformations. This has the consequence that patterns generated by these rules are self similar. Fractal patterns are thus produced by evolution according to rules 90 and 150 from single site seeds, as shown in table 5.

The simulations given in the table occur when only particular blocks occur in the configuration of a cellular automaton. In disordered configurations, all possible blocks can occur. But since a cellular automaton under most rules is irreversible, only a subset of blocks may occur after a sufficiently long time. Often the subset of blocks that occur is, at least approximately, the blocks which correspond to a particular simulation. In this case, the behaviour of one cellular automaton may be considered "attracted" to that of another.

It is common to find "domains" in which only particular blocks occur. Within each such domain, the evolution may correspond to that of a simpler rule. The domains are separated by walls or "defects", whose behaviour is not reproduced by the simpler rule. In some cases, the defects remain stationary; in others, they execute random walks, and, for example, annihilate in pairs. In the latter cases, the sizes of domains grow slowly with time.

While a large subset of possible initial configurations for a cellular automaton may be attracted to a particular form of behaviour, there are usually some special initial states (typically occurring among disordered states with probability zero), for which very different behaviour occurs. Such special initial states may for example consist of blocks which yield a simulation to which the rule is not generically attracted.

The blocking transformations considered in the table represent one form of transformation between rules. Many others can also be considered. A general class, which includes the blocking transformations of the table, are those transformations which can be carried out by arbitrary finite state machines.

The blocking transformations used in the table have the property that they reduce the total number of sites. This is a consequence of the fact that the blocks used are always taken not to overlap. An alternative approach is to perform replacements for overlapping blocks, thus obtaining configurations with the same number of sites. An example of such a replacement is $00 \to 0$, $01 \to 1$, $10 \to 1$, $11 \to 0$. For some rules, the resulting transformed configurations show evolution according to other $k = 2$, $r = 1$ cellular automaton rules. Rules related in this way must have the same global properties, and must yield for example the same entropies. The minimal representative rules from table 1 equivalent under such transformations are:

| | |
|---|---|
| 15, 240 | 240 |
| 23, 232 | 132 |
| 43, 212 | 184 |
| 51, 204 | 204 |
| 77, 178 | 222 |
| 85, 170 | 170 |
| 105, 150 | 150 |
| 113, 142 | 226 |

Main table by John Milnor (*Institute for Advanced Study*). (Original program by S. Wolfram.) Second table by Peter Grassberger.

# Table 8: Factorizations into Compositions of Simpler Rules

| $\phi$ | $\phi_1$ | $\phi_2$ |
|---|---|---|
| 0 | 0 | 0 |
| | 0 | 12 |
| | 0 | 48 |
| | 0 | 60 |
| | 0 | 192 |
| | 0 | 204 |
| | 0 | 240 |
| | 0 | 252 |
| | 17 | 0 |
| | 34 | 0 |
| | 34 | 192 |
| | 51 | 0 |
| | 68 | 0 |
| | 68 | 192 |
| | 85 | 0 |
| | 102 | 0 |
| | 119 | 0 |
| | 136 | 0 |
| | 153 | 0 |
| | 170 | 0 |
| | 187 | 0 |
| | 187 | 3 |
| | 204 | 0 |
| | 221 | 0 |
| | 221 | 3 |
| | 238 | 0 |
| | 255 | 0 |
| | 255 | 3 |
| | 255 | 12 |
| | 255 | 15 |
| | 255 | 48 |
| | 255 | 51 |
| | 255 | 60 |
| | 255 | 63 |

| $\phi$ | $\phi_1$ | $\phi_2$ |
|---|---|---|
| 1 | 17 | 192 |
| | 238 | 3 |
| 2 | 17 | 48 |
| | 238 | 12 |
| 3 | 17 | 240 |
| | 51 | 192 |
| | 204 | 3 |
| | 238 | 15 |
| 8 | 119 | 48 |
| | 136 | 12 |
| 12 | 34 | 48 |
| | 34 | 240 |
| | 51 | 48 |
| | 204 | 12 |
| | 221 | 12 |
| | 221 | 15 |
| 15 | 51 | 240 |
| | 204 | 15 |
| 18 | 17 | 60 |
| | 238 | 60 |
| 19 | 17 | 252 |
| | 238 | 63 |
| 24 | 102 | 48 |
| | 153 | 12 |
| 34 | 34 | 12 |
| | 34 | 204 |
| | 85 | 48 |
| | 170 | 12 |
| | 221 | 48 |
| | 221 | 51 |
| 36 | 102 | 192 |
| | 153 | 3 |

| $\phi$ | $\phi_1$ | $\phi_2$ |
|---|---|---|
| 46 | 34 | 60 |
| | 34 | 252 |
| | 221 | 60 |
| | 221 | 63 |
| 51 | 51 | 204 |
| | 85 | 240 |
| | 170 | 15 |
| | 204 | 51 |
| 60 | 51 | 60 |
| | 102 | 240 |
| | 153 | 15 |
| | 204 | 60 |
| 72 | 119 | 60 |
| | 136 | 60 |
| 90 | 102 | 60 |
| | 153 | 60 |
| 126 | 102 | 252 |
| | 153 | 63 |
| 128 | 119 | 3 |
| | 136 | 192 |
| 136 | 85 | 3 |
| | 119 | 51 |
| | 136 | 204 |
| | 170 | 192 |
| 170 | 85 | 51 |
| | 170 | 204 |
| 200 | 119 | 63 |
| | 136 | 252 |
| 204 | 51 | 51 |
| | 85 | 15 |
| | 170 | 240 |
| | 204 | 204 |

**Factorizations into compositions of simpler rules.**

The 256 rules in table 1 are stated as functions of three site values $\phi(a_{-1}, a_0, a_1)$. Of these, 48 depend only on two of the site values. Some other rules can be formed from compositions of these simpler rules. This table lists rules which can be formed by compositions according to

$$\phi(a_{-1}, a_0, a_1) = \phi_2(\phi_1(-, a_{-1}, a_0), \phi_1(-, a_0, a_1), -),$$

where – indicates that the value is irrelevant. Only minimal representative rules from table 1 are included. In each case, all possible compositions are listed. Note that most of the compositions do not commute.

Table by Erica Jen (*Los Alamos National Laboratory*).

## Table 9: Lengths of Distinct Blocks of Sites Newly Excluded at Time $t$

| rule | $t=1$ | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | — | — | — |
| 1 | 3 | — | — | — |
| 2 | 2 | — | — | — |
| 3 | 3 | — | — | — |
| 4 | 2 | — | — | — |
| 5 | 5 | — | — | — |
| 6 | 3 | 6 | 7 | 7 |
| 7 | 4 | 5 | 5 | 6 |
| 8 | 2 | 1 | — | — |
| 9 | 4 | 7 | 9 | 9 |
| 10 | 3 | — | — | — |
| 11 | 3 | 5 | 7 | 9 |
| 12 | 2 | — | — | — |
| 13 | 4 | 4 | 6 | 6 |
| 14 | 3 | 5 | 7 | 9 |
| 15 | — | — | — | — |
| 18 | 3 | 11 | 12 | 13 |
| 19 | 3 | 3 | — | — |
| 22 | 8 | 7 | 11 | 9 |
| 23 | 5 | 6 | 7 | 8 |
| 24 | 2 | 3 | — | — |
| 25 | 5 | 6 | 8 | 8 |
| 26 | 4 | 10 | 8 | 11 |
| 27 | 4 | 6 | 6 | 9 |
| 28 | 3 | 6 | 6 | 8 |
| 29 | 4 | — | — | — |
| 30 | — | — | — | — |
| 32 | 2 | 4 | 6 | 8 |
| 33 | 4 | 7 | 6 | 6 |
| 34 | 2 | — | — | — |
| 35 | 4 | 6 | 7 | 9 |
| 36 | 3 | 2 | — | — |
| 37 | 9 | 8 | 9 | 8 |
| 38 | 4 | 3 | — | — |
| 40 | 3 | 4 | 5 | 7 |
| 41 | 5 | 9 | 8 | 9 |
| 42 | 3 | — | — | — |
| 43 | 5 | 7 | 9 | 11 |
| 44 | 4 | 4 | 6 | 6 |
| 45 | — | — | — | — |
| 46 | 3 | 3 | — | — |
| 50 | 3 | 5 | 9 | 11 |
| 51 | — | — | — | — |
| 54 | 5 | 9 | 9 | 7 |
| 56 | 3 | 4 | 6 | 8 |
| 57 | 6 | 5 | 5 | 7 |
| 58 | 4 | 5 | 5 | 6 |
| 60 | — | — | — | — |
| 62 | 5 | 7 | 8 | 7 |
| 72 | 3 | 3 | — | — |
| 73 | 6 | 6 | 7 | 14 |
| 74 | 4 | 6 | 6 | 7 |
| 76 | 3 | — | — | — |
| 77 | 5 | 6 | 7 | 8 |
| 78 | 4 | 4 | 6 | 5 |
| 90 | — | — | — | — |
| 94 | 5 | 7 | 11 | 11 |
| 104 | 8 | 8 | 8 | 7 |
| 105 | — | — | — | — |
| 106 | — | — | — | — |
| 108 | 5 | 4 | — | — |
| 110 | 5 | 10 | 11 | 11 |
| 122 | 5 | 7 | 8 | 10 |
| 126 | 3 | 12 | 13 | 14 |
| 128 | 3 | 5 | 7 | 9 |
| 130 | 4 | 6 | 7 | 10 |
| 132 | 4 | 5 | 6 | 7 |
| 134 | 5 | 6 | 6 | 8 |
| 136 | 3 | 4 | 5 | 6 |
| 138 | 3 | — | — | — |
| 140 | 4 | 5 | 6 | 7 |
| 142 | 5 | 7 | 9 | 11 |
| 146 | 6 | 6 | 8 | 8 |
| 150 | — | — | — | — |
| 152 | 5 | 5 | 6 | 6 |
| 154 | — | — | — | — |
| 156 | 6 | 7 | 7 | 9 |
| 160 | 5 | 7 | 9 | 11 |
| 162 | 4 | 6 | 8 | 10 |
| 164 | 9 | 9 | 8 | 9 |
| 168 | 4 | 5 | 6 | 7 |
| 170 | — | — | — | — |
| 172 | 4 | 5 | 6 | 7 |
| 178 | 5 | 6 | 7 | 8 |
| 184 | 4 | 6 | 8 | 10 |
| 200 | 3 | — | — | — |
| 204 | — | — | — | — |
| 232 | 5 | 6 | 7 | 8 |

**Lengths of distinct blocks of sites newly excluded at time *t*.**

Most cellular automaton rules are irreversible, so that even starting from all possible initial configurations, only a subset of configurations can occur after $t$ time steps. In this subset of configurations, only certain blocks of site values can occur. The subset can be specified by giving the blocks which are excluded. In some cases (such as rule 128), the number of distinct excluded blocks is finite; in other cases, it is countably infinite. Irreversibility leads to an increase in the size of the set of excluded blocks with time.

The table gives the lengths of the shortest blocks which are newly excluded after exactly $t$ time steps. Such blocks can occur in configurations up to time $t-1$, but cannot occur at time $t$ or after. The lengths $L(t)$ of the shortest blocks newly excluded at time $t$ obey the inequality (see pages 159–202 in this book) $L(t) \geq L(t-1) - 2$.

The notation — in the table indicates that no blocks are newly excluded at a particular time step. This implies that the rule has reached a stable set of configurations, which can occur after any number of steps. It should be noted, however, that this table takes no account of the probabilities with which different configurations may occur.

Table by Lyman P. Hurd (*Mathematics Department, Princeton University*). (Original program by S. Wolfram.)

# Table 10: Regular Language Complexities

| | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $t>5$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 [1] | . | . | . | . | 1 [1] | 1 [1] |
| 1 | 4 [6] | . | . | . | . | 4 [6] | 4 [6] |
| 2 | 3 [4] | . | . | . | . | 3 [4] | 3 [4] |
| 3 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 4 | 2 [3] | . | . | . | . | 2 [3] | 2 [3] |
| 5 | 9 [15] | . | . | . | . | 9 [15] | 9 [15] |
| 6 | 9 [16] | 13 [22] | 22 [37] | 26 [44] | 31 [52] | | |
| 7 | 4 [7] | 7 [12] | 12 [21] | 14 [24] | 16 [27] | | |
| 8 | 3 [4] | 1 [1] | . | . | . | 1 [1] | 1 [1] |
| 9 | 9 [16] | 22 [40] | 44 [80] | 106 [198] | 266 [500] | | |
| 10 | 4 [6] | . | . | . | . | 4 [6] | 4 [6] |
| 11 | 3 [5] | 7 [12] | 10 [17] | 12 [20] | 14 [23] | | |
| 12 | 2 [3] | . | . | . | . | 2 [3] | 2 [3] |
| 13 | 6 [11] | 10 [17] | 12 [19] | 14 [21] | 16 [23] | | |
| 14 | 3 [5] | 7 [12] | 10 [17] | 12 [20] | 14 [23] | | |
| 15 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 16 | 3 [4] | . | . | . | . | 3 [4] | 3 [4] |
| 17 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 18 | 5 [9] | 47 [91] | 143 [270] | | | | |
| 19 | 3 [5] | 5 [8] | . | . | . | 5 [8] | 5 [8] |
| 20 | 10 [17] | 21 [37] | 32 [57] | 37 [65] | 50 [89] | | |
| 21 | 4 [7] | 9 [16] | 12 [21] | 14 [24] | 16 [27] | | |
| 22 | 15 [29] | 280 [551] | 4506 [8963] | | | | |
| 23 | 11 [20] | 15 [26] | 19 [32] | 23 [38] | 27 [44] | | |
| 24 | 2 [3] | 3 [4] | . | . | . | 3 [4] | 3 [4] |
| 25 | 6 [11] | 26 [50] | 55 [106] | 114 [220] | 333 [649] | | |
| 26 | 13 [25] | 92 [179] | 2238 [4454] | | | | |
| 27 | 10 [18] | 14 [25] | 18 [32] | 21 [37] | 24 [42] | | |
| 28 | 3 [5] | 8 [14] | 10 [17] | 11 [18] | 12 [19] | | |
| 29 | 4 [7] | . | . | . | . | 4 [7] | 4 [7] |
| 30 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 32 | 2 [3] | 5 [7] | 7 [9] | 9 [11] | 11 [13] | $2t+1[2t+3]$ | 4 [6] |
| 33 | 5 [9] | 11 [20] | 26 [47] | 40 [68] | 41 [68] | | |
| 34 | 2 [3] | . | . | . | . | 2 [3] | 2 [3] |
| 35 | 4 [7] | 7 [13] | 9 [16] | 10 [18] | 12 [21] | | |
| 36 | 3 [5] | 3 [4] | . | . | . | 3 [4] | 3 [4] |
| 37 | 15 [29] | 194 [376] | 870 [1698] | 3735 [7290] | | | |
| 38 | 5 [9] | 5 [8] | . | . | . | 5 [8] | 5 [8] |
| 40 | 10 [17] | 12 [19] | 15 [22] | 18 [25] | 21 [28] | | |
| 41 | 14 [27] | 128 [250] | 1049 [2069] | | | | |
| 42 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 43 | 9 [16] | 13 [22] | 17 [28] | 21 [34] | 25 [40] | | |
| 44 | 4 [7] | 11 [20] | 18 [32] | 23 [40] | 27 [46] | | |
| 45 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 46 | 3 [5] | 5 [8] | . | . | . | 5 [8] | 5 [8] |
| 48 | 2 [3] | . | . | . | . | 2 [3] | 2 [3] |
| 49 | 4 [7] | 6 [10] | 7 [11] | 9 [14] | 10 [15] | | |

| | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ | $t = 5$ | $t > 5$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| 50 | 3 [5] | 8 [14] | 10 [17] | 12 [20] | 14 [23] | | |
| 51 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 52 | 4 [7] | 5 [9] | . | . | . | 5 [9] | 5 [9] |
| 53 | 10 [18] | 15 [25] | 17 [28] | 21 [33] | 23 [36] | | |
| 54 | 9 [16] | 17 [32] | 94 [179] | 675 [1316] | | | |
| 56 | 3 [5] | 5 [9] | 7 [12] | 9 [15] | 11 [18] | | |
| 57 | 11 [20] | 15 [27] | 15 [26] | 24 [42] | 32 [55] | | |
| 58 | 10 [18] | 20 [35] | 33 [55] | 55 [88] | 76 [122] | | |
| 60 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 61 | 5 [9] | 16 [30] | 40 [76] | 94 [177] | 185 [350] | | |
| 62 | 5 [9] | 21 [39] | 61 [114] | 81 [150] | 129 [240] | | |
| 64 | 3 [4] | 1 [1] | . | . | . | 1 [1] | 1 [1] |
| 65 | 9 [15] | 20 [35] | 42 [75] | 88 [157] | 220 [401] | | |
| 66 | 2 [3] | 3 [4] | . | . | . | 3 [4] | 3 [4] |
| 68 | 2 [3] | . | . | . | . | 2 [3] | 2 [3] |
| 69 | 5 [8] | 10 [17] | 12 [19] | 14 [23] | 16 [25] | | |
| 70 | 3 [5] | 8 [14] | 9 [15] | 11 [19] | 11 [19] | | |
| 72 | 5 [9] | 5 [8] | . | . | . | 5 [8] | 5 [8] |
| 73 | 15 [29] | 82 [155] | 390 [757] | 1443 [2796] | | | |
| 74 | 13 [25] | 45 [85] | 66 [123] | 69 [125] | 75 [135] | | |
| 76 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 77 | 11 [20] | 15 [26] | 19 [32] | 23 [38] | 27 [44] | | |
| 78 | 10 [18] | 15 [27] | 18 [30] | 20 [34] | 22 [36] | | |
| 80 | 4 [6] | . | . | . | . | 4 [6] | 4 [6] |
| 81 | 3 [5] | 7 [11] | 9 [14] | 11 [16] | 13 [19] | | |
| 82 | 13 [25] | 167 [331] | 3134 [6257] | | | | |
| 84 | 3 [5] | 7 [12] | 9 [14] | 11 [17] | 13 [19] | | |
| 85 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 86 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 88 | 13 [25] | 63 [117] | 114 [210] | 117 [213] | 1288 [2106] | | |
| 89 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 90 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 92 | 10 [18] | 14 [23] | 18 [29] | 18 [27] | 22 [33] | | |
| 94 | 15 [29] | 230 [455] | 3904 [7760] | | | | |
| 96 | 9 [16] | 11 [17] | 14 [20] | 17 [23] | 20 [26] | | |
| 97 | 14 [27] | 99 [195] | 626 [1237] | | | | |
| 98 | 3 [5] | 4 [6] | 6 [9] | 8 [12] | 10 [15] | | |
| 100 | 5 [9] | 11 [19] | 17 [29] | 18 [29] | 22 [34] | | |
| 102 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 104 | 15 [29] | 265 [525] | 2340 [4647] | 1394 [2675] | 1542 [2913] | | |
| 105 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 106 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 108 | 9 [16] | 11 [19] | . | . | . | 11 [19] | 11 [19] |
| 110 | 5 [9] | 20 [38] | 160 [312] | 1035 [2037] | | | |
| 112 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 113 | 9 [16] | 13 [22] | 17 [28] | 21 [34] | 25 [40] | | |
| 114 | 10 [18] | 20 [35] | 33 [56] | 50 [82] | 72 [115] | | |
| 116 | 3 [5] | 5 [8] | . | . | . | 5 [8] | 5 [8] |
| 118 | 5 [9] | 16 [29] | 49 [92] | 74 [139] | 95 [175] | | |

| | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $t>5$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| 120 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 122 | 15 [29] | 179 [347] | 5088 [9933] | | | | |
| 124 | 5 [9] | 20 [38] | 208 [407] | 1356 [2672] | | | |
| 126 | 3 [5] | 13 [23] | 107 [198] | 2867 [5476] | | | |
| 128 | 4 [6] | 6 [8] | 8 [10] | 10 [12] | 12 [14] | $2t+2[2t+4]$ | 3 [5] |
| 130 | 9 [15] | 14 [21] | 18 [25] | 22 [29] | 26 [33] | | |
| 132 | 5 [9] | 7 [12] | 9 [15] | 11 [18] | 13 [21] | | |
| 134 | 14 [27] | 44 [82] | 99 [182] | 125 [224] | | | |
| 136 | 3 [5] | 4 [6] | 5 [7] | 6 [8] | 7 [9] | $t+2[t+4]$ | 3 [5] |
| 138 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 140 | 4 [7] | 5 [9] | 6 [11] | 7 [13] | 8 [15] | | |
| 142 | 9 [16] | 13 [22] | 17 [28] | 21 [34] | 25 [40] | | |
| 144 | 9 [16] | 16 [28] | 20 [34] | 24 [40] | 28 [46] | | |
| 146 | 15 [29] | 92 [177] | 1587 [3126] | | | | |
| 148 | 14 [27] | 68 [127] | 113 [209] | 188 [347] | | | |
| 150 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 152 | 6 [11] | 20 [37] | 30 [55] | 32 [59] | 36 [65] | | |
| 154 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 156 | 11 [20] | 20 [35] | 24 [42] | 28 [47] | 34 [58] | | |
| 160 | 9 [15] | 16 [24] | 25 [35] | 36 [48] | 49 [63] | $(t+2)^2[(t+2)(t+4)]$ | 9 [15] |
| 162 | 5 [8] | 7 [10] | 9 [12] | 11 [14] | 13 [16] | | |
| 164 | 15 [29] | 116 [227] | 667 [1310] | 1214 [2363] | | | |
| 168 | 4 [7] | 5 [8] | 6 [9] | 7 [10] | 8 [11] | $t+3[t+6]$ | 3 [5] |
| 170 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 172 | 10 [18] | 11 [20] | 12 [22] | 13 [24] | 14 [26] | | |
| 176 | 6 [11] | 8 [14] | 10 [17] | 12 [20] | 14 [23] | | |
| 178 | 11 [20] | 15 [26] | 19 [32] | 23 [38] | 27 [44] | | |
| 180 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 184 | 4 [7] | 6 [10] | 8 [13] | 10 [16] | 12 [19] | | |
| 188 | 5 [9] | 14 [25] | 21 [36] | 25 [43] | 33 [56] | | |
| 192 | 3 [5] | 4 [6] | 5 [7] | 6 [8] | 7 [9] | | |
| 196 | 4 [7] | 5 [8] | 6 [9] | 7 [10] | 8 [11] | | |
| 200 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 204 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |
| 208 | 3 [5] | . | . | . | . | 3 [5] | 3 [5] |
| 212 | 9 [16] | 13 [22] | 17 [28] | 21 [34] | 25 [40] | | |
| 216 | 10 [18] | 11 [19] | 12 [20] | 13 [21] | 14 [22] | | |
| 224 | 4 [7] | 5 [8] | 6 [9] | 7 [10] | 8 [11] | $t+3[t+6]$ | 3 [5] |
| 232 | 11 [20] | 15 [26] | 19 [32] | 23 [38] | 27 [44] | | |
| 240 | 1 [2] | . | . | . | . | 1 [2] | 1 [2] |

**Regular language complexities.**

The set of configurations that can appear after $t$ steps in the evolution of a one-dimensional cellular automaton can be shown to form a regular formal language (see pages 159–202 in this book). Possible configurations thus correspond to possible paths through a finite graph which represents the grammar for the regular language.

The table gives the minimum number of nodes in the graphs for such grammars; the number of arcs is given in brackets in each case. The notation . indicates that the regular language is the same as at the preceding time step.

Entries in the last column of the table give sizes of graphs for regular languages representing limiting sets of states that can be reached after any number of steps.

The size of a regular grammar gives a measure of the "complexity" of the set of configurations it describes. Notice that the grammar specifies merely which configurations can possibly occur; it does not account for the probabilities of different configurations.

The graphs used for the table represent possible sequences of site values that occur in configurations read from left to right. Rules related by reflection may in general yield different regular languages. The table thus includes minimal representatives for all rules from table 1 not related by complementation.

Entries in the table for $t \leq 5$ that have been left blank were not found. They are probably $\gtrsim 20000$. The growth of regular language complexities is bounded by $2^{2^{4t}} - 1$.

For some rules, it has been possible to find explicit forms for the regular languages produced after any number of time steps. Formulae for complexities in these cases are listed in the table. In many cases, it is however suspected that the limiting set does not form a regular language, and may in fact be non-recursive.

Table by Lyman P. Hurd (*Mathematics Department, Princeton University*). (Original program by S. Wolfram.)

## Table 11: Measure Theoretical Complexities

| rule | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0.8223 | 0.8223 | 0.8223 |
| 2 | 0.7356 | 0.7356 | 0.7356 |
| 3 | 0.9003 | 0.9003 | 0.9003 |
| 4 | 0.3768 | 0.3768 | 0.3768 |
| 5 | 1.8005 | 1.8005 | 1.8005 |
| 6 | 1.783 | 1.964 | 1.968 |
| 7 | 1.2707 | 1.670 | 1.933 |
| 8 | 0.7356 | 0 | 0 |
| 9 | 1.9135 | 2.598 | 3.303 |
| 10 | 1.1247 | 1.1247 | 1.1247 |
| 11 | 1.0434 | 1.3442 | 1.973 |
| 12 | 0.5623 | 0.5623 | 0.5623 |
| 13 | 1.4756 | 1.7879 | 1.713 |
| 14 | 0.9026 | 1.3607 | 1.927 |
| 15 | 0 | 0 | 0 |
| 16 | 0.7356 | 0.7356 | 0.7356 |
| 17 | 0.9003 | 0.9003 | 0.9003 |
| 18 | 0.9026 | 2.129 | 3.933 |
| 19 | 0.9026 | 1.1539 | 1.1539 |
| 20 | 1.7756 | 2.759 | 3.059 |
| 21 | 1.2707 | 1.8266 | 1.931 |
| 22 | 2.591 | 4.601 | 6.213 |
| 23 | 1.9862 | 2.153 | 2.330 |
| 24 | 0.5623 | 0.9216 | 0.9216 |
| 25 | 1.643 | 2.665 | 3.231 |
| 26 | 2.244 | 2.659 | 2.945 |
| 27 | 1.666 | 2.128 | 2.392 |
| 28 | 0.8305 | 1.6009 | 1.6645 |
| 29 | 1.2652 | 1.2652 | 1.2652 |
| 30 | 0 | 0 | 0 |
| 32 | 0.3768 | 0.4957 | 0.2346 |
| 33 | 1.2930 | 1.941 | 2.529 |
| 34 | 0.5623 | 0.5623 | 0.5623 |
| 35 | 1.1034 | 1.748 | 2.006 |
| 36 | 0.9003 | 0.4634 | 0.4634 |
| 37 | 2.518 | 4.435 | 5.410 |
| 38 | 1.298 | 1.256 | 1.256 |
| 40 | 1.775 | 1.547 | 1.127 |
| 41 | 2.332 | 4.134 | 5.471 |
| 42 | 0.9003 | 0.9003 | 0.9003 |
| 43 | 1.9584 | 2.269 | 2.483 |
| 44 | 0.9003 | 1.574 | 1.748 |
| 45 | 0 | 0 | 0 |
| 46 | 0.5623 | 0.9216 | 0.9216 |
| 48 | 0.5623 | 0.5623 | 0.5623 |
| 49 | 1.2512 | 1.451 | 1.455 |
| 50 | 0.8305 | 1.589 | 1.775 |
| 51 | 0 | 0 | 0 |

| rule | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|
| 52 | 0.9003 | 0.9216 | 0.9216 |
| 53 | 1.906 | 2.057 | 2.016 |
| 54 | 1.7707 | 2.609 | 3.921 |
| 56 | 0.8305 | 1.3503 | 1.732 |
| 57 | 2.086 | 2.190 | 1.946 |
| 58 | 1.9132 | 2.132 | 2.106 |
| 60 | 0 | 0 | 0 |
| 61 | 1.430 | 2.065 | 3.076 |
| 62 | 1.341 | 2.406 | 3.720 |
| 64 | 0.7356 | 0 | 0 |
| 65 | 1.5310 | 2.268 | 2.970 |
| 66 | 0.5623 | 0.9216 | 0.9216 |
| 68 | 0.5623 | 0.5623 | 0.5623 |
| 69 | 1.0562 | 1.790 | 1.842 |
| 70 | 0.8305 | 1.7802 | 1.815 |
| 72 | 0.9003 | 0.4634 | 0.4634 |
| 73 | 2.604 | 3.685 | 4.473 |
| 74 | 2.461 | 2.713 | 2.755 |
| 76 | 0.8305 | 0.8305 | 0.8305 |
| 77 | 1.9862 | 2.153 | 2.330 |
| 78 | 1.7553 | 2.111 | 2.029 |
| 80 | 1.1247 | 1.1247 | 1.1247 |
| 81 | 1.0434 | 1.5837 | 1.716 |
| 82 | 2.460 | 3.823 | 5.375 |
| 84 | 0.8992 | 1.740 | 1.984 |
| 85 | 0 | 0 | 0 |
| 86 | 0 | 0 | 0 |
| 88 | 2.2441 | 3.081 | 3.605 |
| 89 | 0 | 0 | 0 |
| 90 | 0 | 0 | 0 |
| 92 | 1.9132 | 1.768 | 1.735 |
| 94 | 2.599 | 3.682 | 5.311 |
| 96 | 1.782 | 1.3689 | 0.995 |
| 97 | 2.491 | 3.470 | 4.846 |
| 98 | 0.8305 | 0.8932 | 0.979 |
| 100 | 1.298 | 1.667 | 1.632 |
| 102 | 0 | 0 | 0 |
| 104 | 2.591 | 4.379 | 4.969 |
| 105 | 0 | 0 | 0 |
| 106 | 0 | 0 | 0 |
| 108 | 1.7707 | 1.5093 | 1.5093 |
| 110 | 1.344 | 2.435 | 3.407 |
| 112 | 0.9003 | 0.9003 | 0.9003 |
| 113 | 1.957 | 2.266 | 2.482 |
| 114 | 1.754 | 2.344 | 2.825 |
| 116 | 0.5623 | 0.9215 | 0.9215 |
| 118 | 1.342 | 1.945 | 2.827 |
| 120 | 0 | 0 | 0 |
| 122 | 2.600 | 4.307 | 5.981 |

| rule | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|
| 124 | 1.343 | 2.321 | 3.933 |
| 126 | 0.9003 | 2.049 | 3.914 |
| 128 | 0.8223 | 0.457 | 0.1986 |
| 130 | 1.533 | 1.263 | 1.025 |
| 132 | 1.292 | 1.459 | 1.637 |
| 134 | 2.496 | 3.050 | 3.010 |
| 136 | 0.9003 | 0.8223 | 0.641 |
| 138 | 1.0434 | 1.0434 | 1.0434 |
| 140 | 1.2512 | 1.5808 | 1.7551 |
| 142 | 1.957 | 2.266 | 2.482 |
| 144 | 1.913 | 1.988 | 2.056 |
| 146 | 2.604 | 3.742 | 5.350 |
| 148 | 2.328 | 3.589 | 3.815 |
| 150 | 0 | 0 | 0 |
| 152 | 1.644 | 2.626 | 3.031 |
| 154 | 0 | 0 | 0 |
| 156 | 2.083 | 2.506 | 2.563 |
| 160 | 1.805 | 1.633 | 1.281 |
| 162 | 1.0562 | 0.8654 | 0.7355 |

| rule | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|
| 164 | 2.520 | 3.343 | 3.353 |
| 168 | 1.2707 | 1.3676 | 1.369 |
| 170 | 0 | 0 | 0 |
| 172 | 1.909 | 2.080 | 2.242 |
| 176 | 1.4757 | 1.723 | 1.863 |
| 180 | 0 | 0 | 0 |
| 184 | 1.2652 | 1.575 | 1.788 |
| 188 | 1.433 | 1.962 | 1.733 |
| 192 | 0.9003 | 0.8113 | 0.6415 |
| 196 | 1.1034 | 1.0302 | 0.9170 |
| 200 | 0.9003 | 0.9003 | 0.9003 |
| 204 | 0 | 0 | 0 |
| 208 | 1.0434 | 1.0434 | 1.0434 |
| 212 | 1.9584 | 2.269 | 2.483 |
| 216 | 1.667 | 2.033 | 2.065 |
| 224 | 1.2707 | 1.368 | 1.369 |
| 232 | 1.9862 | 2.153 | 2.330 |
| 240 | 0 | 0 | 0 |

**Measures of the information content of regular grammars for sets of configurations generated by evolution from disordered initial states.**

This table gives values of a probabilistic analogue of the regular language complexity of table 10, in which the nodes of regular language graphs are weighted with the probabilities that they are visited.

Starting from a disordered state in which all possible configurations occur with equal probability, irreversible cellular automaton evolution can lead to ensembles in which different configurations occur with different probabilities. These ensembles can be described by probabilistic analogues of regular languages.

All the configurations that can occur after $t$ steps correspond to possible paths through the standard regular language graphs of table 10. To account for the different probabilities of different configurations, one may weight the nodes of the graph according to the probabilities $P_i$ that they are visited. In terms of these probabilities, one may then compute a measure theoretical complexity $-\sum P_i \log_2 P_i$, where the sum runs over all nodes in the regular language graph. The table gives estimated values for this quantity. The last digit in each estimate is subject to statistical errors.

Table and concept by Peter Grassberger (*Physics Department, University of Wuppertal*).

# Table 12: Iterated Rule Expression Sizes

| rule | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ |
|---|---|---|---|---|---|
| 0 | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
| 1 | 1 (1) | 6 (4) | 1 (1) | 6 (4) | 1 (1) |
| 2 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 3 | 1 (1) | 3 (2) | 1 (1) | 3 (2) | 1 (1) |
| 4 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 5 | 1 (1) | 2 (2) | 1 (1) | 2 (2) | 1 (1) |
| 6 | 2 (2) | 4 (4) | 13 (10) | 25 (21) | 110 (50) |
| 7 | 2 (2) | 7 (4) | 6 (6) | 18 (8) | 10 (10) |
| 8 | 1 (1) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
| 9 | 2 (2) | 5 (5) | 18 (11) | 43 (31) | 138 (53) |
| 10 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 11 | 2 (2) | 5 (4) | 10 (6) | 26 (12) | 50 (16) |
| 12 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 13 | 2 (2) | 6 (4) | 9 (5) | 13 (7) | 17 (8) |
| 14 | 2 (2) | 5 (5) | 17 (10) | 51 (24) | 144 (48) |
| 15 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 18 | 2 (2) | 4 (4) | 18 (18) | 35 (26) | 140 (108) |
| 19 | 2 (2) | 8 (5) | 3 (3) | 8 (5) | 3 (3) |
| 22 | 3 (3) | 7 (7) | 27 (26) | 80 (62) | 308 (206) |
| 23 | 3 (3) | 8 (5) | 7 (7) | 33 (9) | 11 (11) |
| 24 | 2 (2) | 4 (4) | 4 (4) | 4 (4) | 4 (4) |
| 25 | 2 (2) | 4 (4) | 8 (8) | 20 (16) | 42 (27) |
| 26 | 2 (2) | 6 (6) | 21 (17) | 56 (43) | 192 (100) |
| 27 | 3 (2) | 4 (4) | 7 (5) | 7 (6) | 15 (7) |
| 28 | 2 (2) | 4 (4) | 11 (7) | 15 (11) | 30 (12) |
| 29 | 3 (2) | 4 (4) | 3 (2) | 4 (4) | 3 (2) |
| 30 | 3 (3) | 9 (7) | 23 (17) | 76 (41) | 185 (105) |
| 31 | 2 (2) | 6 (4) | 7 (6) | 12 (10) | 11 (9) |
| 32 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 33 | 2 (2) | 7 (7) | 12 (12) | 44 (23) | 38 (24) |
| 34 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 35 | 2 (2) | 4 (3) | 8 (4) | 13 (8) | 30 (11) |
| 36 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 37 | 2 (2) | 8 (7) | 25 (17) | 75 (47) | 238 (109) |
| 38 | 2 (2) | 4 (3) | 4 (4) | 4 (3) | 4 (4) |
| 39 | 3 (2) | 3 (3) | 6 (5) | 5 (5) | 15 (7) |
| 40 | 2 (2) | 3 (3) | 5 (5) | 8 (8) | 13 (13) |
| 41 | 3 (3) | 8 (8) | 26 (24) | 92 (69) | 283 (218) |
| 42 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 43 | 3 (3) | 7 (6) | 24 (12) | 62 (27) | 176 (55) |
| 44 | 2 (2) | 3 (3) | 4 (4) | 8 (5) | 10 (6) |
| 45 | 3 (3) | 9 (8) | 24 (20) | 72 (53) | 219 (118) |
| 46 | 3 (2) | 6 (4) | 6 (4) | 6 (4) | 6 (4) |
| 47 | 2 (2) | 5 (4) | 13 (6) | 28 (8) | 64 (16) |
| 50 | 2 (2) | 6 (4) | 15 (6) | 31 (8) | 64 (10) |
| 51 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 54 | 3 (3) | 7 (6) | 18 (15) | 59 (38) | 165 (85) |
| 55 | 2 (2) | 5 (3) | 5 (5) | 5 (3) | 5 (5) |

| rule | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ |
|---|---|---|---|---|---|
| 56 | 2 (2) | 6 (4) | 14 (9) | 38 (20) | 103 (45) |
| 57 | 3 (3) | 7 (6) | 17 (12) | 41 (23) | 130 (50) |
| 58 | 2 (2) | 6 (5) | 15 (10) | 34 (18) | 80 (32) |
| 59 | 2 (2) | 4 (4) | 9 (8) | 14 (10) | 34 (17) |
| 60 | 2 (2) | 2 (2) | 8 (8) | 2 (2) | 8 (8) |
| 61 | 3 (3) | 4 (4) | 14 (10) | 21 (17) | 60 (30) |
| 62 | 3 (3) | 6 (6) | 20 (12) | 56 (27) | 137 (48) |
| 63 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 72 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 73 | 3 (3) | 8 (7) | 36 (20) | 90 (46) | 276 (118) |
| 74 | 2 (2) | 5 (5) | 13 (11) | 30 (22) | 77 (45) |
| 75 | 3 (3) | 8 (8) | 24 (20) | 81 (52) | 241 (118) |
| 76 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 77 | 3 (3) | 7 (5) | 14 (7) | 32 (9) | 57 (11) |
| 78 | 2 (2) | 5 (4) | 8 (7) | 20 (10) | 21 (12) |
| 79 | 2 (2) | 7 (4) | 9 (5) | 13 (6) | 20 (7) |
| 90 | 2 (2) | 2 (2) | 8 (8) | 2 (2) | 8 (8) |
| 91 | 3 (3) | 9 (6) | 26 (19) | 82 (47) | 255 (107) |
| 94 | 3 (3) | 8 (8) | 26 (19) | 106 (46) | 276 (106) |
| 95 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 104 | 3 (3) | 6 (6) | 15 (14) | 27 (26) | 49 (45) |
| 105 | 4 (4) | 4 (4) | 16 (16) | 4 (4) | 256 (256) |
| 106 | 3 (3) | 5 (5) | 25 (21) | 46 (37) | 192 (126) |
| 107 | 4 (4) | 10 (9) | 37 (28) | 108 (70) | 390 (210) |
| 108 | 3 (3) | 5 (5) | 9 (8) | 5 (5) | 9 (8) |
| 109 | 4 (4) | 10 (8) | 31 (20) | 91 (54) | 268 (118) |
| 110 | 3 (3) | 7 (6) | 15 (15) | 40 (28) | 95 (60) |
| 111 | 3 (3) | 7 (6) | 21 (14) | 57 (25) | 139 (56) |
| 122 | 3 (3) | 9 (8) | 27 (20) | 88 (48) | 264 (136) |
| 123 | 3 (3) | 8 (8) | 22 (13) | 51 (28) | 81 (30) |
| 126 | 3 (3) | 8 (8) | 22 (19) | 103 (67) | 221 (116) |
| 127 | 3 (3) | 3 (3) | 3 (3) | 3 (3) | 3 (3) |
| 128 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 129 | 2 (2) | 9 (8) | 26 (20) | 93 (78) | 250 (120) |
| 130 | 2 (2) | 2 (2) | 5 (4) | 5 (4) | 8 (6) |
| 131 | 2 (2) | 6 (5) | 13 (10) | 36 (25) | 88 (45) |
| 132 | 2 (2) | 3 (3) | 4 (4) | 5 (5) | 6 (6) |
| 133 | 2 (2) | 8 (7) | 23 (17) | 74 (41) | 216 (111) |
| 134 | 3 (3) | 6 (6) | 20 (17) | 46 (34) | 174 (90) |
| 135 | 3 (3) | 9 (7) | 22 (17) | 66 (41) | 202 (107) |
| 136 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 137 | 2 (2) | 8 (6) | 14 (14) | 39 (25) | 111 (60) |
| 138 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 139 | 2 (2) | 5 (4) | 6 (4) | 6 (4) | 6 (4) |
| 140 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 141 | 2 (2) | 7 (4) | 10 (6) | 22 (8) | 28 (9) |
| 142 | 3 (3) | 7 (6) | 18 (12) | 52 (27) | 151 (55) |
| 143 | 2 (2) | 5 (4) | 12 (8) | 32 (15) | 86 (34) |

| rule | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ |
|---|---|---|---|---|---|
| 146 | 3 (3) | 6 (6) | 29 (29) | 61 (48) | 224 (193) |
| 147 | 3 (3) | 8 (7) | 21 (18) | 69 (39) | 207 (79) |
| 150 | 4 (4) | 4 (4) | 16 (16) | 4 (4) | 256 (256) |
| 151 | 4 (4) | 10 (7) | 33 (29) | 104 (63) | 372 (201) |
| 152 | 2 (2) | 4 (4) | 7 (7) | 12 (11) | 20 (19) |
| 153 | 2 (2) | 2 (2) | 8 (8) | 2 (2) | 8 (8) |
| 154 | 3 (3) | 4 (4) | 28 (15) | 6 (6) | 42 (19) |
| 155 | 3 (3) | 7 (4) | 8 (5) | 8 (4) | 8 (5) |
| 156 | 3 (3) | 3 (3) | 14 (8) | 12 (8) | 43 (13) |
| 157 | 3 (3) | 7 (4) | 11 (7) | 24 (8) | 23 (10) |
| 158 | 4 (4) | 12 (8) | 34 (20) | 106 (37) | 330 (92) |
| 159 | 3 (3) | 9 (7) | 18 (14) | 63 (24) | 139 (55) |
| 160 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 161 | 2 (2) | 6 (6) | 20 (18) | 65 (43) | 236 (140) |
| 162 | 2 (2) | 3 (3) | 4 (4) | 5 (5) | 6 (6) |
| 163 | 2 (2) | 5 (5) | 15 (9) | 32 (15) | 73 (25) |
| 164 | 2 (2) | 5 (4) | 10 (10) | 16 (13) | 27 (21) |
| 165 | 2 (2) | 2 (2) | 8 (8) | 2 (2) | 8 (8) |
| 166 | 3 (3) | 4 (4) | 24 (15) | 6 (6) | 32 (19) |
| 167 | 3 (3) | 8 (7) | 26 (19) | 82 (43) | 218 (104) |
| 168 | 2 (2) | 5 (4) | 10 (8) | 23 (16) | 49 (32) |
| 169 | 3 (3) | 6 (5) | 28 (21) | 76 (37) | 244 (124) |
| 170 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 171 | 3 (2) | 3 (2) | 3 (2) | 3 (2) | 3 (2) |
| 172 | 2 (2) | 4 (3) | 6 (4) | 7 (5) | 9 (6) |
| 173 | 3 (3) | 10 (7) | 32 (14) | 70 (27) | 206 (46) |
| 174 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 175 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 178 | 3 (3) | 7 (5) | 16 (7) | 32 (9) | 65 (11) |
| 179 | 2 (2) | 4 (4) | 8 (6) | 15 (8) | 31 (10) |
| 182 | 4 (4) | 11 (9) | 47 (33) | 103 (55) | 466 (162) |
| 183 | 3 (3) | 7 (7) | 24 (22) | 55 (22) | 203 (69) |

| rule | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ |
|---|---|---|---|---|---|
| 184 | 2 (2) | 5 (5) | 15 (13) | 48 (37) | 161 (111) |
| 185 | 3 (3) | 5 (5) | 24 (10) | 51 (22) | 149 (45) |
| 186 | 3 (2) | 8 (3) | 20 (4) | 43 (5) | 88 (6) |
| 187 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 188 | 3 (3) | 6 (5) | 24 (10) | 39 (15) | 125 (23) |
| 189 | 3 (3) | 10 (5) | 8 (5) | 8 (5) | 8 (5) |
| 190 | 4 (3) | 5 (4) | 23 (6) | 21 (7) | 91 (9) |
| 191 | 3 (3) | 3 (3) | 3 (3) | 3 (3) | 3 (3) |
| 200 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 201 | 3 (3) | 10 (5) | 23 (10) | 10 (5) | 23 (10) |
| 202 | 2 (2) | 4 (4) | 7 (6) | 11 (8) | 16 (10) |
| 203 | 3 (3) | 7 (5) | 17 (9) | 41 (13) | 66 (19) |
| 204 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 205 | 3 (2) | 3 (2) | 3 (2) | 3 (2) | 3 (2) |
| 206 | 3 (2) | 5 (3) | 8 (4) | 11 (5) | 15 (6) |
| 207 | 2 (2) | 2 (2) | 2 (2) | 2 (2) | 2 (2) |
| 218 | 3 (3) | 9 (6) | 40 (16) | 92 (24) | 158 (38) |
| 219 | 3 (3) | 10 (5) | 10 (5) | 10 (5) | 10 (5) |
| 222 | 4 (3) | 10 (5) | 19 (7) | 31 (9) | 46 (11) |
| 223 | 3 (3) | 3 (3) | 3 (3) | 3 (3) | 3 (3) |
| 232 | 3 (3) | 8 (5) | 19 (7) | 33 (9) | 58 (11) |
| 233 | 4 (4) | 10 (7) | 39 (20) | 112 (34) | 307 (63) |
| 234 | 2 (2) | 6 (4) | 17 (7) | 44 (12) | 106 (21) |
| 235 | 4 (3) | 11 (6) | 28 (10) | 62 (14) | 134 (19) |
| 236 | 3 (2) | 3 (2) | 3 (2) | 3 (2) | 3 (2) |
| 237 | 4 (3) | 7 (5) | 7 (5) | 7 (5) | 7 (5) |
| 238 | 2 (2) | 4 (3) | 6 (4) | 9 (5) | 12 (6) |
| 239 | 3 (3) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |
| 250 | 2 (2) | 3 (3) | 4 (4) | 5 (5) | 6 (6) |
| 251 | 3 (3) | 6 (5) | 10 (7) | 19 (9) | 28 (11) |
| 254 | 4 (3) | 11 (5) | 24 (7) | 45 (9) | 76 (11) |
| 255 | 1 (1) | 1 (1) | 1 (1) | 1 (1) | 1 (1) |

**Sizes of Boolean expressions representing functions corresponding to iterations of cellular automaton rules.**

Cellular automaton rules with $k = 2$ and $r = 1$ can be expressed as Boolean functions of three variables, as in table 1. Iterations of these rules for $t$ steps correspond to functions of $2t + 1$ variables, which may be expressed as Boolean expressions.

The minimal Boolean expressions obtained after one step were given in table 1. This table gives the numbers of terms in Boolean expressions obtained after $t$ time steps. An increase in these numbers potentially reflects increasing difficulty of computing the outcome of more steps of cellular automaton evolution.

The first number in each case gives the number of prime implicants in the corresponding Boolean expression. The possible values of a set of $n$ Boolean variables correspond to the vertices of a Boolean $n$-cube. The cases in which a Boolean function has value 1 then correspond to a region of the Boolean $n$-cube. The number of prime implicants is essentially the number of hyperplanes of various dimensions which must be combined to form this region.

Boolean expressions can conveniently be stated in a disjunctive normal form (DNF), in which they are written as a disjunction (OR) of conjunctions (ANDs). The number of prime implicants gives an upper bound on the number of terms needed in such a form.

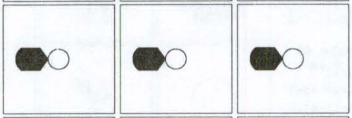
Notice that complementation of a function has no simple effect on its DNF expression. As a result, the table includes minimal representatives for all rules from table 1 not related by reflection.

The general problem of finding an absolutely minimal DNF representation for a function appears to be computationally intractable. The table gives in parentheses the numbers of terms in minimal DNF expressions found by the *espresso* computer program (R. Rudell, Computer Science Department, University of California, Berkeley, 1985) which incorporates known algebraic and heuristic techniques. In most cases, the results given are probably absolutely minimal.

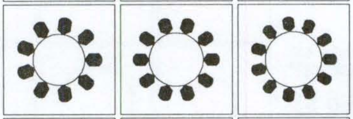
# Table 13: Finite Lattice State Transition Diagrams

rule 0:
N=9:1x1
N=10:1x1
N=11:1x1

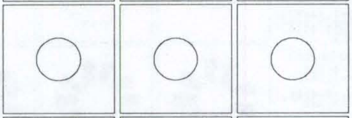
rule 1:
N=9:37x2
N=10:61x2
N=11:100x2



rule 2:
N=9:3x9, 1x3,1x1
N=10:4x10, 1x5,1x1
N=11:6x11, 1x1

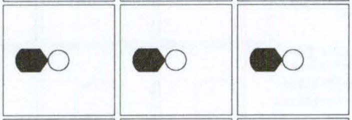


rule 3:
N=9:8x18, 1x9,1x3, 1x2
N=10:13x20, 1x10,1x5, 1x2
N=11:21x22, 2x11,1x2

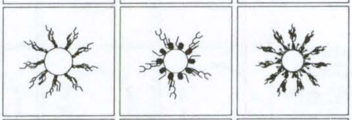


rule 4:
N=9:76x1
N=10:123x1
N=11:199x1

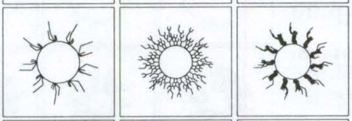
rule 5:
N=9:73x2, 12x1
N=10:136x2, 17x1
N=11:232x2, 22x1



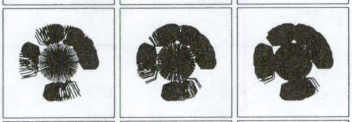
rule 6:
N=9:3x18, 1x1
N=10:11x10, 4x5,3x1
N=11:5x22, 1x1



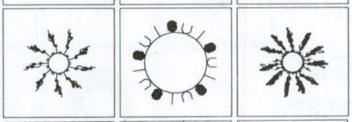
rule 7:
N=9:2x18, 1x9,1x2
N=10:3x20, 1x5,1x2, 2x1
N=11:4x22, 1x11,1x2



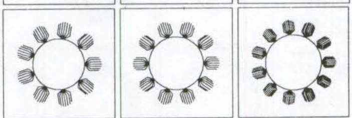
rule 8:
N=9:1x1
N=10:1x1
N=11:1x1



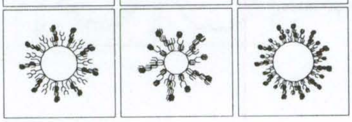
rule 9:
N=9:1x18, 3x3,1x2
N=10:2x15, 2x10,2x5, 1x2
N=11:2x22, 1x2



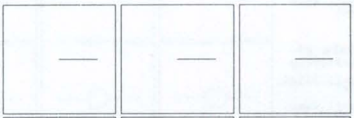
rule 10:
N=9:8x9, 1x3,1x1
N=10:11x10, 2x5,1x1
N=11:18x11, 1x1



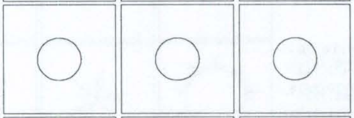
rule 11:
N=9:4x18, 1x3,1x2
N=10:12x10, 2x5,1x2
N=11:9x22, 1x11,1x2



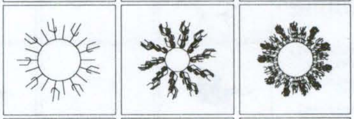
rule 12:
N=9:76x1
N=10:123x1
N=11:199x1



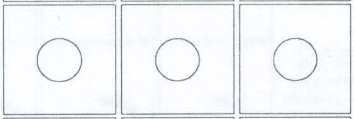
rule 13:
N=9:1x2, 12x1
N=10:1x2, 17x1
N=11:1x2, 22x1



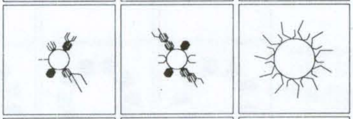
rule 14:
N=9:4x18, 2x9,1x6, 1x1
N=10:13x10, 3x5,3x1
N=11:9x22, 3x11,1x1



rule 15:
N=9:28x18, 1x6,1x2
N=10:99x10, 6x5,1x2, 2x1
N=11:93x22, 1x2

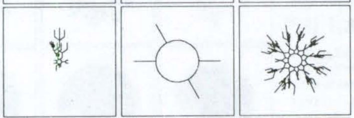


rule 18:
N=9:9x6, 9x2,1x1
N=10:5x6, 5x4,15x2, 1x1
N=11:2x11, 11x4,11x2, 1x1

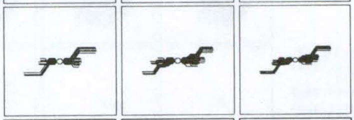


rule 19:
N=9:37x2
N=10:61x2
N=11:100x2

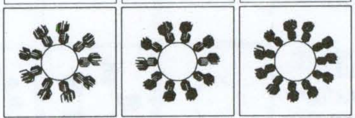
rule 22:
N=9:9x4, 1x1
N=10:10x6, 15x4,3x1
N=11:2x11, 11x5,11x4, 1x1



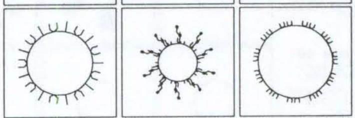
rule 23:
N=9:37x2
N=10:61x2, 2x1
N=11:100x2



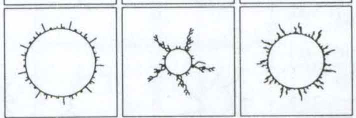
rule 24:
N=9:3x9, 1x3,1x1
N=10:4x10, 1x5,1x1
N=11:6x11, 1x1



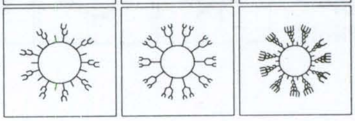
rule 25:
N=9:1x18, 1x9,1x3, 1x2
N=10:2x30, 2x20,1x15, 1x2
N=11:1x33, 2x22,1x11, 1x2

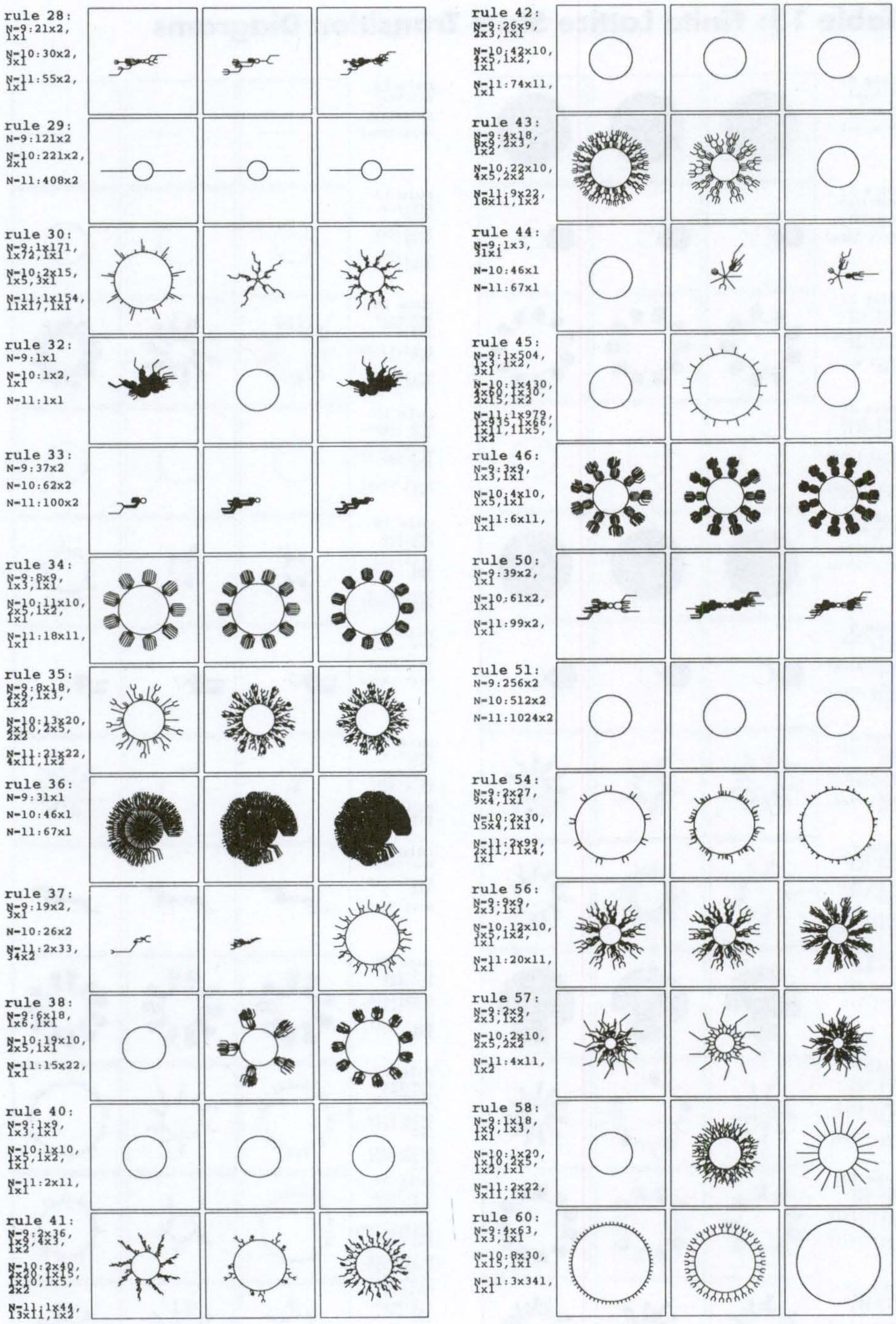


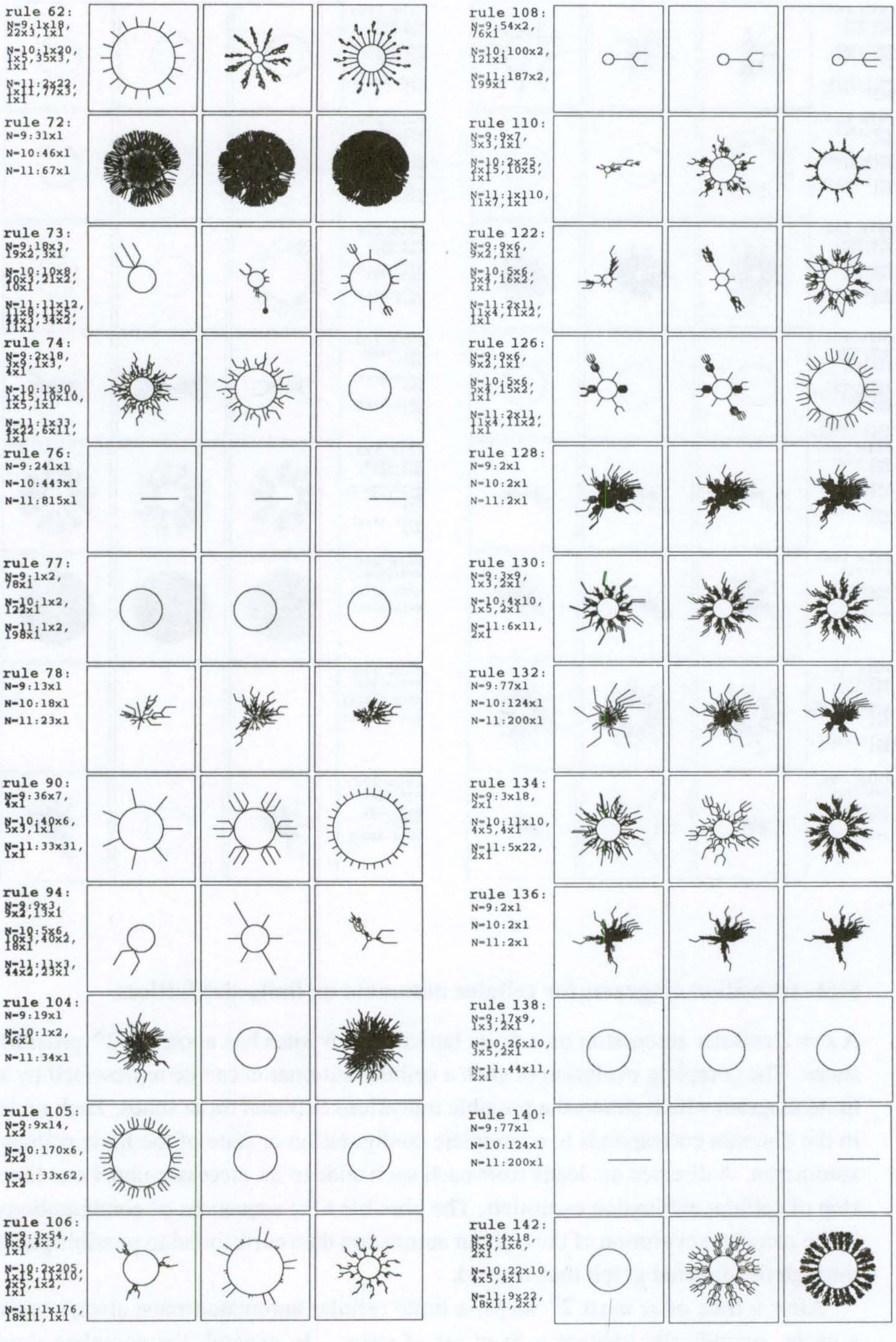
rule 26:
N=9:2x72, 1x18,1x6, 1x1
N=10:12x20, 5x6,1x5, 1x1
N=11:4x88, 4x44,1x33, 1x22,1x11, 1x1

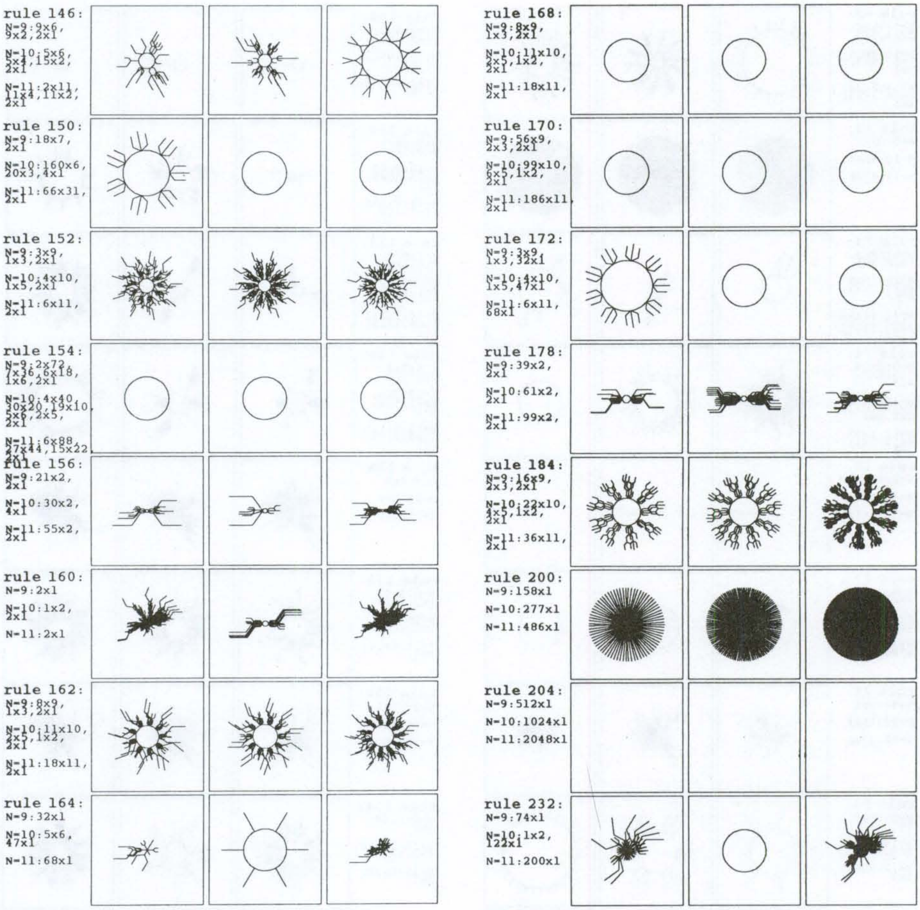


rule 27:
N=9:9x18, 1x9,1x6, 1x2
N=10:13x20, 1x10,1x5, 1x2
N=11:21x22, 2x11,1x2




rule 28: N=9:21x2, 1x1 N=10:30x2, 3x1 N=11:55x2, 1x1
rule 29: N=9:121x2 N=10:221x2, 2x1 N=11:408x2
rule 30: N=9:1x171, 1x72, 1x1 N=10:2x15, 1x5, 3x1 N=11:1x154, 11x17, 1x1
rule 32: N=9:1x1 N=10:1x2, 1x1 N=11:1x1
rule 33: N=9:37x2 N=10:62x2 N=11:100x2
rule 34: N=9:8x9, 1x3, 1x1 N=10:11x10, 2x5, 1x2, 1x1 N=11:18x11, 1x1
rule 35: N=9:8x18, 2x9, 1x3, 1x2 N=10:13x20, 2x10, 2x5, 2x2 N=11:21x22, 4x11, 1x2
rule 36: N=9:31x1 N=10:46x1 N=11:67x1
rule 37: N=9:19x2, 3x1 N=10:26x2 N=11:2x33, 34x2
rule 38: N=9:6x18, 1x6, 1x1 N=10:19x10, 2x5, 1x1 N=11:15x22, 1x1
rule 40: N=9:1x9, 1x3, 1x1 N=10:1x10, 1x5, 1x2, 1x1 N=11:2x11, 1x1
rule 41: N=9:2x36, 1x9, 4x3, 1x2 N=10:2x40, 2x20, 1x15, 1x10, 1x5, 2x2 N=11:1x44, 13x11, 1x2
rule 42: N=9:26x9, 2x3, 1x1 N=10:42x10, 4x5, 1x2, 1x1 N=11:74x11, 1x1
rule 43: N=9:4x18, 8x9, 2x3, 1x2 N=10:22x10, 4x5, 2x2 N=11:9x22, 18x11, 1x2
rule 44: N=9:1x3, 31x1 N=10:46x1 N=11:67x1
rule 45: N=9:1x504, 1x3, 1x2, 3x1 N=10:1x430, 4x60, 1x30, 2x15, 1x2 N=11:1x979, 1x935, 1x66, 1x11, 11x5, 1x2
rule 46: N=9:3x9, 1x3, 1x1 N=10:4x10, 1x5, 1x1 N=11:6x11, 1x1
rule 50: N=9:39x2, 1x1 N=10:61x2, 1x1 N=11:99x2, 1x1
rule 51: N=9:256x2 N=10:512x2 N=11:1024x2
rule 54: N=9:2x27, 9x4, 1x1 N=10:2x30, 15x4, 1x1 N=11:2x99, 2x11, 11x4, 1x1
rule 56: N=9:9x9, 2x3, 1x1 N=10:12x10, 3x5, 1x2, 1x1 N=11:20x11, 1x1
rule 57: N=9:2x9, 2x3, 1x2 N=10:2x10, 2x5, 2x2 N=11:4x11, 1x2
rule 58: N=9:1x18, 1x9, 1x3, 1x1 N=10:1x20, 1x10, 2x5, 1x2, 1x1 N=11:2x22, 3x11, 1x1
rule 60: N=9:4x63, 1x3, 1x1 N=10:8x30, 1x15, 1x1 N=11:3x341, 1x1


rule 62:
N=9:1x18, 22x3,1x1
N=10:1x20, 1x5,35x3, 1x1
N=11:2x22, 1x11,77x3, 1x1
rule 72:
N=9:31x1
N=10:46x1
N=11:67x1
rule 73:
N=9:18x3, 19x2,3x1
N=10:10x8, 20x3,21x2, 10x1
N=11:11x12, 11x8,11x5, 44x3,34x2, 11x1
rule 74:
N=9:2x18, 3x9,1x3, 4x1
N=10:1x30, 1x15,10x10, 1x5,1x1
N=11:1x33, 4x22,6x11, 1x1
rule 76:
N=9:241x1
N=10:443x1
N=11:815x1
rule 77:
N=9:1x2, 78x1
N=10:1x2, 122x1
N=11:1x2, 198x1
rule 78:
N=9:13x1
N=10:18x1
N=11:23x1
rule 90:
N=9:36x7, 4x1
N=10:40x6, 5x3,1x1
N=11:33x31, 1x1
rule 94:
N=9:9x3, 9x2,13x1
N=10:5x6, 10x3,40x2, 18x1
N=11:11x3, 44x2,23x1
rule 104:
N=9:19x1
N=10:1x2, 26x1
N=11:34x1
rule 105:
N=9:9x14, 1x2
N=10:170x6, 2x2
N=11:33x62, 1x2
rule 106:
N=9:3x54, 8x9,4x3, 1x1
N=10:2x205, 1x15,11x10, 2x5,1x2, 1x1
N=11:1x176, 18x11,1x1
rule 108:
N=9:54x2, 76x1
N=10:100x2, 121x1
N=11:187x2, 199x1
rule 110:
N=9:9x7, 3x3,1x1
N=10:2x25, 2x15,10x5, 1x1
N=11:1x110, 11x7,1x1
rule 122:
N=9:9x6, 9x2,1x1
N=10:5x6, 5x4,16x2, 1x1
N=11:2x11, 11x4,11x2, 1x1
rule 126:
N=9:9x6, 9x2,1x1
N=10:5x6, 5x4,15x2, 1x1
N=11:2x11, 11x4,11x2, 1x1
rule 128:
N=9:2x1
N=10:2x1
N=11:2x1
rule 130:
N=9:3x9, 1x3,2x1
N=10:4x10, 1x5,2x1
N=11:6x11, 2x1
rule 132:
N=9:77x1
N=10:124x1
N=11:200x1
rule 134:
N=9:3x18, 2x1
N=10:11x10, 4x5,4x1
N=11:5x22, 2x1
rule 136:
N=9:2x1
N=10:2x1
N=11:2x1
rule 138:
N=9:17x9, 1x3,2x1
N=10:26x10, 3x5,2x1
N=11:44x11, 2x1
rule 140:
N=9:77x1
N=10:124x1
N=11:200x1
rule 142:
N=9:4x18, 8x9,1x6, 2x1
N=10:22x10, 4x5,4x1
N=11:9x22, 18x11,2x1




**State transition diagrams for cellular automata on finite size lattices.**

A $k = 2$ cellular automaton on a finite lattice with $N$ sites has a total of $2^N$ possible states. The complete evolution of such a cellular automaton can be represented by a finite diagram which shows the possible transitions between these states. Each node in the diagram corresponds to a complete configuration or state of the finite cellular automaton. A directed arc leads from each such node to its successor under one time step of cellular automaton evolution. The possible time sequences of configurations in the complete evolution of the cellular automaton then correspond to possible paths through the directed graph thus formed.

After a time of at most $2^N$ steps, a finite cellular automaton must always enter a cycle, periodically visiting a fixed set of states. In general, the complete state transition diagram contains a number of distinct cycles.

The table shows the fragment of the state transition diagram associated with the longest cycle, for all inequivalent $k = 2$, $r = 1$ rules. Results are given for lattices of sizes $N = 9$, $N = 10$ and $N = 11$. In all cases, the lattices are taken to have periodic boundary conditions, as if their sites were arrranged in a circle.

The table also gives the lengths and multiplicities of all the cycles for each rule. (The notation used is $g \times L$, representing $g$ cycles of length $L$.) Notice that the state transition diagram fragments associated with different cycles of the same length may not be identical. When there are several cycles of maximal length, the fragment shown is the one involving the largest total number of states.

State transition diagram fragments have the general form of cycles fed by trees. The cellular automaton always reaches the cycle after a sufficiently long time. The trees represent transients, and contain states which can occur only after a limited number of time steps. Such transient phenomena are a manifestation of irreversibility in the cellular automaton evolution.

Some finite cellular automata, such as rule 13, are reversible, so that their state transition diagrams contain no transients, and all states are on cycles.

In some other cases, such as rule 90, highly regular state transition diagrams are obtained, containing for example only balanced trees (see pages 159–202 in this book). Many rules, however, yield complicated state transition diagrams.

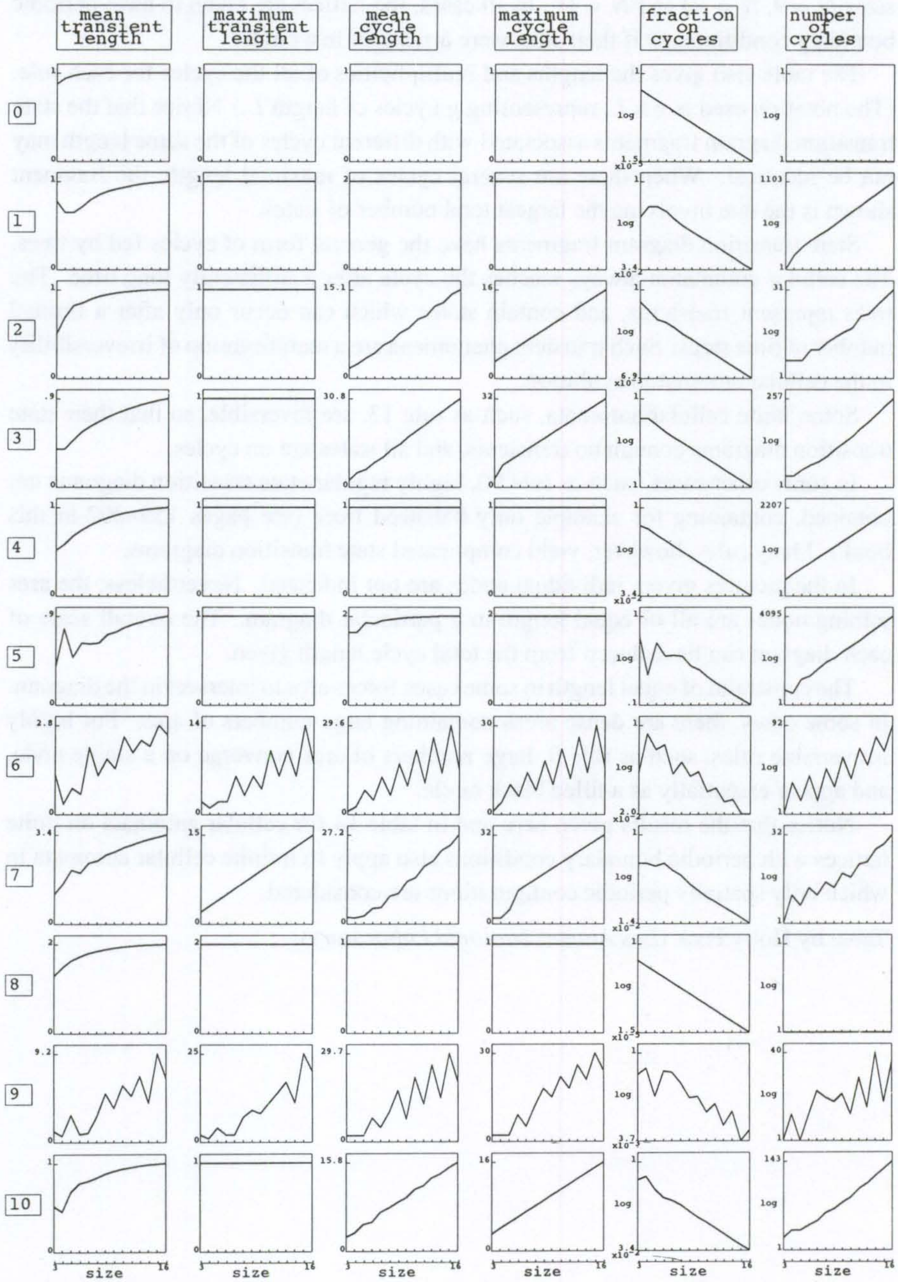
In the pictures given, individual nodes are not indicated. Nevertheless, the arcs joining nodes are all of equal length in a particular diagram. The overall scale of each diagram can be deduced from the total cycle length given.

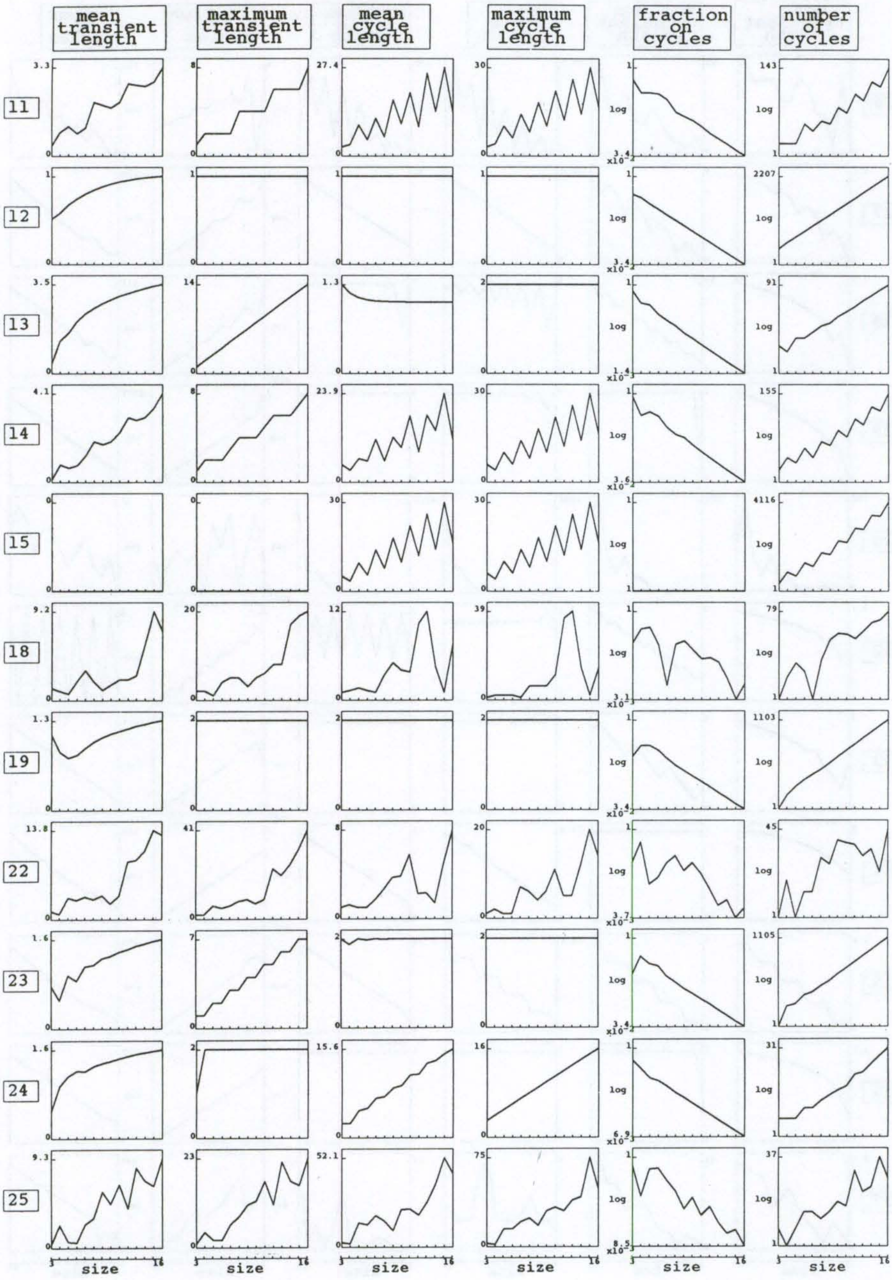
The constraint of equal length in some cases forces arcs to intersect in the diagram. In some cases, there are dense areas containing large numbers of arcs. For highly irreversible rules, such as rule 0, large numbers of arcs converge on a single node, and appear essentially as a filled black circle.

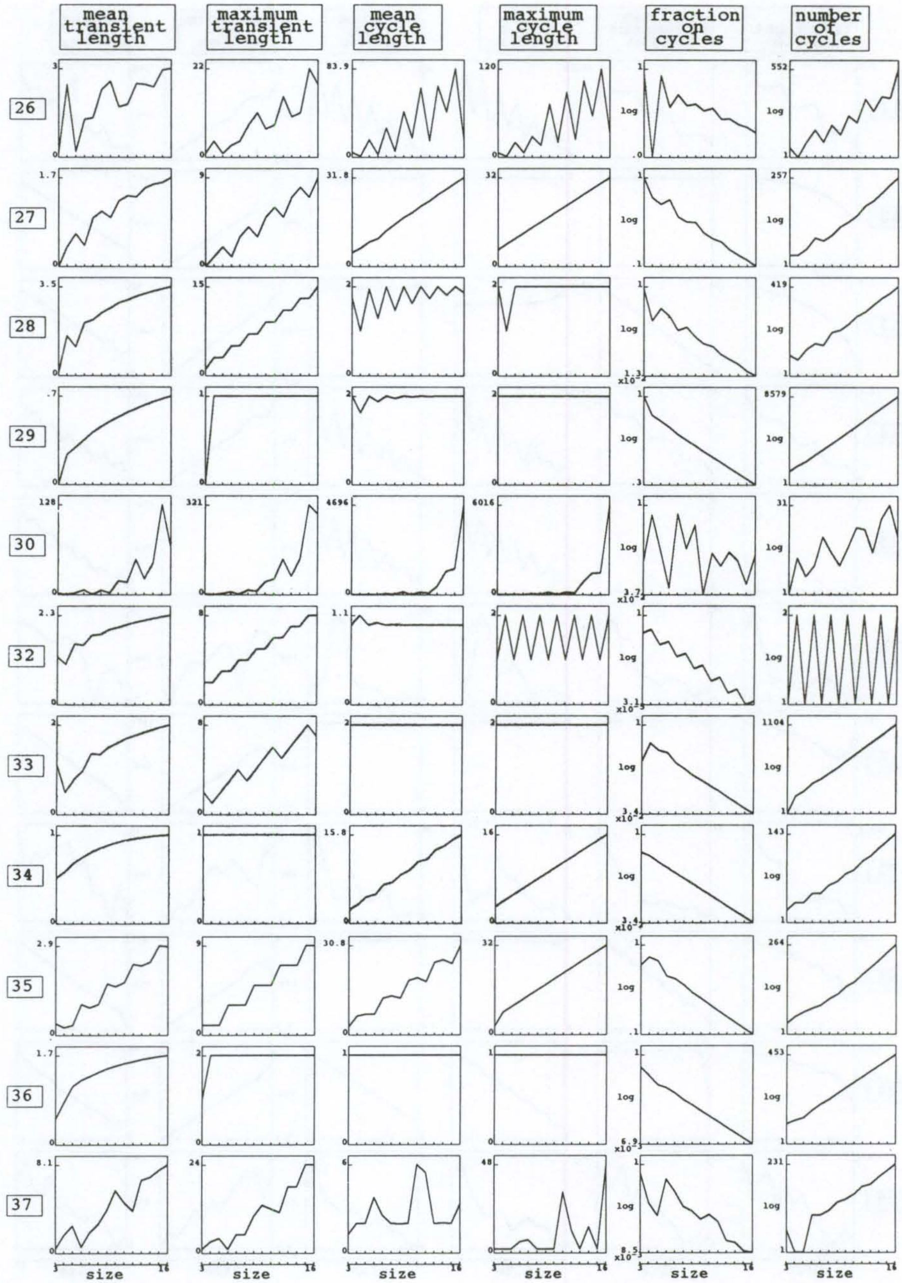
Notice that the results given here and in table 14 for cellular automata on finite lattices with periodic boundary conditions also apply to infinite cellular automata in which only spatially periodic configurations are considered.

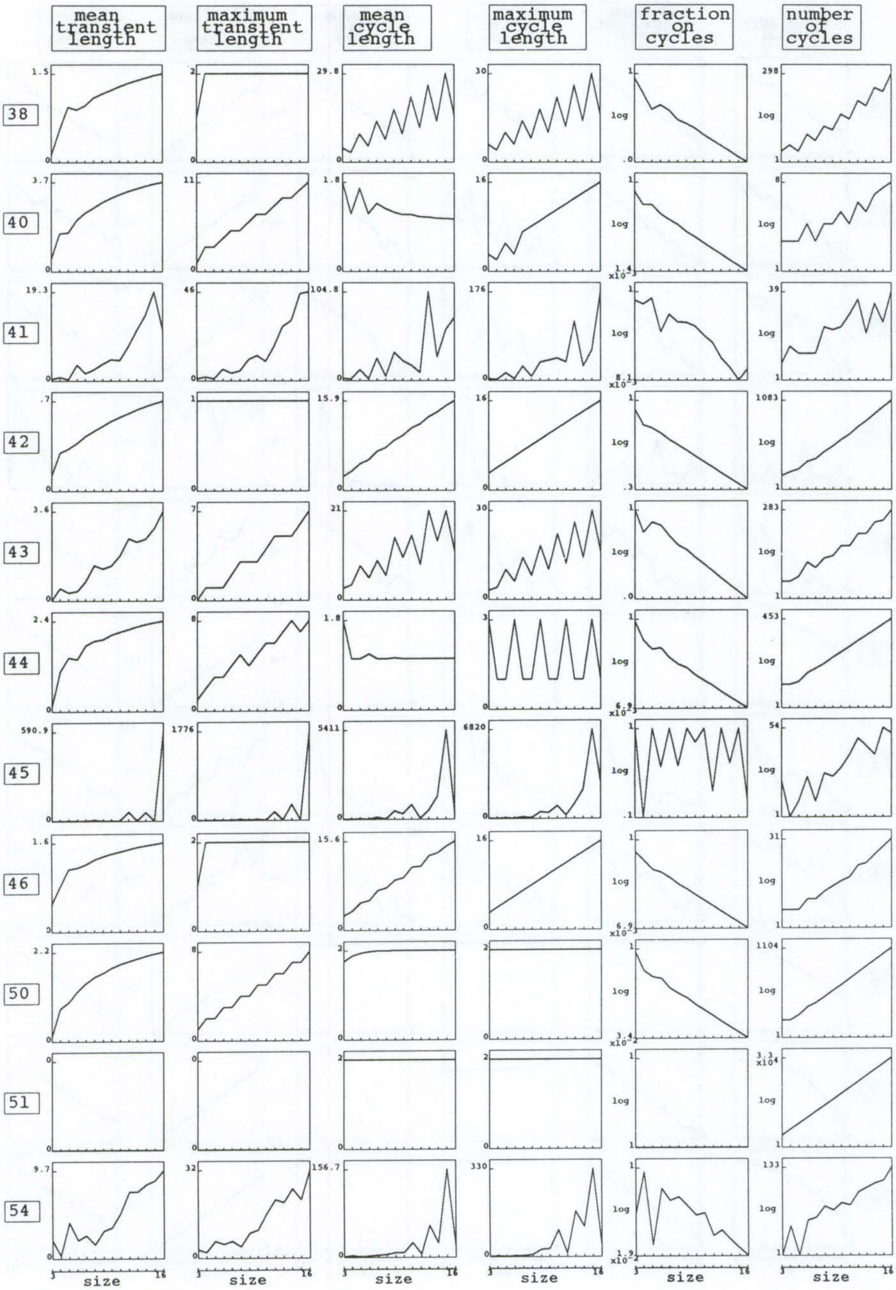
Table by Holly Peck (*Los Alamos National Laboratory*).

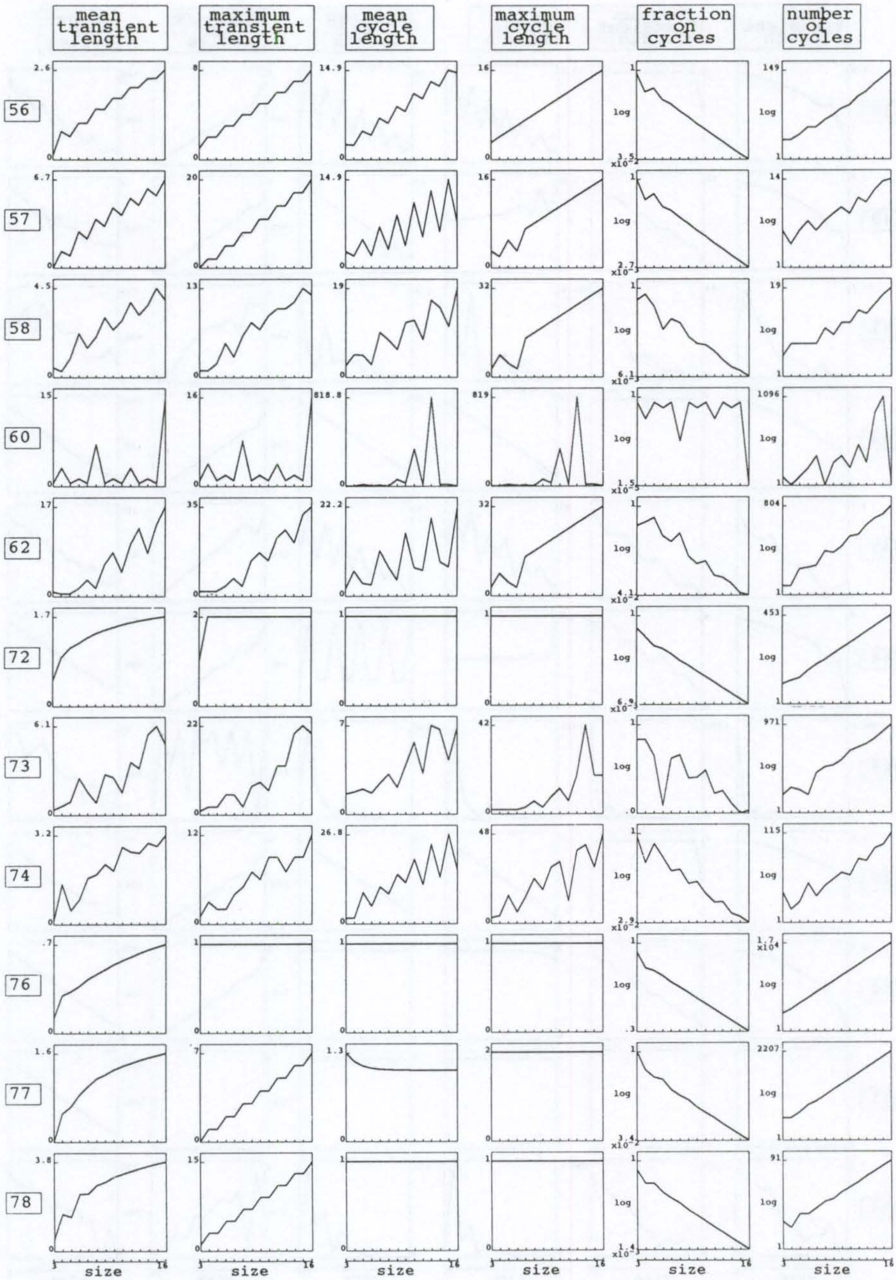
# Table 14: Global Properties for Finite Lattices

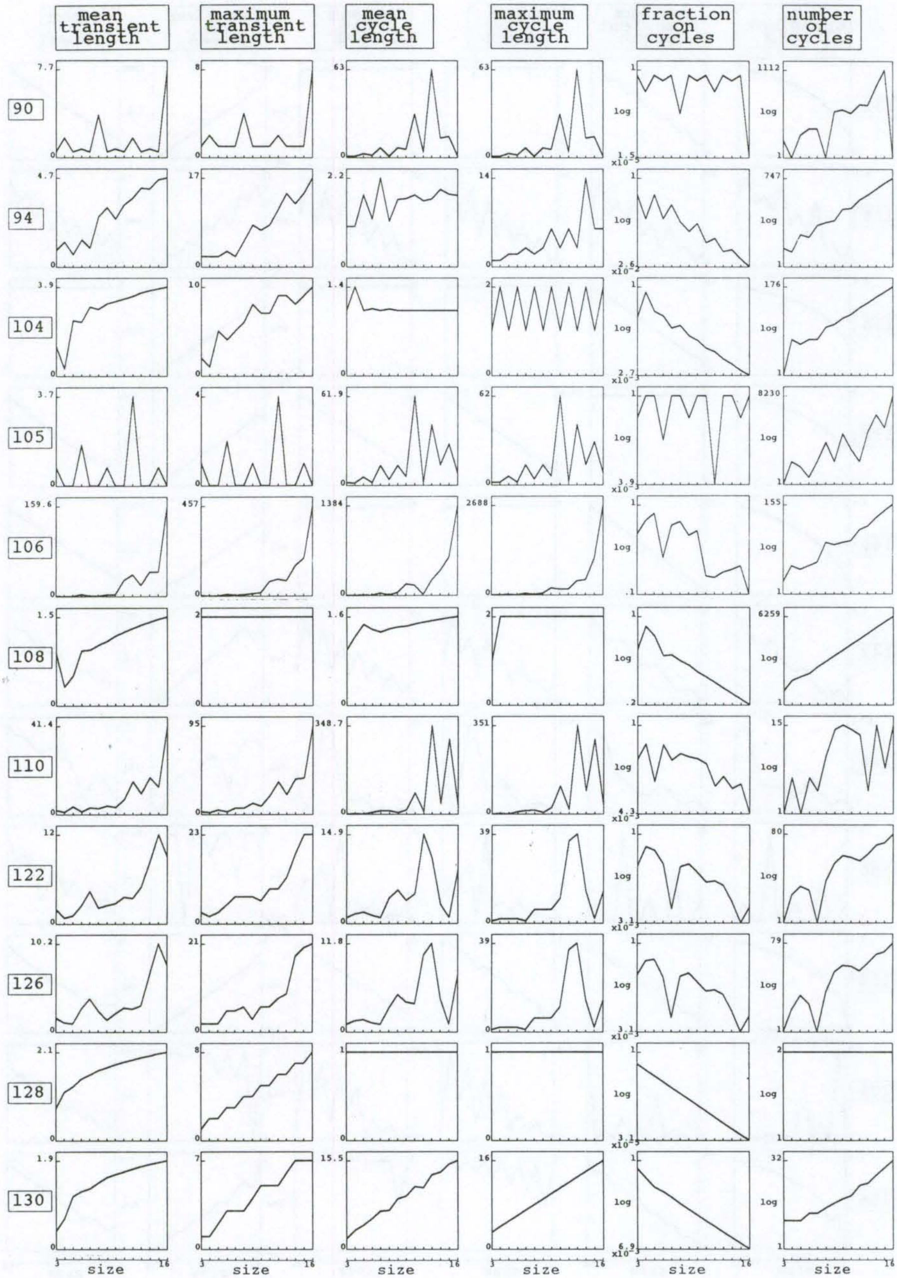



mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
11
12
13
14
15
18
19
22
23
24
25
log
size

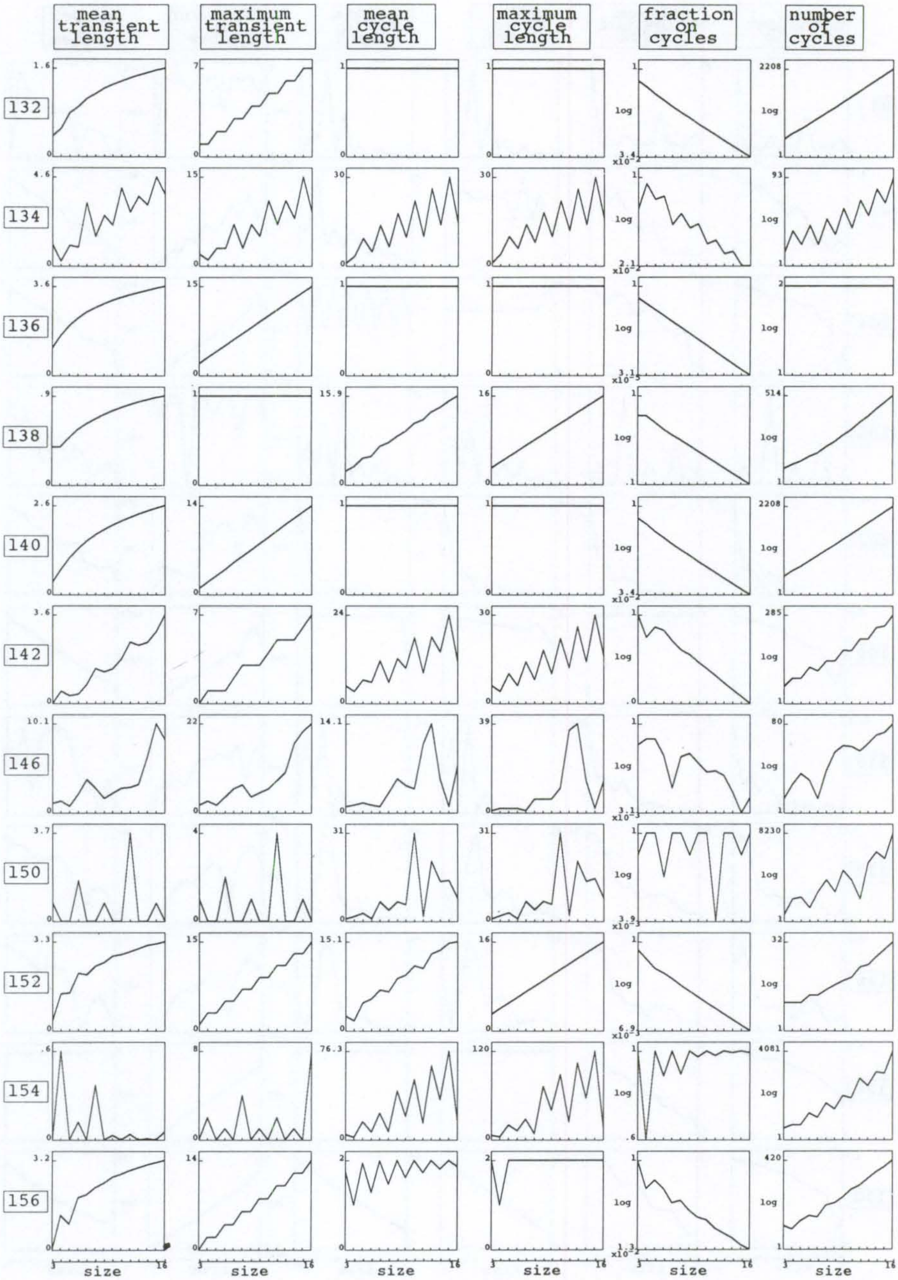

mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
26
27
28
29
30
32
33
34
35
36
37
log
size

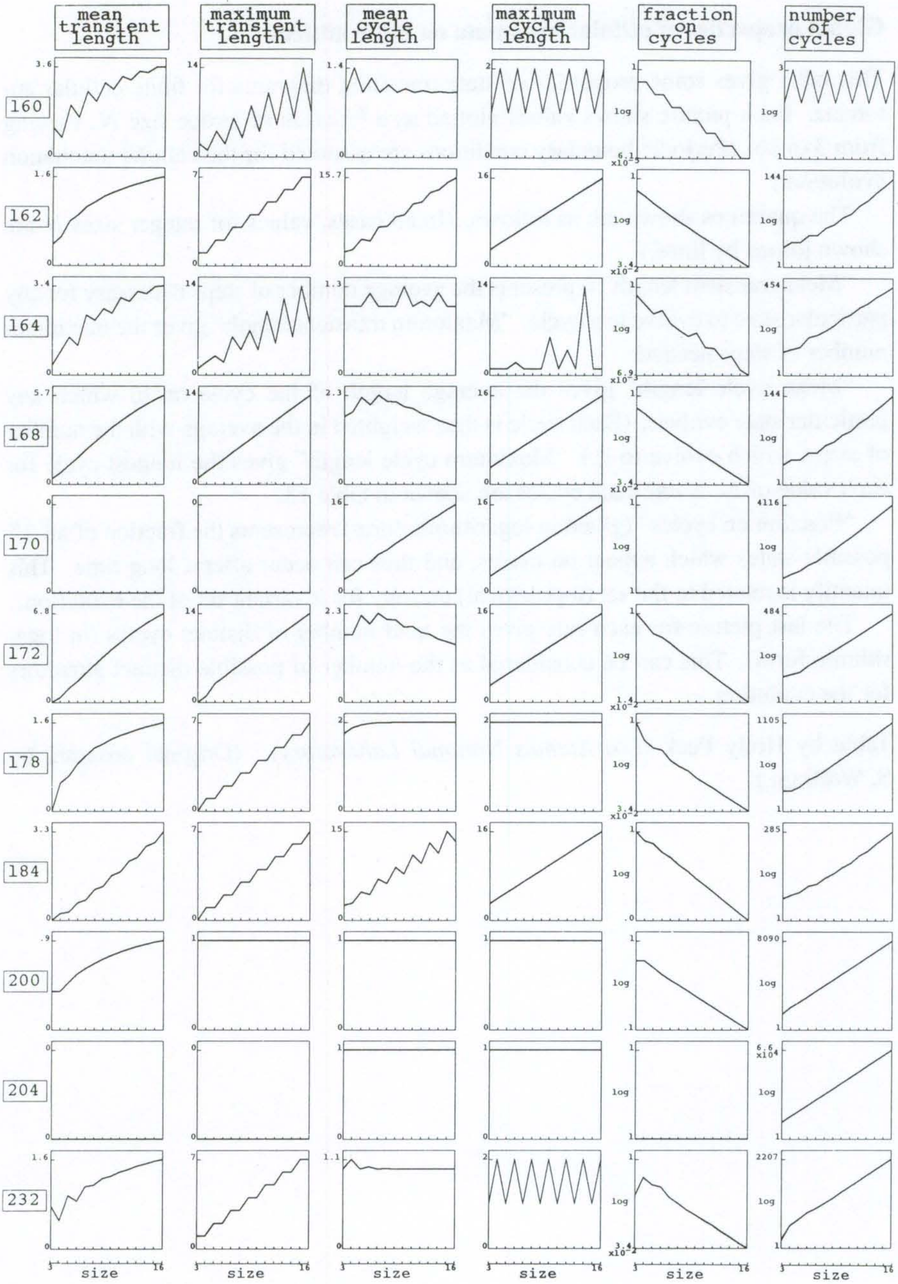

mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
38
40
41
42
43
44
45
46
50
51
54
log
size


mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
56
57
58
60
62
72
73
74
76
77
78
size
size
size
size
size
size



mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
90
94
104
105
106
108
110
122
126
128
130
size


mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
132
134
136
138
140
142
146
150
152
154
156
log
size


mean transient length
maximum transient length
mean cycle length
maximum cycle length
fraction on cycles
number of cycles
160
162
164
168
170
172
178
184
200
204
232
size
size
size
size
size
size

**Global properties of cellular automata on finite lattices.**

This table gives some properties of state transition diagrams for finite cellular automata. Each picture shows values plotted as a function of lattice size $N$, varying from 3 to 16. (Periodic boundary conditions are assumed for the cellular automaton evolution.)

The quantities shown are as follows. (In all cases, values for integer sizes $N$ are shown joined by lines.)

"Mean transient length" represents the average number of steps necessary for any particular state to evolve to a cycle. "Maximum transient length" gives the maximum number of steps needed.

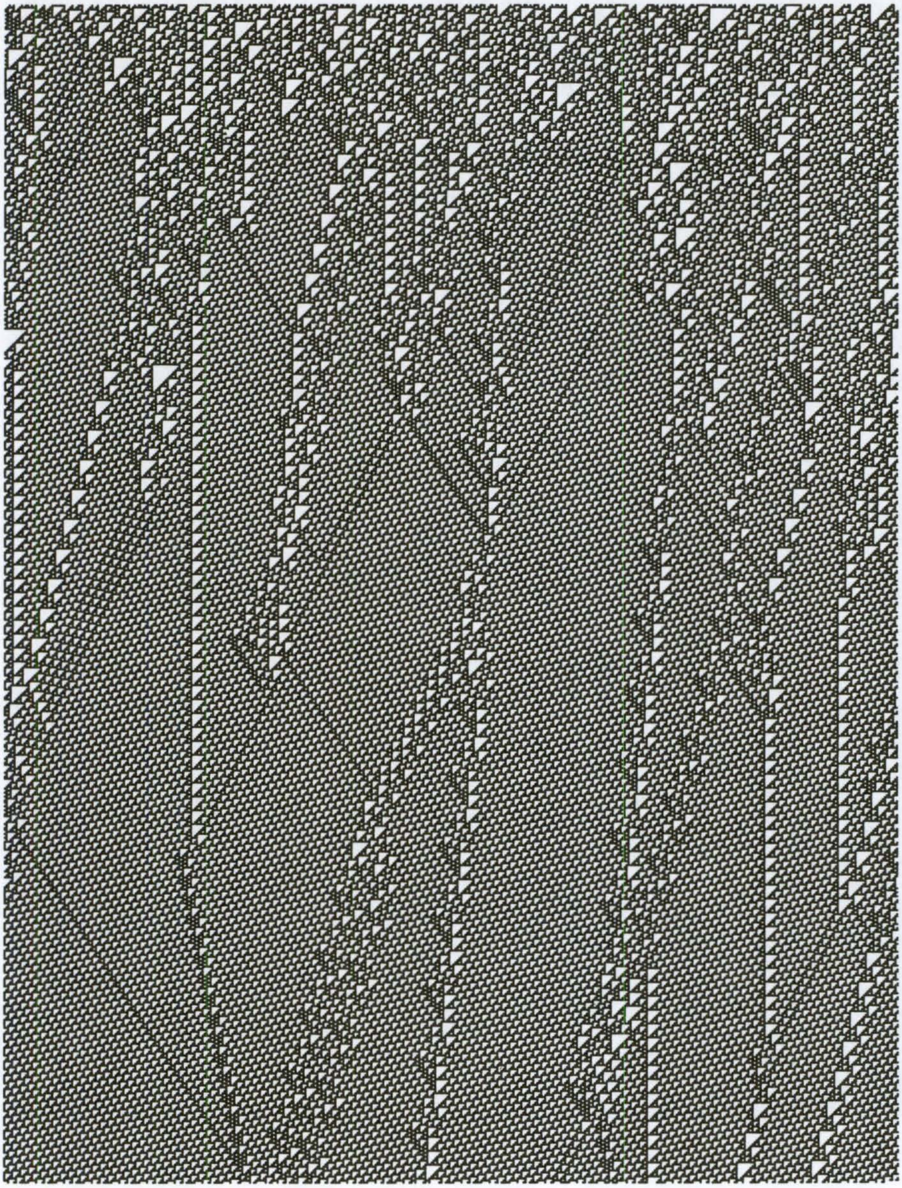
"Mean cycle length" gives the average length of the cycle on to which any particular state evolves. (Each cycle is thus weighted in the average with the number of states which evolve to it.) "Maximum cycle length" gives the longest cycle for each value of $N$. Some such cycles are shown in table 13.

"Fraction on cycles" (given in logarithmic form) represents the fraction of all $2^N$ possible states which appear on cycles, and thus can occur after a long time. This quantity is related to the set (topological) entropy for invariant set of the evolution.

The last picture for each rule gives the total number of distinct cycles (in logarithmic form). This can be considered as the number of possible distinct attractors for the evolution.

Table by Holly Peck (*Los Alamos National Laboratory*). (Original program by S. Wolfram.)

## Table 15: Structures in Rule 110



**Structures in rule 110.**

The previous two pages show patterns produced by evolution according to rule 110, starting from a disordered initial configuration. The first picture shows all sites on a size 400 lattice. The second picture shows every other site in space and time on a size 800 lattice.

The configurations produced after many steps can be represented in terms of particle-like structures superimposed on a periodic background. The background is found to have spatial period 14 and temporal period 7, and corresponds to repetitions of the block **B** = 10011011111000. The configurations are then of the form $\cdots$ **BBBBPBBB** $\cdots$ , where the particles **P** that have been found so far are:

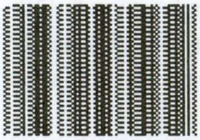
| velocity | **P** |
|---|---|
| −6/12 | 1000110011101111111000 |
| −2/4 | 11111000 |
| −14/42 | 111000011101111111111000 |
| −8/30 | 1001100110001111111000 |
| −4/15 | 00000 |
| −4/36 | 111011111111000 |
| −8/20 | 1111000011000 |
| 0/7 | 11111111000 |
| 0/7 | 100011000 |
| 0/7 | 10011011111111000 |
| 2/10 | 11101011000 |
| 2/10 | 111010001101111000 |
| 2/3 | 111000 |

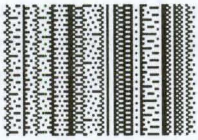
The "velocity" is written as (spatial period)/(temporal period).

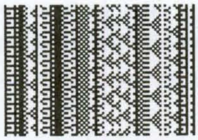
One may speculate that the behaviour of rule 110 is sophisticated enough to support universal computation.

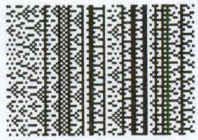
Table of particles by Doug Lind (*Mathematics Department, University of Washington, Seattle*).

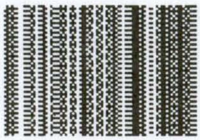
## Table 16: Patterns Generated by Second-Order Rules

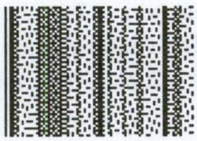

rule 0 (00000000)


rule 1 (00000001)

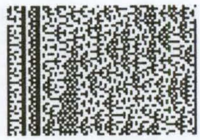

rule 2 (00000010)

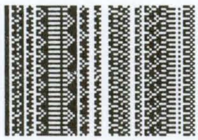

rule 3 (00000011)

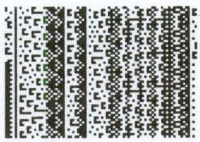

rule 4 (00000100)

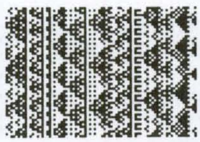

rule 5 (00000101)

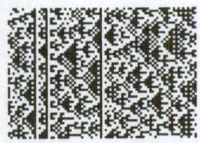

rule 6 (00000110)


rule 7 (00000111)


rule 8 (00001000)


rule 9 (00001001)

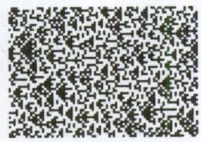

rule 10 (00001010)


rule 11 (00001011)

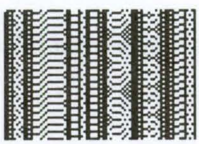

rule 12 (00001100)

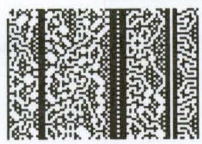

rule 13 (00001101)

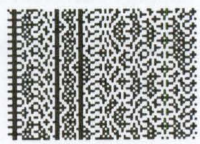

rule 14 (00001110)

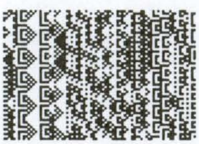

rule 15 (00001111)


rule 18 (00010010)

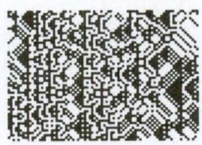

rule 19 (00010011)

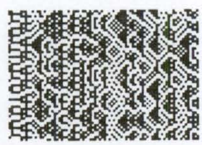

rule 22 (00010110)

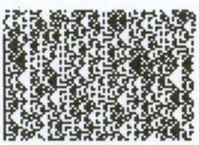

rule 23 (00010111)

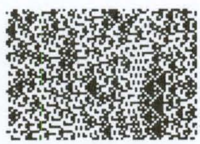

rule 24 (00011000)

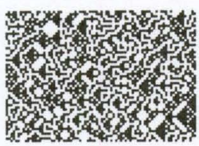

rule 25 (00011001)

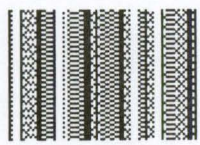

rule 26 (00011010)


rule 27 (00011011)

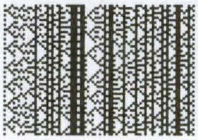

rule 28 (00011100)

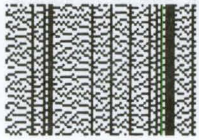

rule 29 (00011101)

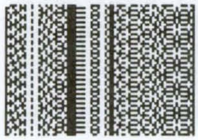

rule 30 (00011110)

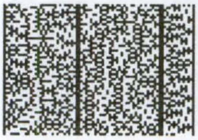

rule 32 (00100000)

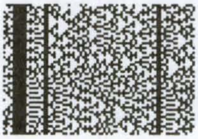

rule 33 (00100001)

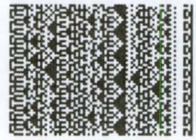

rule 34 (00100010)


rule 35 (00100011)


rule 36 (00100100)

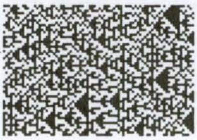

rule 37 (00100101)

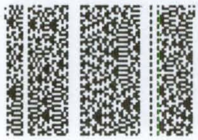

rule 38 (00100110)

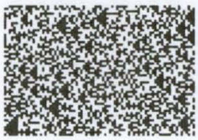

rule 40 (00101000)

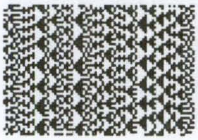

rule 41 (00101001)


rule 42 (00101010)

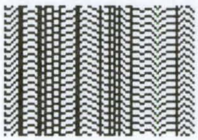

rule 43 (00101011)

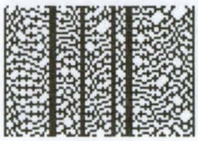

rule 44 (00101100)

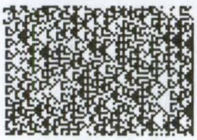

rule 45 (00101101)

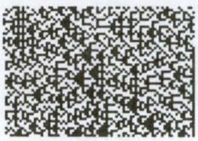

rule 46 (00101110)

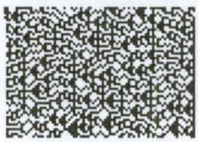

rule 50 (00110010)

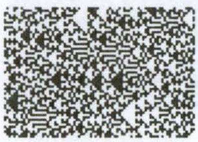

rule 51 (00110011)

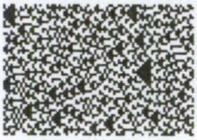

rule 54 (00110110)

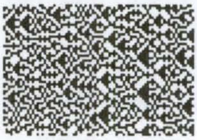

rule 56 (00111000)


rule 57 (00111001)

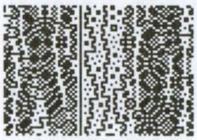

rule 58 (00111010)

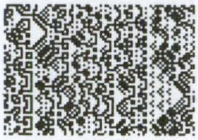

rule 60 (00111100)

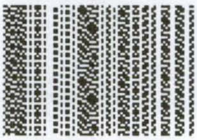

rule 61 (00111101)


rule 62 (00111110)


rule 72 (01001000)

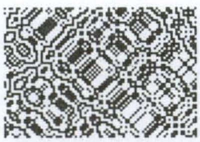

rule 73 (01001001)


rule 74 (01001010)


rule 76 (01001100)


rule 77 (01001101)

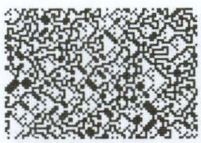

rule 78 (01001110)

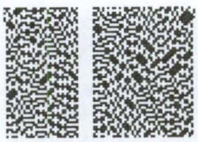

rule 90 (01011010)

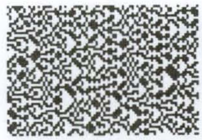

rule 94 (01011110)

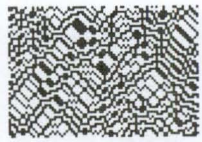

rule 104 (01101000)

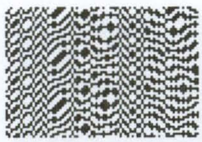

rule 105 (01101001)


rule 106 (01101010)

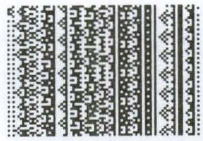

rule 108 (01101100)


rule 110 (01101110)


rule 122 (01111010)


rule 126 (01111110)

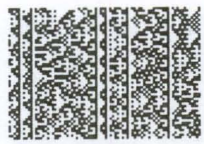

rule 128 (10000000)


rule 130 (10000010)


rule 132 (10000100)

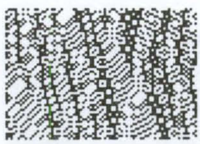

rule 134 (10000110)

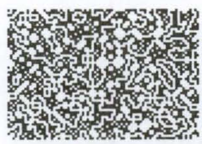

rule 136 (10001000)


rule 138 (10001010)

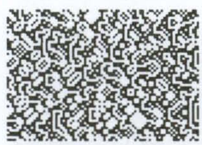

rule 140 (10001100)


rule 142 (10001110)

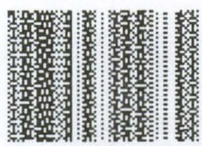

rule 146 (10010010)

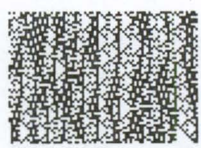

rule 150 (10010110)

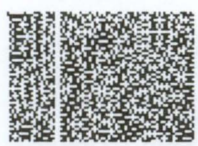

rule 152 (10011000)


rule 154 (10011010)

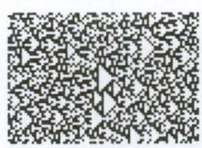

rule 156 (10011100)

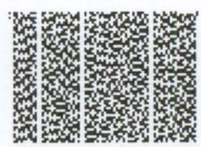

rule 160 (10100000)

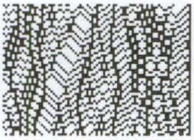

rule 162 (10100010)

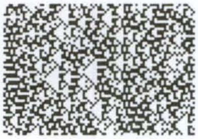

rule 164 (10100100)

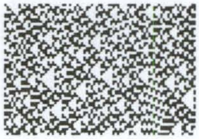

rule 168 (10101000)

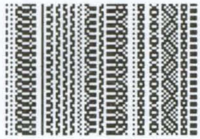

rule 170 (10101010)

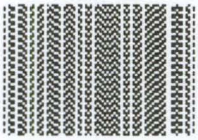

rule 172 (10101100)

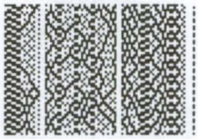

rule 178 (10110010)


rule 184 (10111000)

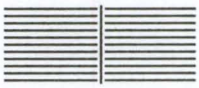

rule 188 (10111100)

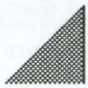

rule 200 (11001000)

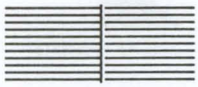

rule 204 (11001100)


rule 232 (11101000)

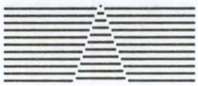

rule 0 (00000000)


rule 1 (00000001)


rule 2 (00000010)


rule 3 (00000011)

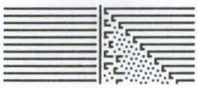

rule 4 (00000100)


rule 5 (00000101)


rule 6 (00000110)


rule 7 (00000111)


rule 8 (00001000)


rule 9 (00001001)


rule 10 (00001010)


rule 11 (00001011)

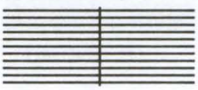

rule 12 (00001100)


rule 13 (00001101)

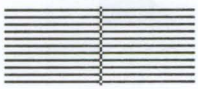

rule 14 (00001110)

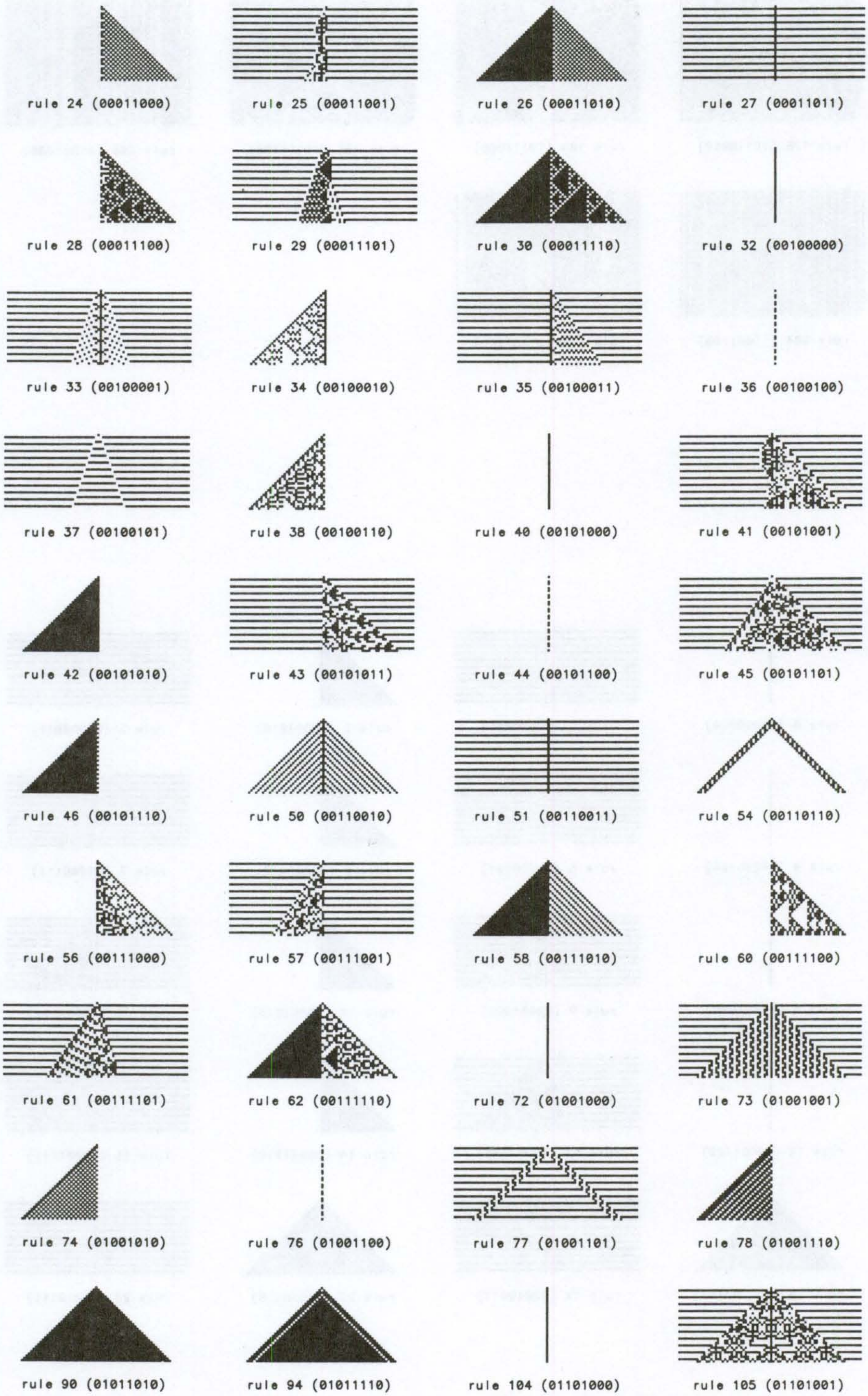

rule 15 (00001111)


rule 18 (00010010)


rule 19 (00010011)


rule 22 (00010110)


rule 23 (00010111)


rule 24 (00011000)
rule 25 (00011001)
rule 26 (00011010)
rule 27 (00011011)
rule 28 (00011100)
rule 29 (00011101)
rule 30 (00011110)
rule 32 (00100000)
rule 33 (00100001)
rule 34 (00100010)
rule 35 (00100011)
rule 36 (00100100)
rule 37 (00100101)
rule 38 (00100110)
rule 40 (00101000)
rule 41 (00101001)
rule 42 (00101010)
rule 43 (00101011)
rule 44 (00101100)
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rule 46 (00101110)
rule 50 (00110010)
rule 51 (00110011)
rule 54 (00110110)
rule 56 (00111000)
rule 57 (00111001)
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rule 60 (00111100)
rule 61 (00111101)
rule 62 (00111110)
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rule 73 (01001001)
rule 74 (01001010)
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rule 77 (01001101)
rule 78 (01001110)
rule 90 (01011010)
rule 94 (01011110)
rule 104 (01101000)
rule 105 (01101001)


rule 106 (01101010)


rule 108 (01101100)


rule 110 (01101110)


rule 122 (01111010)


rule 126 (01111110)


rule 128 (10000000)


rule 130 (10000010)


rule 132 (10000100)


rule 134 (10000110)


rule 136 (10001000)


rule 138 (10001010)


rule 140 (10001100)


rule 142 (10001110)


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rule 150 (10010110)


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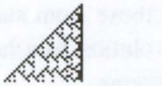
rule 154 (10011010)


rule 156 (10011100)


rule 160 (10100000)


rule 162 (10100010)


rule 164 (10100100)


rule 168 (10101000)


rule 170 (10101010)


rule 172 (10101100)


rule 178 (10110010)


rule 184 (10111000)


rule 188 (10111100)


rule 200 (11001000)


rule 204 (11001100)


rule 232 (11101000)

**Patterns generated by second-order reversible rules.**

This table shows patterns produced by second-order generalizations of the $k = 2$, $r = 1$ cellular automata considered above. The rules are of the form

$$a_i^{(t+1)} = \phi(a_{i-1}^{(t)}, a_i^{(t)}, a_{i+1}^{(t)}) + a_i^{(t-1)} \bmod 2,$$

where $\phi$ is a standard $k = 2$, $r = 1$ function, as listed in table 1. Such rules determine the configuration at time $t + 1$ in terms of the configurations both at time $t$ and time $t - 1$. The rules have the special feature that they are reversible: given configurations at times $t$ and $t + 1$, the configuration at time $t - 1$ can be deduced uniquely according to the rule

$$a_i^{(t-1)} = \phi(a_{i-1}^{(t)}, a_i^{(t)}, a_{i+1}^{(t)}) + a_i^{(t+1)} \bmod 2.$$

The first set of patterns were generated with disordered initial configurations at times 0 and −1. In the second set of patterns, the configurations at time 0 and −1 were both taken to be 1.

The forms of behaviour produced by these reversible rules are qualitatively similar to those from standard $k = 2$, $r = 1$ cellular automata, shown in tables 2 and 5. Evolution to a homogeneous, fixed, pattern is however impossible for reversible systems.