Inference in fuzzy systems

Artificial intelligence
Kristóf Karacs
PPKE-ITK

Fuzzy relation

- Given two universes $\mathcal{X}$ and $\mathcal{Y}$, a fuzzy relation $\mathcal{R}$ is

$$\mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$$

where $\subset$ denotes a fuzzy subset

- $\mathcal{R}$ is defined by $\mu_{\mathcal{R}}(x, y)$

Composition

- Given two fuzzy relations

$$\mathcal{R}\colon \mathcal{X} \times \mathcal{Y} \to [0,1] \qquad \mathcal{S}\colon \mathcal{Y} \times \mathcal{Z} \to [0,1]$$

their composition is defined by

$$\mathcal{T} = \mathcal{R} \circ \mathcal{S}\colon \mathcal{X} \times \mathcal{Z} \to [0,1]$$

$$\mu_{\mathcal{R}\circ\mathcal{S}}(x,z) = \max_{y\in\mathcal{Y}}\{\min\{\mu_{\mathcal{R}}(x,y), \mu_{\mathcal{S}}(y,z)\}\}$$

- Called an inner or - and product

Composition example

$$R(X,Y) = \begin{bmatrix} 0.8 & 1 & 0.1 & 0.7 \\ 0 & 0.8 & 0 & 0 \end{bmatrix} \qquad S(Y,Z) = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0 & 0.4 & 0 \\ 0.3 & 0.5 & 0.8 \\ 0.6 & 0.7 & 0.5 \end{bmatrix}$$

$$(x_i, z_j) = \max_{y_k}\left(\min\left((x_i, y_k), (y_k, z_j)\right)\right)$$

$$R \circ S(X,Z) = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0 & 0.4 & 0 \end{bmatrix}$$

Form of reasoning

- Fuzzy version of generalized modus ponens
- Antecedent (premise): *x* is A'
  - Implication: if *x* is A then *y* is B
  - Consequence: *y* is B'

$$\frac{A',\quad A \longrightarrow B}{B'}$$

$$A' \circ R_{A \to B} = B'$$

Implication

- Let A and B be two fuzzy sets in U_1 and U_2, respectively
- Implication is a relation defined by

$$A \rightarrow B \triangleq A \otimes B,$$

  - where $\otimes$ is the tensor (outer) product of the vectors using the logical operator and ($\wedge$)

- Implication functions
  - I(x,y) = min(x, y) — Mamdani
  - I(x,y) = max(1-x, y) — Dilne, Zadeh
  - I(x,y) = xy — Larsen

Implication example

- Rule
  - “If temperature is high, then humidity is fairly high.”
- Fuzzy variables
  - $t \in U_t = \{20,30,40\}$ $h \in U_h = \{20,50,70,90\}$
- Fuzzy sets
  - $HT \subseteq U_t$ $\mu_{HT}(t) = [0.1, 0.5, 0.9]^T$
  - $FHH \subseteq U_h$ $\mu_{FHH}(h) = [0.2, 0.6, 0.7\ 1]^T$
- Fuzzy rule
  - R(*t*,*h*): if *t* is HT then *h* is FHH

Implication example

- $\mu_{HT}(t) = [0.1, 0.5, 0.9]^T$
- $\mu_{FHH}(h) = [0.2, 0.6, 0.7\ 1]^T$
- $R_{HT \to FHH} = \mathrm{HT} \otimes \mathrm{FHH}$
- $R_{HT \to FHH} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.5 & 0.5 \\ 0.2 & 0.6 & 0.7 & 0.9 \end{bmatrix}$

Implication example

- According to the rule what is the humidity if temperature is fairly high?
  - t = FHT, FHT $\subseteq U_t$
- $\mu_{FHT}(t) = \mu_{HT}^2(t) = [0.01, 0.25, 0.81]^T$

Implication example

- $R(h) = R(t) \circ R_{HT \to FHH} = FHT \circ R_{HT \to FHH}$

$$= [0.01 \quad 0.25 \quad 0.81] \circ \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.5 & 0.5 \\ 0.2 & 0.6 & 0.7 & 0.9 \end{bmatrix}$$

$$= [0.2 \quad 0.6 \quad 0.7 \quad 0.81]$$

Single rule

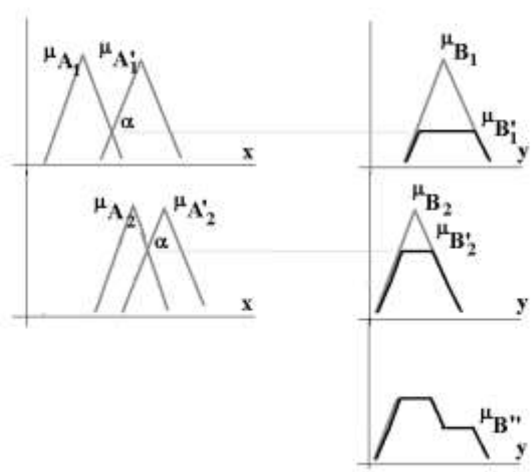
General fuzzification

$A \rightarrow B$
A'

B'

Singleton fuzzification

Superposition of rules



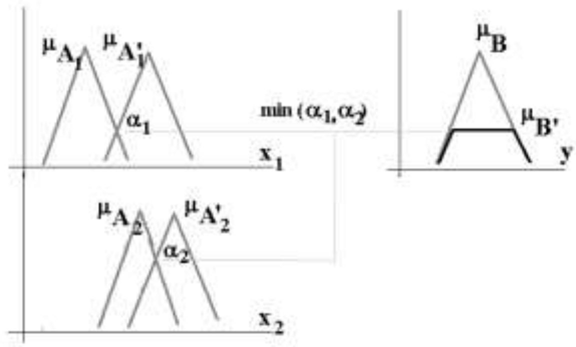
$A_1 \rightarrow B_1$
A'_1

$A_2 \rightarrow B_2$
A'_2

B'_1, B'_2

B''

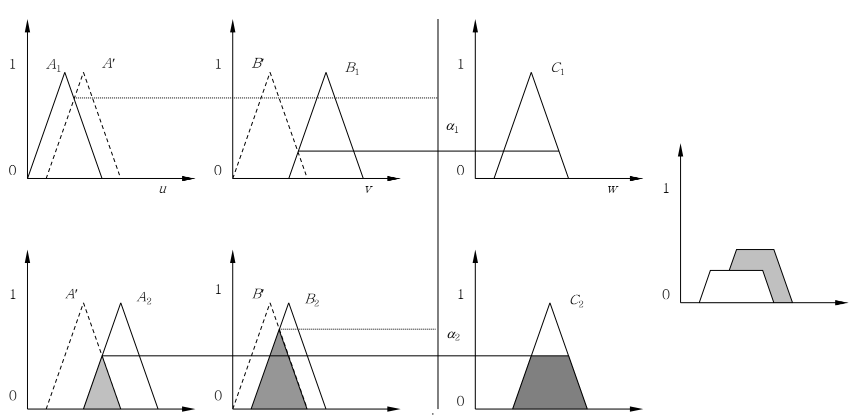
Multiple antecedents



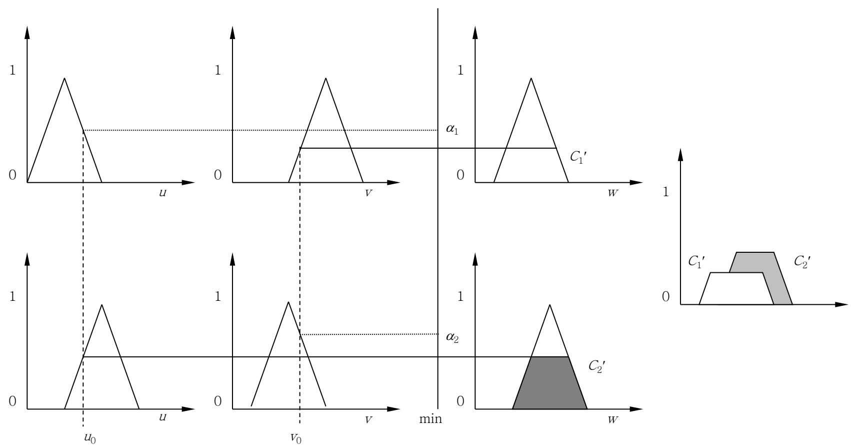
$A_1 \wedge A_2 \rightarrow B$
A'_1, A'_2

B'

Multiple Rules – Fuzzy-Fuzzy



Multiple Rules – Crisp-Fuzzy



Defuzzification

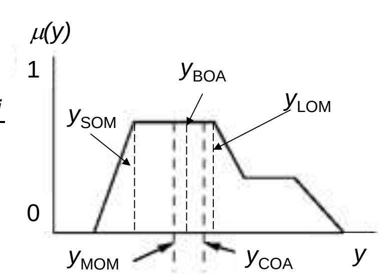
- Converting fuzzy set to crisp data
  - Mean of maxima (MOM)
    - y_{mj}: set of points with maximum membership value

$$y_{MOM} = \frac{\sum_{j=1}^{l} y_{mj}}{l}$$

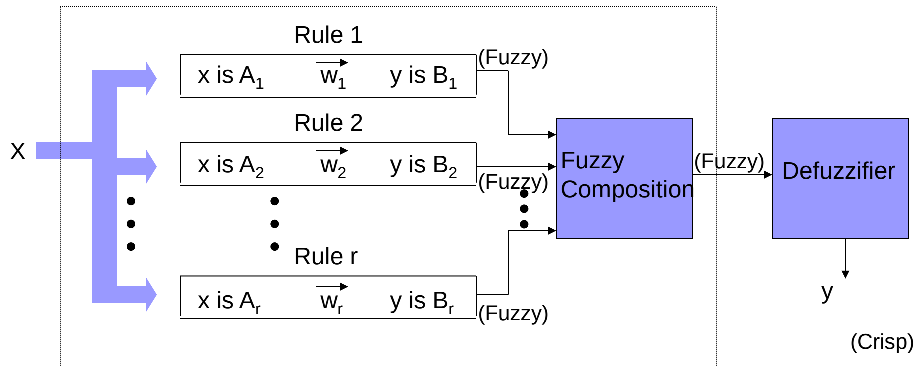
  - Center of area (COA)

$$y_{COA} = \frac{\sum_{j=1}^{l} \mu(y_{mj}) y_{mj}}{\sum_{j=1}^{l} \mu(y_{mj})}$$

  - Bisector of Area (BOA)
  - Smallest of Maximum (SOM)
  - Largest of Maximum (LOM)



Model of a fuzzy system



Ingredients of a fuzzy system

- Normalization of universes
- Fuzzification of crisp input data
- Fuzzy inference
- Defuzzification
- Denormalization