Error control coding – binary linear codes example

Developing linear codes

C(5,2)

$$\mathbf{G}_{2x5} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \mathbf{H}_{3x5} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Message vectors

$$\mathbf{c}^{(0)} = \mathbf{u}^{(0)}\mathbf{G} = (00)\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = (00000)$$

$$\mathbf{c}^{(1)} = \mathbf{u}^{(1)}\mathbf{G} = (01)\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = (01111)$$

$$\mathbf{c}^{(2)} = \mathbf{u}^{(2)}\mathbf{G} = (10)\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = (10110)$$

$$\mathbf{c}^{(3)} = \mathbf{u}^{(3)}\mathbf{G} = (11)\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = (11001)$$

The error group for syndrome 001

$$\mathbf{G}_{2x5}=\begin{pmatrix}1&0&1&1&0\\0&1&1&1&1\end{pmatrix}\rightarrow\mathbf{H}_{3x5}=\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}$$

$$\mathbf{s}^{(1)}=(001)\rightarrow E_1=E_{001}=\left\{\mathbf{e}:\mathbf{He}^T=\mathbf{s}^{(1)T}\right\}=\left\{\mathbf{e}:\mathbf{He}^T=\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$\mathbf{e}=(00001)\rightarrow\mathbf{He}^T=\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\mathbf{s}^{(1)}=(001)\rightarrow E_1=E_{001}=\left\{\mathbf{e}:\mathbf{He}^T=\mathbf{s}^{(1)T}\right\}=\left\{(00001),(01110),(10111),(11000)\right\}$$

Checking the error group property

$$\mathbf{s}^{(1)} = (001) \rightarrow E_1 = E_{001} = \left\{\mathbf{e} : \mathbf{H}\mathbf{e}^T = \mathbf{s}^{(1)T}\right\} = \left\{(00001),(01110),(10111),(11000)\right\}$$

$$\mathbf{e} = (00001) \rightarrow \mathbf{H}\mathbf{e}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{e} = (01110) \rightarrow \mathbf{H}\mathbf{e}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Selecting the group leader
(P_b=0.01)

$$\mathbf{s}^{(1)} = (001) \rightarrow E_1 = E_{001} = \left\{ \mathbf{e} : \mathbf{He}^T = \mathbf{s}^{(1)T} \right\} = \left\{ (00001), (01110), (10111), (11000) \right\}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$w=1 \qquad w=3 \qquad w=4 \qquad w=2$$

$$9{,}6*10^{-3} \qquad 9{,}8*10^{-7} \qquad 9{,}9*10^{-9} \qquad 9{,}7*10^{-5}$$

The group leader is $\mathbf{e}_{\mathbf{s}=(001)} = (00001)$ it occurs with the largest probability

The syndrome detection table

Based on the syndrome vector we identify the correpsonding most likely error vector and we store these pairs in an LUT !

For example:

Syndrome vector	Maximum likely error vector (the group leader)
000	00000
001	00001
010	00010
011	00011
100	00100
101	00101
110	10000
111	01000

Constructing the syndrome decoding table

1. List the numbers in decimal from $0,\dots,2^{n-1}$

2. Convert this decimal numbers to *n* bit binary numbers (the possible error vectors $\mathbf{e}\in\{0,1\}^n$)

3. Carry out the multiplications $\mathbf{He}^T=\mathbf{s}^T \quad \forall\mathbf{e}\in\{0,1\}^n$

4. Group the results with respect to **s** (collect all **e** vectors into the same group if they belong to the same **s)**

5. Determine the minimum weight **e** in each group

6. Construct an LUT by entering the "**s** and the corresponding minimum weight **e**" pairs

E.g.: constructing the syndrome decoding table of a C(5,2) code

List of possible error vectors

$0 \rightarrow \mathbf{e} = (00000)$

$1 \rightarrow \mathbf{e} = (00001)$

$2 \rightarrow \mathbf{e} = (00010)$

$3 \rightarrow \mathbf{e} = (00011)$

$4 \rightarrow \mathbf{e} = (00100)$

$5 \rightarrow \mathbf{e} = (00101)$

$6 \rightarrow \mathbf{e} = (00110)$

$7 \rightarrow \mathbf{e} = (00111)$

$8 \rightarrow \mathbf{e} = (01000)$

$9 \rightarrow \mathbf{e} = (01001)$

$10 \rightarrow \mathbf{e} = (01010)$

$11 \rightarrow \mathbf{e} = (01011)$

$12 \rightarrow \mathbf{e} = (01100)$

$13 \rightarrow \mathbf{e} = (01101)$

$14 \rightarrow \mathbf{e} = (01110)$

$15 \rightarrow \mathbf{e} = (01111)$

$16 \rightarrow \mathbf{e} = (10000)$

$17 \rightarrow \mathbf{e} = (10001)$

$18 \rightarrow \mathbf{e} = (10010)$

$19 \rightarrow \mathbf{e} = (10011)$

$20 \rightarrow \mathbf{e} = (10100)$

$21 \rightarrow \mathbf{e} = (10101)$

$22 \rightarrow \mathbf{e} = (10110)$

$23 \rightarrow \mathbf{e} = (10110)$

$24 \rightarrow \mathbf{e} = (11000)$

$25 \rightarrow \mathbf{e} = (11001)$

$26 \rightarrow \mathbf{e} = (11010)$

$27 \rightarrow \mathbf{e} = (11011)$

$28 \rightarrow \mathbf{e} = (11100)$

$29 \rightarrow \mathbf{e} = (11101)$

$30 \rightarrow \mathbf{e} = (11110)$

$31 \rightarrow \mathbf{e} = (11111)$

E.g.: constructing the syndrome decoding table of a C(5,2) code

Multiplication with the generator matrix

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \boxed{\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \qquad \boxed{\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

E.g.: constructing the syndrome decoding table of a C(5,2) code

Multiplication with the generator matrix

$$\boxed{\mathbf{He}^T = \begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}}$$

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E.g.: constructing the syndrome decoding table of a C(5,2) code

Multiplication with the generator matrix

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

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E.g.: constructing the syndrome decoding table of a C(5,2) code

Multiplication with the generator matrix

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$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Constructing the groups and assigning the group leaders

$$E_{(000)} = \{(00000),(01111),(10110),(11001)\} \to \mathbf{e}_{000} = (00000)$$

$$E_{(001)} = \{(00001),(01110),(10111),(11000)\} \to \mathbf{e}_{001} = (00001)$$

$$E_{(010)} = \{(00010),(01101),(10100),(11011)\} \to \mathbf{e}_{010} = (00010)$$

$$E_{(011)} = \{(00011),(01100),(10101),(11010)\} \to \mathbf{e}_{011} = (00011)$$

$$E_{(100)} = \{(00100),(01011),(10010),(11101)\} \to \mathbf{e}_{100} = (00100)$$

$$E_{(101)} = \{(00101),(01010),(10011),(11100)\} \to \mathbf{e}_{101} = (00101)$$

$$E_{(110)} = \{(00110),(01001),(10000),(11111)\} \to \mathbf{e}_{110} = (10000)$$

$$E_{(111)} = \{(00111),(01000),(10001),(11110)\} \to \mathbf{e}_{111} = (01000)$$

The syndrome decoding table

Syndrome vector	Group leader error vector
000	00000
001	00001
010	00010
011	00011
100	00100
101	00101
110	10000
111	01000

Another way of constructing the error groups

If $\mathbf{e} \in E_{\mathbf{s}}$ and $\mathbf{e}' = \mathbf{e} + \mathbf{c}$ then $\mathbf{e}' \in E_{\mathbf{s}}$

1. Pick an error vector **e**

2. Calculate the corresponding syndrome vector $\mathbf{He}^T = \mathbf{s}^T$

3. Construct the error group as follows

$$E_{\mathbf{s}} := \left\{ \mathbf{e}, \mathbf{e} + \mathbf{c}^{(1)}, \mathbf{e} + \mathbf{c}^{(2)}, \ldots, \mathbf{e} + \mathbf{c}^{\left(2^k - 1\right)} \right\}$$

4. Pick another error vector **e”** for which $\mathbf{e}'' \notin E_{\mathbf{s}}$

and go back to Step 1.

Example

Pick $\mathbf{e} = (00001)$

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{c}^{(0)} = (00000); \mathbf{c}^{(1)} = (01111); \mathbf{c}^{(2)} = (10110); \mathbf{c}^{(3)} = (11001)$$

$$E_{(001)} = \{(00001);(00001)+(01111);(00001)+(10110);(00001)+(11001)\}$$

$$E_{(001)} = \{(00001);(01110);(10111);(11000)\}$$

$$\mathbf{e}_{(001)} = (00001)$$

Example (cont')

Pick $\mathbf{e} = (00010)$

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{c}^{(0)} = (00000); \mathbf{c}^{(1)} = (01111); \mathbf{c}^{(2)} = (10110); \mathbf{c}^{(3)} = (11001)$$

$$E_{(010)} = \{(00010); (00010) + (01111); (00010) + (10110); (00010) + (11001)\}$$

$$E_{(001)} = \{(00010); (01101); (10100); (11011)\}$$

$$\mathbf{e}_{(001)} = (00010)$$

Example

Pick $\mathbf{e}=(00100)$

$$\mathbf{He}^T=\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\mathbf{c}^{(0)}=(00000);\mathbf{c}^{(1)}=(01111);\mathbf{c}^{(2)}=(10110);\mathbf{c}^{(3)}=(11001)$$

$$E_{(100)}=\{(00100);(00100)+(01111);(00100)+(10110);(00100)+(11001)\}$$

$$E_{(100)}=\{(00100);(01011);(10010);(11101)\}$$

$$\mathbf{e}_{(100)}=(00100)$$

Example

Pick $\mathbf{e}=(01000)$

$$\mathbf{He}^T=\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\mathbf{c}^{(0)}=(00000);\mathbf{c}^{(1)}=(01111);\mathbf{c}^{(2)}=(10110);\mathbf{c}^{(3)}=(11001)$$

$$E_{(111)}=\{(01000);(01000)+(01111);(01000)+(10110);(01000)+(11001)\}$$

$$E_{(111)}=\{(01000);(00111);(11110);(10001)\}$$

$$\mathbf{e}_{(111)}=(01000)$$

Example

Pick $\mathbf{e}=(10000)$

$$\mathbf{He}^T=\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\mathbf{c}^{(0)}=(00000);\mathbf{c}^{(1)}=(01111);\mathbf{c}^{(2)}=(10110);\mathbf{c}^{(3)}=(11001)$$

$$E_{(110)}=\{(10000);(10000)+(01111);(10000)+(10110);(10000)+(11001)\}$$

$$E_{(110)}=\{(10000);(11111);(00110);(01001)\}$$

$$\mathbf{e}_{(100)}=(10000)$$

Example

Pick $\mathbf{e}=(00101)$

$$\mathbf{He}^T=\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\mathbf{c}^{(0)}=(00000);\mathbf{c}^{(1)}=(01111);\mathbf{c}^{(2)}=(10110);\mathbf{c}^{(3)}=(11001)$$

$$E_{(101)}=\{(00101);(00101)+(01111);(00101)+(10110);(00101)+(11001)\}$$

$$E_{(101)}=\{(00101);(01010);(10011);(11100)\}$$

$$\mathbf{e}_{(001)}=(00101)$$

Example

Pick $\mathbf{e} = (00011)$

$$\mathbf{He}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{c}^{(0)} = (00000); \mathbf{c}^{(1)} = (01111); \mathbf{c}^{(2)} = (10110); \mathbf{c}^{(3)} = (11001)$$

$$E_{(011)} = \{(00011); (00011) + (01111); (00011) + (10110); (00011) + (11001)\}$$

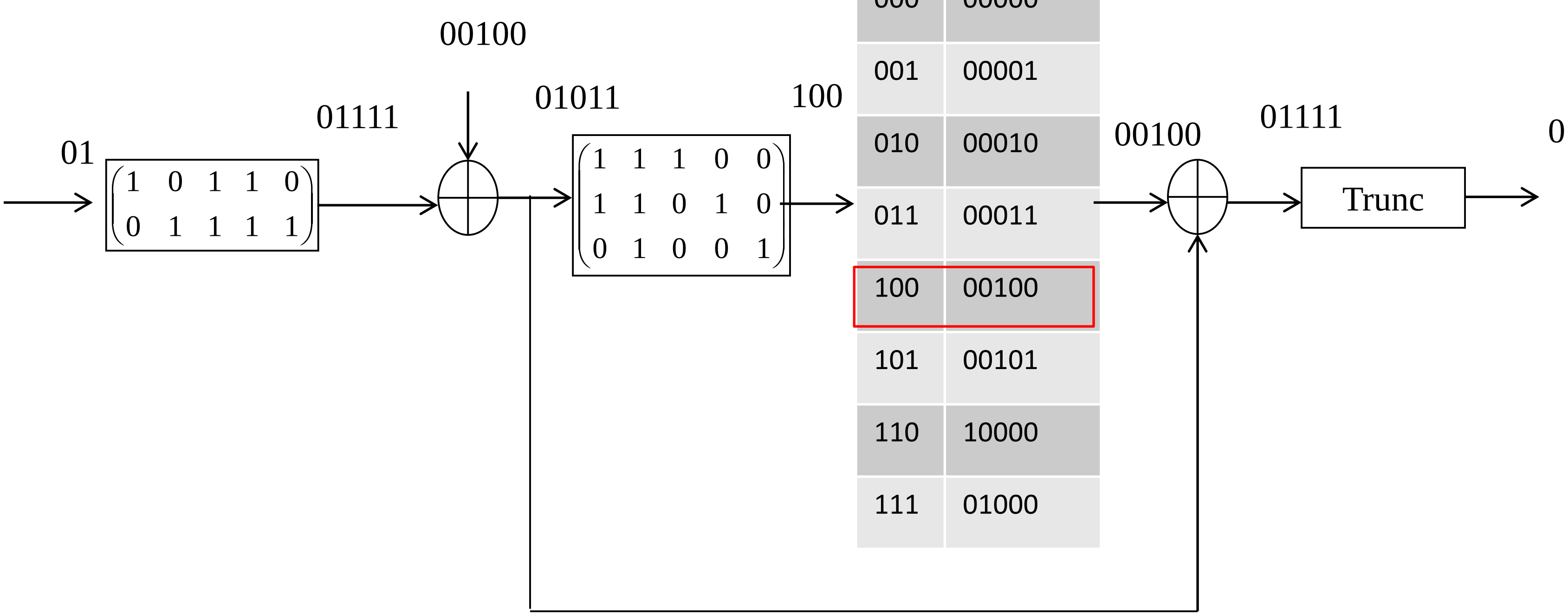
$$E_{(011)} = \{(00011); (01100); (10101); (11010)\}$$

$$\mathbf{e}_{(011)} = (00011)$$

The syndrome decoding table

Syndrome vector	Group leader error vector
000	00000
001	00001
010	00010
011	00011
100	00100
101	00101
110	10000
111	01000

The coding scheme



s	e
000	00000
001	00001
010	00010
011	00011
100	00100
101	00101
110	10000
111	01000

The standard array

Syndrome vector	**0**	**1**	**2**	**3**
000	00000	01111	10110	11001
001	00001	01110	10111	11000
010	00010	01101	10100	11011
011	00011	01100	10101	11010
100	00100	01011	10010	11101
101	00101	01010	10011	11100
110	00110	01001	10000	11111
111	00111	01000	10001	11110