Computer controlled systems

Lecture 1

version: 2018.10.02. – 10:51:55

1. Mathematical summary

1.1. Abelian group

1. **Group** $(H, +)$ – set H, and an operation defined on H: $+$
   - closure – for any a and b in H their „sum" $a + b$ is also an element of H
   - commutativity – $\forall a, b \in H \quad a + b = b + a$
   - associativity – $\forall a, b, c \in H \quad (a + b) + c = a + (b + c)$
   - there is an identity element – $\exists 0 \in H \ \forall a \in H \quad a + 0 = a$
   - for any element there exists an inverse element – $\forall a \in H \ \exists a^{-1} \in H \quad a + a^{-1} = 0$
2. **Field** $K = (H, +, \cdot)$ – set H and to operations (additive) $+$ and multiplicative $\cdot$
   - $(H, +)$ Abelian group
   - $(H \backslash \{0\}, \cdot)$ Abelian group
   - multiplication is distributive with respect to addition – $\forall a, b, c \in H \quad a \cdot (b + c) = a \cdot b + a \cdot c$
3. **Vector space V defined on a field K**

   (a) $(V, +)$ is an Abelian group

   (b) Multiplication with a scalar $K \times V \to V$ – $\alpha \in K$, $\boldsymbol{v} \in V$, $\alpha \cdot \boldsymbol{v} \in V$
   - $\forall \boldsymbol{v} \in V$ and $\forall \alpha, \beta \in K \quad (\alpha + \beta) \cdot \boldsymbol{v} = \alpha \cdot \boldsymbol{v} + \beta \cdot \boldsymbol{v}$
   - $\forall \boldsymbol{u}, \boldsymbol{v} \in V$ and $\forall \alpha \in K \quad \alpha \cdot (\boldsymbol{u} + \boldsymbol{v}) = \alpha \cdot \boldsymbol{u} + \alpha \cdot \boldsymbol{v}$
   - $\forall \boldsymbol{v} \in V$ and $\forall \alpha, \beta \in K \quad (\alpha \cdot \beta)\boldsymbol{v} = \alpha \cdot (\beta \cdot \boldsymbol{v})$
   - $1 \cdot \boldsymbol{v} = \boldsymbol{v}$, where $1 \in K$ the identity element for multiplication

   Examples for vector spaces:

   (a) $\mathbb{R}^n$

   (b) $K^{n \times k}$: $n \times k$ matrices defined on field K

   (c) $K[x]$: polynomials defined on field K

   (d) $K[x]$: where $\forall f \in K[x]$, $\deg(f) \leq n$

   (e) $C[0, 1]$: real valued continuous function defined on the interval $[0, 1]$

   (f) $V = P(H)$: set of every subset of finite set H,
   defined on a finite field $K = \{0, 1\}$ with two elements
   $\forall A, B \in H$: $A + B = (A \setminus B) \cup (B \setminus A)$ and $0 \cdot A = \emptyset$, $1 \cdot A = A$

1.2. Homogeneous linear transformation

V and W vector space defined on field K
In case of $\mathcal{A}: V \to W$ linear transformation, ha $\forall \boldsymbol{x}, \boldsymbol{y} \in V$ and $a \in K$

- $\mathcal{A}(\boldsymbol{x}+\boldsymbol{y}) = \mathcal{A}(\boldsymbol{x}) + \mathcal{A}(\boldsymbol{y})$ (linearity)
- $\mathcal{A}(a \cdot \boldsymbol{x}) = a \cdot \mathcal{A}(\boldsymbol{x})$ (homogeneity)

Examples for linear transformations

- identity transformation
- differentiation
- counter-example: $\mathcal{B}: \mathbb{R} \to \mathbb{R}$, $\mathcal{B}(x) = x^2$

1.2.1. Matrix of a linear transformation

Given a linear transformation $\mathcal{A}: V \to W$. We say that A is a matrix of $\mathcal{A}$ (for a fixed coordinates system for vector spaces V and W) if $\mathcal{A}(\boldsymbol{v}) = A \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in V$

Let $[v] = [\boldsymbol{v}_1, .., \boldsymbol{v}_n]$ denote a fixed basis of vector space V, and $[w] = [\boldsymbol{w}_1, .., \boldsymbol{w}_m]$ denote a fixed basis of vector space W, such that $\mathcal{A}(\boldsymbol{v}_i) = \boldsymbol{w}_i$. Then the columns of matrix A are the coordinate matrices of $\boldsymbol{w}_i$ (column vectors).

1.3. Image and kernel of a linear transformation

$\mathcal{A}: V \to W$ is a linear mapping

- its image is: $\mathrm{Im}(\mathcal{A}) = \left\{\boldsymbol{w} \in W \mid \exists \boldsymbol{v} \in V \ \mathcal{A}(\boldsymbol{v}) = \boldsymbol{w}\right\}$
- its kernel is: $\mathrm{Ker}(\mathcal{A}) = \left\{\boldsymbol{v} \in V \mid \mathcal{A}(\boldsymbol{v}) = \boldsymbol{0}\right\}$

Example 1. Calculation of the kernel and image of a linear mapping

Let $\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping, which, in the canonical basis, is given by matrix A. Determine the image and the kernel space of $\mathcal{A}$. First of all, we solve the linear equation system $A\boldsymbol{x} = 0$.

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & 1 & -2 \end{pmatrix}, \quad A\boldsymbol{x} = 0 \to \underbrace{\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & -1 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\text{Gauss elimination is necessary for determining both the image and kernel space of a matrix}} \tag{1}$$

We obtained that $x_1 = -x_3$ and $x_2 = x_3$, therefore, any $\boldsymbol{x} = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ will satisfy the linear equation system $A\boldsymbol{x} = 0$, for all $x_3 \in \mathbb{R}$.
- the image space is spanned by the linearly independent columns of matrix A
- the kernel space is spanned by the linearly independent nontrivial solutions of equation system $A\boldsymbol{x} = 0$.

$$\begin{gathered} A\boldsymbol{x}_i = 0, \ \forall i = \overline{1, r}, \ \text{where } r = \dim(\mathrm{Ker}(\mathcal{A})) \\ v = \sum_{i=1}^{r} c_i \boldsymbol{x}_i = 0 \quad \Leftrightarrow \quad c_i = 0 \ \ \forall i = \overline{1, r} \end{gathered} \tag{2}$$

$$\mathrm{Im}(\mathcal{A}) = \mathrm{span}\left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\rangle = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \middle| (\alpha, \beta) \in \mathbb{R}^2 \right\} \tag{3}$$

$$\mathrm{Ker}(\mathcal{A}) = \mathrm{span}\left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle = \left\{ \alpha \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\} \tag{4}$$

Theorem 1. Dimension theorem

For an arbitrary linear mapping $\mathcal{A} : V \to W$: $\dim(\mathrm{Ker}(\mathcal{A})) + \dim(\mathrm{Im}(\mathcal{A})) = \dim(V)$

Example 2. (General solution of a linear transformation) Determine the *general* solution of a system of linear equations $A\boldsymbol{x} = b$.

First of all, we compute the general solution of the Homogeneous equation $A\boldsymbol{x} = 0$:

$$\boldsymbol{x}_h \in \left\{ \sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i \;\middle|\; \alpha_i \in \mathbb{R},\ \boldsymbol{v}_i \in \mathbb{R}^n,\ i = \overline{1,k} \right\} = \mathrm{span}\langle \boldsymbol{v}_1, .., \boldsymbol{v}_k \rangle = \mathrm{Ker}(\mathcal{A}) \tag{5}$$

Then we need to find a special solution $\boldsymbol{x}_{0,ih}$ for the inhomogeneous equation $A\boldsymbol{x} = \boldsymbol{b}$. Finally, the the sum of the two will give the general solution of the inhomogeneous equation:

$$\boldsymbol{x}_{ih} \in \left\{ \boldsymbol{x}_{0,ih} + \sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i \;\middle|\; \alpha_i \in \mathbb{R},\ \boldsymbol{v}_i \in \mathbb{R}^n,\ i = \overline{1,k} \right\} =: "\boldsymbol{x}_{0,ih} + \mathrm{Ker}(\mathcal{A})" \tag{6}$$

Matlab 1. null, orth, rank

The kernel (or null) space of a linear transformation given by its matrix A can be computed in Matlab as follows:

```
>> A = [ 1 2 -1 ; 2 3 -1 ; -1 1 -2 ];
>> null(A)
ans =
   -0.5774
    0.5774
    0.5774
```

The image space can also be computed by function `orth`, but note, that Matlab gives an orthonormal basis for the image space, therefore, in general, it gives other vectors compared to those computed by hand.

```
>> orth(A)
ans =
   -0.5345   -0.0000
   -0.8018    0.3162
   -0.2673   -0.9487
```

You can check that the two vectors are indeed orthogonal.
Due to the fact that $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear mapping into the *same* space, $\mathrm{Ker}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A}) = \mathbb{R}^3$. In Matlab we can check this as follows:

```
>> rank([null(A) orth(A)])
ans = 3
```

1.4. Determinant and rank of a matrix

Theorem 2. (Minor expansion theorem) The determinant of a matrix A can be computed as the sum of an elements of a row or column multiplied by the corresponding signed minor determinant. Expansion of $\det(A)$ along the ith row: $\det(A) = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} \cdot D_{ij}$

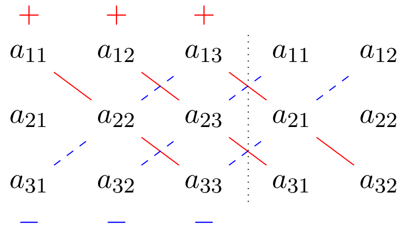
where a_{ij} denotes an element of A in the ith row and jth column, and D_{ij} is the corresponding minor determinant, the sign of the minors varies as the black and white colors on the chess table: $(-1)^{i+j}$.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

Source: Wikipedia.org, see also Wolfram.com

Sarrus rule: a fast method the compute a 3×3 determinant



$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

The **column rank** of a matrix gives the number of linearly independent columns of the matrix. The **row rank** of a matrix gives the number of linearly independent rows of the matrix.

- For any matrix, the two types of ranks are always the same and it gives the dimension of the larges nonzero minor determinant of the same matrix. We denote it as: $\mathrm{rank}(A)$.
- Matrix A is called full-rank if its rank is $\mathrm{rank}(A) = \min(\text{nr. of rows}, \text{nr. of columns})$.
- A square matrix A is called full-rank if its determinant is nonzero.

1.5. Matrix inversion

Matrix $A \in T^{n\times n}$ is **invertible** $\quad\Leftrightarrow\quad \det(A) \neq 0$

Matrix inversion by using the adjugate matrix $\mathrm{adj}(A)$: $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

In $A \in \mathbb{R}^{2\times 2}$ case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{7}$$

In $A \in \mathbb{R}^{3\times 3}$ case:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}^T \tag{8}$$

Example 3. (Determine the inverse of the following matrix)

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} \quad \det(A) = 4 \quad \mathrm{adj}(A) = \begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0.25 \end{pmatrix}$$

1.6. Eigen values and eigen vectors (eigen value/spectral decomposition)

$\mathcal{A} : V \to V$ (V egy K test feletti vektortér) lineáris leképezés
sajátvektora az $\boldsymbol{x} \in V$, $\boldsymbol{x} \neq \mathbf{0}$ vektor, ha $\exists\lambda \in K \quad \mathcal{A}(\boldsymbol{x}) = \lambda \cdot \boldsymbol{x}$
Ekkor az $\boldsymbol{x}$ vektort az $\mathcal{A}$ leképezés λ sajátértékhez tartozó sajátvektorának nevezzük.

A sajátértékek a leképezés mátrixából számolhatók. Nem számít, hogy milyen bázisokra vonatkozóan van felírva a mátrix, a sajátértékek mindig ugyanazok lesznek, mivel a sajátérték a leképezés tulajdonsága. A sajátvektorok is ugyanazok lesznek, azonban a kiindulási tér bázisában koordinátázva kapjuk ezeket.

$Ax = \lambda\boldsymbol{x} \quad\to\quad \lambda\boldsymbol{x} - A\boldsymbol{x} = \mathbf{0} \quad\to\quad (\lambda I - A)\boldsymbol{x} = \mathbf{0}$
Ez egy homogén lineáris egyenletrendszer, melynek az $\boldsymbol{x} \neq \mathbf{0}$ vektor a megoldása, tehát létezik nem triviális megoldás, mely azzal ekvivalens, hogy az együtthatómátrix oszlopai lineárisan összefüggőek, vagyis a determinánsa nulla.

Figyelem! A karakterisztikus polinom a Lineáris algebra tárgyból tanult alaktól eltérő, azonban azzal abszolútértékben megegyező, ezért azonos λ értékek esetén nulla. A most használt alaknál az A mátrixhoz képest a determinánsban nem csak a főátlóbeli elemek változnak, hanem a főátlóbeli elemek is, a (-1)-szeresükre.

- **Characteristic polynomial**: $\det(\lambda I - A)$
- A sajátértékek a karakterisztikus egyenlet gyökei: $\det(\lambda I - A) = 0$
- A sajátvektorok a $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ lineáris egyenletrendszer megoldásai a megfelelő λ értékekre külön-külön számolva

Example 4.

Determine the eigen values and eigen vector of the following matrix!

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \quad \det\left(\lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}\right) = \begin{vmatrix} \lambda - 1 & 0 \\ -2 & \lambda + 3 \end{vmatrix} = (\lambda - 1)(\lambda + 3) \tag{9}$$

Eigen values: $\lambda_1 = 1, \lambda_2 = -3$

Eigen vector(s) for $\lambda_1 = 1$:

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 4 & 0 \end{array}\right) \quad \to \quad -2x + 4y = 0 \quad \to \quad y = x/2 \quad \to \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cdot p \quad p \in \mathbb{R} \setminus \{0\}$$

Eigen vector(s) for $\lambda_2 = -3$

$$\left(\begin{array}{cc|c} 4 & 0 & 0 \\ -2 & 0 & 0 \end{array}\right) \quad \to \quad 4x = -2x = 0 \quad \to \quad x = 0 \quad \to \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot q \quad q \in \mathbb{R} \setminus \{0\}$$

Theorem 3. Cayley-Hamilton

Every matrix satisfies its own characteristic equation In the special case, when $A \in \mathbb{R}^{2\times 2}$:

$$\text{Characteristic (polynomial) equation: } \lambda^2 + \lambda\,\mathrm{tr}(A) + \det(A) = 0 \tag{10}$$

$$\text{According to this theorem we have: } A^2 + A\,\mathrm{tr}(A) + I\det(A) = 0 \tag{11}$$

Example 5. Cayley-Hamilton

Legyen adott a következő mátrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) - 6 = \lambda^2 - 5\lambda - 2 \tag{12}$$

$$A^2 - 5 \cdot A - 2 \cdot I = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{13}$$

1.7. Basis transformation, diagonalization

Egy vektortérben az éppen tekintett bázis határozza meg a vektorok koordinátamátrixát, ettől azonban néha el kell térni. Ez egy lineáris leképezés alkalmazásával valósítható meg, melynek mátrixát úgy kapjuk, hogy a mátrix oszlopaiba írjuk az új bázisvektorokat a régi bázis szerinti koordinátázva, majd a kapott mátrixon invertáljuk. A mátrix, és ezáltal a leképezés mindig invertálható, mivel bázist bázisba visz, melyeknek vektorai lineárisan függetlenek. Jelölje egy $\boldsymbol{x} \in V$ vector $[e]$ bázis szerinti koordináta mátrixát $x_{[e]}$, egy új $[v]$ bázis szerinti koordinátamátrixát pedig $x_{[v]}$. Legyen S mátrix, melynek oszlopaiban az új bázisvektorok találhatók $S = \left[\boldsymbol{v}_1, ..., \boldsymbol{v}_n\right]$. Pontosabban fogalmazva, az S mátrix oszlopai az új bázisvektorok $[e]$ bázisban felírt koordináta mátrixai legyenek: $S = \left[v_{1[e]}, ..., v_{n[e]}\right]$. Ekkor

$S^{-1} \cdot x_{[e]} = x_{[v]} \qquad S \cdot x_{[v]} = x_{[e]}$

Tehát az S mátrix által leírt leképezés valójában az új bázisra vonatkozó koordinátázást a régire transzformálja, és az S^{-1} mátrix transzformál az új bázisba.

Example 6. (Basis transformations) This will be important at state-space transformations

We consider two vector spaces $V = W$ (for simplicity let be both $\mathbb{R}^2$), and let be the following vectors in $\mathbb{R}^2$ (their coordinates matrices being given in the canonical basis $[e] = [\boldsymbol{e}_1, \boldsymbol{e}_2] = [\boldsymbol{i}, \boldsymbol{j}]$ of $\mathbb{R}^2$):

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{[e]}, \ \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{[e]}, \ \boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{[e]}, \ \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}_{[e]}, \ \boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{[e]}, \ \boldsymbol{w}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}_{[e]} \tag{14}$$

$[e] = [\boldsymbol{e}_1, \boldsymbol{e}_2]$ is the canonical basis of $V = W = \mathbb{R}^2$, Furthermore, we consider $[v] = [\boldsymbol{v}_1, \boldsymbol{v}_2]$ and $[w] = [\boldsymbol{w}_1, \boldsymbol{w}_2]$ be an alternative basis of vector spaces V and W, which are not canonical. Now we consider a linear mapping $\mathcal{A} : V \to W$, If we use on both sides (V and W) the canonical basis, this mapping can be given by matrix A: $(V_{[e]} \xrightarrow{A} W_{[e]})$:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{15}$$

In the starting vector space (V, from which $\mathcal{A}$ maps), we want to switch from the canonical basis $[e]$ to basis $[v]$, similarly, in the image space (W, into which $\mathcal{A}$ maps), we want to switch to basis $[w]$. We wonder, what would be the matrix of the linear mapping $\mathcal{A}$, which maps from $V_{[v]}$ into $V_{[w]}$. First, let us define the following matrices:

$$S := \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} v_{1[e]} & v_{2[e]} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

$$T := \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} w_{1[e]} & w_{2[e]} \end{pmatrix}}_{\text{this is the correct prescription}} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \qquad T^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \tag{17}$$

then we have that

$$\boldsymbol{v}_i = S\boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{e}_i = S^{-1}\boldsymbol{v}_i. \tag{18}$$

The same is true for $[w]$, as well.
Let be $\boldsymbol{x}$ be a point in vector space V. The coordinate matrix of $\boldsymbol{x}$ in the canonical basis $[e]$ is $\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right)$, in other words:

$$\boldsymbol{x} = 3\boldsymbol{e}_1 = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} =: \begin{pmatrix} 3 \\ 0 \end{pmatrix}_{[e]} \tag{19}$$

Now, we want to compute the coordinate matrix of $\boldsymbol{x}$ in basis $[v]$

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} S^{-1}\boldsymbol{v}_1 & S^{-1}\boldsymbol{v}_2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = S^{-1} \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ &= S^{-1} S \begin{pmatrix} 3 \\ 0 \end{pmatrix} = S S^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} S^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \end{pmatrix} =: \begin{pmatrix} 0 \\ -3 \end{pmatrix}_{[v]} \end{aligned} \tag{20}$$

What is important that we have a linear mapping: $\mathcal{A} : V \to W$, $\mathcal{A}(\boldsymbol{x}) = \boldsymbol{y}$, furthermore

$$\begin{matrix} x_{[e]} = Sx_{[v]} \\ x_{[v]} = S^{-1}x_{[e]} \end{matrix}, \quad \begin{matrix} y_{[e]} = Ty_{[w]} \\ y_{[w]} = T^{-1}y_{[e]} \end{matrix}, \quad \begin{matrix} \boldsymbol{x} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \cdot x_{[e]} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} \cdot x_{[v]} \\ \boldsymbol{y} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \cdot y_{[e]} = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 \end{pmatrix} \cdot y_{[w]} \end{matrix} \tag{21}$$

The coordinates matrices $x_{[e]}$, $x_{[v]}$, $y_{[e]}$ and $y_{[w]}$ are intentionally NOT denoted as vectors, since they depend on the choice of the basis vectors (i.e. coordinates system). On the other hand, vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are two well-defined elements of V and W (eg. $\boldsymbol{x}$ is the left upper corner of the blackboard), which are independent of the choice of the coordinates system.

$$\mathcal{A}(\boldsymbol{x}) = \boldsymbol{y} \begin{array}{l} \nearrow y_{[e]} = A x_{[e]} \\ \searrow y_{[w]} = \hat{A} x_{[v]} \end{array} \qquad \begin{array}{ccc} V_{[e]} & \xrightarrow{A} & W_{[e]} \\ S \downarrow & \circlearrowleft & \downarrow T \\ V_{[v]} & \xRightarrow[\hat{A}]{} & W_{[w]} \end{array} \quad \Rightarrow \quad \boxed{\hat{A} = T^{-1} A S} \tag{22}$$

$$y_{[w]} = \hat{A} x_{[v]} = T^{-1} A \underbrace{S x_{[v]}}_{x_{[e]}} = T^{-1} \underbrace{A x_{[e]}}_{y_{[e]}} = T^{-1} y_{[e]} = y_{[w]} \tag{23}$$

Egy $\mathcal{A} : V \to V$ lineáris transzformáció mátrixa diagonális, ha a sajátvektorok bázisára vonatkozóan írjuk fel. A sajátvektorok azonban nem minden esetben alkotnak bázist, ezért nem minden leképezés írható le diagonális mátrixszal.

Egy lineáris leképezés mátrixa diagonalizálható akkor és csak akkor, ha minden sajátértékére annak algebrai és geometriai multiplicitása megegyezik.

- egy λ sajátérték algebrai multiplicitása k, ha λ k-szoros gyöke a karakterisztikus polinomnak.
- egy λ sajátérték geometriai multiplicitása k, ha a λ-hoz tartozó sajátaltér (λ-hoz tartozó sajátvektorok által alkotott altér) k dimenziós.

Example 7. $A = \begin{pmatrix} 0 & -3 & -3 \\ -3 & 0 & -3 \\ -3, & -3 & 0 \end{pmatrix} \quad \to \quad \lambda_{1,2} = 3,\ \lambda_3 = -6$

A 3 sajátérték algebrai multiplicitása tehát 2, a 6 sajátértéké 1.

$\lambda_{1,2} = 3$ sajátértékhez tartozó sajátvektorok: $\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot p,\ \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot q \quad p, q \in \mathbb{R} \setminus \{0\}$

$\lambda_3 = -6$ sajátértékhez tartozó sajátvektorok: $\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot r \quad r \in \mathbb{R} \setminus \{0\}$

A 3 sajátérték geometriai multiplicitása tehát 2, a 6 sajátértéké 1, mindkét sajátértékre a multiplicitások páronként megegyeznek, ezért a mátrix diagonalizálható. Ha a sajátvektorokat oszlopaiban tartalmazó mátrix S, akkor a leképezés diagonális mátrixa

$$D = S^{-1} \cdot A \cdot S \quad \to \quad A = S \cdot D \cdot S^{-1}$$

$$S = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad S^{-1} = \frac{1}{3} \cdot \begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad \to \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

1.8. Quadratic forms

A kvadratikus alakok $Q(\boldsymbol{x}) : V \to T$ alakú leképezések, ahol V egy T test feletti vektortér. Mi speciálisan $Q(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ leképezésekkel foglalkozunk. Általános alakja:

$$Q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

A kvadratikus alak felírható mátrixszorzat alakban is, ahol $a_{ij} = A_{ij}$ a mátrix megfelelő elemei. A mátrixszorzás definíciójából adódóan ez skalárszorzat alakban is írható.

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} = \langle A\boldsymbol{x}, \boldsymbol{x}\rangle$$

A kvadratikus alakhoz tartozó mátrix nem egyértelmű. A főátlóbeli elemek egyértelműek, ezek az x_i^2 alakú tagok együtthatói, azon kívül azonban csak az $a_{ij} + a_{ji}$ összegek ismertek $(i \neq j)$.

Példa:

$Q(\boldsymbol{x}) = 3x_1^2 + 4x_1x_2 - x_2^2$ kvadratikus alak többféle mátrixszal leírható

$$A_1 = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix}$$

A szimmetrikus mátrix egyértelműen tartozik a kvadratikus alakhoz, ezért ezt nevezzük a kvadratikus alak mátrixának (a példában A_2 mátrix).

Egy kvadratikus alak egy vektorhoz egy valós számot rendel. Ennek lehetséges előjele alapján osztályozzuk a kvadratikus alakokat, illetve ezzel ekvivalensen a szimmetrikus mátrixokat. (Később ezt sokszor fogjuk használni stabilitási vizsgálatok során.)
A $Q(x)$ kvadratikus alak, illetve az ezt leíró A szimmetrikus mátrix

- **pozitív definit**, ha $\langle A\boldsymbol{x}, \boldsymbol{x}\rangle \ > 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$
  Megjegyzés: minden kvadratikus alak a nullvektor esetén nulla értéket vesz fel, $\langle A \cdot \boldsymbol{0}, \boldsymbol{0}\rangle = 0$
- **pozitív szemidefinit**, ha $\langle A\boldsymbol{x}, \boldsymbol{x}\rangle \ \geq 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^n$
  és $\exists \boldsymbol{y} \neq \boldsymbol{0}$, hogy $\langle A\boldsymbol{y}, \boldsymbol{y}\rangle = 0$
- **negatív definit**, ha $\langle A\boldsymbol{x}, \boldsymbol{x}\rangle \ < 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$
- **negatív szemidefinit**, ha $\langle A\boldsymbol{x}, \boldsymbol{x}\rangle \ \leq 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^n$ és $\exists \boldsymbol{y} \neq \boldsymbol{0}$, hogy $\langle A\boldsymbol{y}, \boldsymbol{y}\rangle = 0$
- **indefinit**, ha pozitív és negatív értékeket egyaránt felvesz a kvadratikus alak.

A kvadratikus alakhoz tartozó A szimmetrikus mátrix λ_i sajátértékei valósak, melyekkel a fentiekkel ekvivalens feltételek fogalmazhatók meg. A $Q(\boldsymbol{x})$ kvadratikus alak

- pozitív definit $\Leftrightarrow$ $\forall i \ \lambda_i > 0$
- pozitív szemidefinit $\Leftrightarrow$ $\forall i \ \lambda_i \geq 0$ és $\exists j \ \lambda_j = 0$
- negatív definit $\Leftrightarrow$ $\forall i \ \lambda_i < 0$
- negatív szemidefinit $\Leftrightarrow$ $\forall i \ \lambda_i \leq 0$ és $\exists j \ \lambda_j = 0$
- indefinit $\Leftrightarrow$ $\exists i \ \lambda_i > 0$ és $\exists j \ \lambda_j < 0$

A kvadratikus alakot leíró mátrix is diagonalizálható. Mivel egy szimmetrikus mátrix különböző sajátértékeihez tartozó sajátvektorok ortogonálisak, ha diagonalizálható, akkor ortonormált sajátbázis szerint is diagonalizálható. Amennyiben valamelyik sajátérték geometriai multiplicitása nem 1, és a számolás során nem ortogonális sajátvektorokat kaptunk, akkor valamilyen ortogonalizációs eljárással (pl. Gram-Schmidt ortogonalizáció) elérhető, hogy a sajátalteret ortogonális vektorok generálják. Emellett a vektorokat minden esetben normálni kell.

Ekkor ha S az ortonormált bázis vektorait tartalmazó mátrix, akkor $S^{-1} = S^T$

$A = S \cdot D \cdot S^{-1}$

$\boldsymbol{x}^T \cdot A \cdot \boldsymbol{x} = \boldsymbol{x}^T \cdot S \cdot D \cdot S^{-1} \cdot \boldsymbol{x} = (\boldsymbol{x}^T \cdot (S^T)^T) \cdot D \cdot (S^{-1} \cdot \boldsymbol{x}) = (S^T \cdot \boldsymbol{x})^T \cdot D(\cdot S^{-1} \cdot \boldsymbol{x})$

$\boldsymbol{y} = S^{-1} \cdot \boldsymbol{x} = S^T \cdot \boldsymbol{x} \quad \longrightarrow \quad \boldsymbol{x}^T \cdot A \cdot \boldsymbol{x} = \boldsymbol{y}^T \cdot D \cdot \boldsymbol{y}$

1.9. Matrix calculus

Az exponenciális függvény ($\exp(x) = e^x$) az alábbi hatványsorral definiálható.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{(Taylor sorfejtés)}$$

Elképzelhető, hogy egy fix paramétere is van a függvénynek, például: $a \in \mathbb{R}$ konstans, és t az idő változó

$$e^{at} = 1 + at + \frac{a^2t^2}{2} + ... = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}$$

Az exponenciális függvényt kiterjeszthetjük mátrixokra (mátrix exponenciális),
ekkor $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$.

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + ... = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Tulajdonságok:

- $e^{\mathbf{0}} = I$, ahol $\mathbf{0} \in \mathbb{R}^{n\times n}$ nullmátrix
- Ha $AB = BA$ akkor $e^A e^B = e^{A+B}$. Általában nem igaz a képlet, mivel a mátrix szorzás nem kommutatív.
- $e^A e^{-A} = I$
- $e^{At} e^{As} = e^{A(t+s)}$, ahol $A \in \mathbb{R}^{n\times n}$, $t, s \in \mathbb{R}$

$$\begin{aligned}
e^{At}e^{As} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \sum_{k=0}^{\infty} \frac{A^k s^k}{k!} \\
&= (I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + ...)(I + As + \frac{A^2s^2}{2} + \frac{A^3s^3}{6} + ...) = \\
&= I + A(t+s) + \frac{1}{2}A^2(t^2 + 2ts + s^2) + \frac{1}{6}A^3(t^3 + 3ts^2 + 3t^2s + s^3) + ... \\
&= I + A(t+s) + \frac{1}{2}A^2(t+s)^2 + \frac{1}{6}A^3(t+s)^3 + ... = \sum_{k=0}^{\infty} \frac{A^k (t+s)^k}{k!}
\end{aligned}$$

- $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$

$$\begin{aligned}
\frac{d}{dt}(e^{At}) &= A + A^2t + \frac{1}{2}A^3t^2 + ... + \frac{1}{(k-1)!}A^k t^{k-1} \\
&= A(I + At + \frac{1}{2}A^2t^2 + ...) = (I + At + \frac{1}{2}A^2t^2 + ...)A
\end{aligned}$$

Example 8. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$e^A = I + A + \frac{A^2}{2!} + ... = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$e^B = I + B + \frac{B^2}{2!} + ... = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad e^B e^A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$e^{A+B} = I + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{2!} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{3!} + ... = I \cdot (1 + \frac{1}{2!} + \frac{1}{4!} + ...) + I' \cdot (1 + \frac{1}{3!} + \frac{1}{5!} + ...) =$
$= \begin{pmatrix} ch(1) & sh(1) \\ sh(1) & ch(1) \end{pmatrix}$

Proposition 4. tetszőleges diagonalizálható mátrix exponenciális függvénye

$$A = SDS^{-1} \Rightarrow e^A = Se^D S^{-1} = S \exp \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} S^{-1} = S \operatorname{diag}\left(e^{\lambda_1}, .., e^{\lambda_n}\right) S^{-1} \tag{24}$$

Proof (inkább csak levezetés).

$$A^k = SD^k S^{-1} \quad , \quad D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} = \operatorname{diag}(\lambda_1^k, .., \lambda_n^k) \tag{25}$$

Teljes indukció:

$$A^k = A^{k-1} A = \left(SD^{k-1} S^{-1}\right)\left(SDS^{-1}\right) = SD^k S^{-1} \tag{26}$$

Ekkor a következőket csinálhatjuk:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = S \sum_{k=0}^{\infty} \frac{D^k}{k!} S^{-1} = S \operatorname{diag}\left(\sum_{k=0}^{\infty} \lambda_1^k, .., \sum_{k=0}^{\infty} \lambda_n^k\right) S^{-1} = S \operatorname{diag}\left(e^{\lambda_1}, .., e^{\lambda_n}\right) S^{-1} \tag{27}$$

Tehát, valóban $e^A = Se^D S^{-1}$. □

Matlab 2. Mátrixexponenciális kiszámítása – diagonalizáció ($A = SDS^{-1}$) eig,expm,diag

```
>> A = rand(3,3)
A =
    0.8147    0.9134    0.2785
    0.9058    0.6324    0.5469
    0.1270    0.0975    0.9575
>> [S,D] = eig(A)
S =
    0.6752   -0.7134   -0.5420
   -0.7375   -0.6727   -0.2587
   -0.0120   -0.1964    0.7996
```

```
D =
   -0.1879         0         0
         0    1.7527         0
         0         0    0.8399

>> expm(A)
ans =
    3.2881    2.2290    1.3802
    2.2617    2.8712    1.7128
    0.4615    0.3910    2.7554
```

```
>> S * expm(D) / S
ans =
    3.2881    2.2290    1.3802
    2.2617    2.8712    1.7128
    0.4615    0.3910    2.7554

>> S * diag(exp(diag(D))) / S
ans =
    (... ugyanaz )
```

1.10. Állandó Együtthatós Lineáris Differenciálegyenletek

A továbbiakban a vektorértékű változók, függvények (pl. $\boldsymbol{x}$) nem lesznek kiemelve.

A tárgy keretében főleg elsőrendű lineáris differenciálegyenletekkel foglalkozunk.

$$\dot{x}(t) = Ax(t) \quad x(0) = x_0 \in \mathbb{R}^n$$

1. Ha $A \in \mathbb{R}$ (skaláris eset), akkor a változók szétválasztásának módszerével meghatározható a megoldás.

Megoldás: $x(t) = e^{At}x_0$, ahol $x_0 \in \mathbb{R}$ skalár

2. Ha $A \in \mathbb{R}^{n\times n}$, akkor szükséges az e^{At} mátrix kiszámítása.
   Megoldás: $x(t) = e^{At}x_0 \quad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$
   Közelítés: $I + At + \frac{A^2t^2}{2} + \ldots$

Példa: Adott az alábbi csatolatlan differenciálegyenlet rendszer

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= 2x_2 \end{aligned}$$

Amely átírható a következő alakra

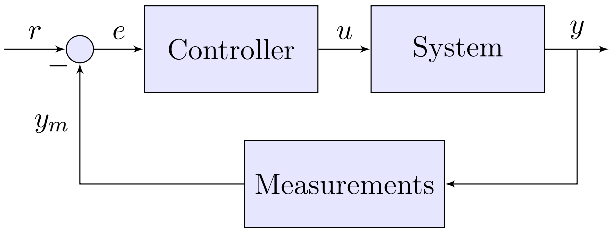
$$\dot{x} = Ax \qquad A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Mivel csak a diagonális elemek nem nullák, a rendszer csatolatlan, ezért a változók szétválasztásának módszerével a differenciálegyenlet megoldható.

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned}$$

ahol $c_1 = x_{1_0}$ és $c_2 = x_{2_0}$

2. Általános Példák Szabályzott Rendszerekre



Rendszer: Olyan fizikai vagy logikai eszköz, amely jeleken végez valamilyen műveletet. (Bemenő jeleket dolgoz fel, és kimenő jeleket állít elő.)
Példa rendszerekre, és szabályozásra

- Autó sebessége: beavatkozás: gázpedál, érzékelés: sebességmérő, szabályozás: humán vagy tempomat
- Hűtőszekrény hőmérséklet: beavatkozás: hűtőközeg mozgatása - kompresszor, érzékelés: hőmérő, szabályozás: szabályozó automatika beállított hőmérséklet elérése érdekében
- További példák találhatóak az FSB könyv első fejezetében, lásd a tárgy honlapján!

Computer controlled systems

Lecture 2

version: 2018.10.02. – 10:52:09

Problem types

- Convolution of two casual signals
- Solution of initial value probles using Laplace transformation, eg. $\ddot{y} + a\dot{y} + by = e^{-at}$, $y(0)$, $\dot{y}(0)$, $y(t) = ?$
- Partial fractional decomposition
- Determine the transfer function of $\ddot{y} + a\dot{y} + by = u(t)$.
- Solution of initial values problem in the state-space form using Laplace tr, eg. $\dot{x} = Ax,\ x(0)$
- Compute the transfer function ($H(s)$) of the state space model ($\dot{x} = Ax + Bu,\ y = Cx$)
- Solution of the state space model, given both the input and the initial values – impulse response ($h(t)$), response to the unit step function

1. Causal convolution

Definition 1. A signal $f(t)$ is called causal if $f(t < 0) = 0$.

Definition 2. (Causal convolution) The convolution of two causal signals $f(t)$ and $g(t)$ reduces to:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{0}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{0}^{t} f(\tau)g(t-\tau)\mathrm{d}\tau$$

2. Laplace transformation

Definition 3. The Laplace transform of a signal $f(t)$ is denoted by $F(s)$, $s \in \mathbb{C}$ and it is defined as follows:

$$F(s) = \mathfrak{L}\{f(t), s\} = \int_{0}^{\infty} f(t)e^{-st}\mathrm{d}t \tag{1}$$

Due to the fact that the integral operator is linear, the Laplace transformation is inherently linear, i.e. it preserves the order of addition and scaling operations.

Based on the properties of the integral the laplace transform is a linear mapping.

2.1. Rules

1. Convolution in time domain: $\boxed{\mathfrak{L}\{(f*g)(t), s\} = F(s)G(s),}$

   ahol $F(s) = \mathfrak{L}\{f(t), s\}$, $G(s) = \mathfrak{L}\{g(t), s\}$, $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$. Derivation:

$$\begin{aligned}\mathfrak{L}\{(f*g)(t), s\} &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\, e^{-st}\mathrm{d}t = \int_0^\infty \int_0^\infty f(\tau)g(t-\tau)e^{-st}\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_0^\infty \int_0^\infty g(t-\tau)e^{-s(t-\tau)}\mathrm{d}t\, f(\tau)e^{-s\tau}\mathrm{d}\tau = \int_0^\infty \int_{-\tau}^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta\, f(\tau)e^{-s\tau}\mathrm{d}\tau \\ &= \int_0^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta \int_0^\infty f(\tau)e^{-s\tau}\mathrm{d}\tau = \mathfrak{L}\{f(t), s\}\,\mathfrak{L}\{g(t), s\}\end{aligned} \tag{2}$$

   We will deal with functions for which $f(t) = g(t) = 0$ for all $t < 0$, hence

$$\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = \int_0^\infty f(\tau)g(t-\tau)\mathrm{d}\tau \quad \text{mivel } g(t-\tau) = 0 \text{ bármely } \tau > t \tag{3}$$

   It was also used during the above derivation (change of variables: $\vartheta = t - \tau$):

$$\int_0^\infty g(t-\tau)e^{-s(t-\tau)}\mathrm{d}t = \int_{-\tau}^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta = \int_0^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta \quad \text{since } g(t<0) = 0 \tag{4}$$

2. Time derivative:

   $\boxed{\mathfrak{L}\{\dot{y}(t), s\} = sY(s) - y(0),}$ ahol $Y(s) = \mathfrak{L}\{y(t), s\}$. Derivation:

$$\int_0^\infty \dot{y}(t)e^{-st}\mathrm{d}t = y(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty y(t)e^{-st}\mathrm{d}t = -y(0) + s\mathfrak{L}\{y(t), s\} \tag{5}$$

3. Second derivative according to the time variable:

   $\boxed{\mathfrak{L}\{\ddot{y}(t), s\} = s^2Y(s) - \dot{y}(0) - sy(0).}$ Derivation:

$$\mathfrak{L}\{\ddot{y}(t), s\} = s\mathfrak{L}\{\dot{y}(t), s\} - \dot{y}(0) = s^2Y(s) - sy(0) - \dot{y}(0) \tag{6}$$

2.2. Limit theorems

1. Initial value theorem. If $e^{-st}y(t)$ is Lebesgue integrible on $t \in \mathbb{R}_+$, then $\boxed{y(0) = \lim_{s\to\infty} sY(s)}$

   Proof. Let us take the limit of both the left and right sides of the rule of derivation $s \to \infty$:

$$\lim_{s\to\infty}\int_0^\infty e^{-st}\dot{y}(t)\mathrm{d}t \overset{(*)}{=} \int_0^\infty \underbrace{\lim_{s\to\infty} e^{-st}}_{=0}\dot{y}(t)\mathrm{d}t = \lim_{s\to\infty}(sY(s) - y(0)) \quad\Rightarrow\quad y(0) = \lim_{s\to\infty} sY(s) \tag{7}$$

   $^{(*)}$Due to Lebesgue's dominated convergence theorem, since $e^{-st}\dot{y}(t)$ is (by assumption) Lebesgue integrible on $\mathbb{R}_+$.

2. Limit value theorem. If $e^{-st}y(t)$ is Lebesgue integrible on $t \in \mathbb{R}_+$, then $\boxed{y(\infty) = \lim_{s\to 0} sY(s)}$

   Proof. Let us take the derivation rule

$$\int_0^\infty \dot{y}(t)e^{-st}\mathrm{d}t = sY(s) - y(0) \tag{8}$$

and consider the limit of both sides $s \to 0$:

$$\lim_{s\to 0}\int_0^\infty e^{-st}\underbrace{\dot{y}(t)\mathrm{d}t}_{\mathrm{d}y(t)} \overset{(*)}{=} \int_0^\infty \underbrace{\lim_{s\to 0} e^{-st}}_{\to 1}\underbrace{\dot{y}(t)\mathrm{d}t}_{\mathrm{d}y(t)} = \lim_{s\to 0}\left(sY(s) - y(0)\right) \tag{9}$$

$$\int_0^\infty \mathrm{d}y(t) = \lim_{s\to 0} sY(s) - y(0) \tag{10}$$

$$y(\infty) - y(0) = \lim_{s\to 0} sY(s) - y(0) \quad \Rightarrow \quad y(\infty) = \lim_{s\to 0} sY(s) \tag{11}$$

$^{(*)}$Due to Lebesgue's dominated convergence theorem, since $e^{-st}\dot{y}(t)$ is (by assumption) Lebesgue integrible on $\mathbb{R}_+$.

Counterexample if $e^{-st}y(t)$ is not absolute integrible for all $\mathrm{Re}\{s\} > 0$. Let $y(t) = e^t$. Its Laplace transformation is $Y(s) = \frac{1}{s-1}$. $\lim_{s\to 0}\frac{s}{s-1} = 0$, however $y(t) \to \infty$.

2.3. Laplace transformation of some important functions

1. $\boxed{\mathfrak{L}\{\delta(t), s\} = 1}$ derivation: $\displaystyle\int_0^\infty \delta(t)e^{-st}\mathrm{d}t = \lim_{T\to 0}\frac{1}{T}\int_0^T e^{-st}\mathrm{d}t = \frac{1}{s}\underbrace{\lim_{T\to 0}\frac{1-e^{-sT}}{T}}_{\text{L'Hospital: } s} = 1$

   Note that in this derivation Dirac's function is estimated as follows:

$$\delta(t) = \lim_{T\to 0}\begin{cases}\frac{1}{T} & t \in [0, T) \\ 0 & \text{otherwise}\end{cases} \tag{12}$$

2. $\boxed{\mathfrak{L}\{1(t), s\} = \frac{1}{s}}$ derivation: $\int_0^\infty 1(t)e^{-st}\mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$,
   where $1(t)$ unit step function, for this the $u(t)$ notation is laso commonly used, but in this course $u(t)$ denotes the input of the system.

3. $\mathfrak{L}\{t \cdot 1(t), s\} = \frac{1}{s^2}$ (unit step velocity function)

4. $\boxed{\mathfrak{L}\{e^{-at}, s\} = \frac{1}{s+a},}$ it is the most commonly used when trying to determine inverse Laplace transformations of complex rational functions.

5. $\mathfrak{L}\{e^{-t/T}, s\} = \frac{1}{s+1/T} = \frac{T}{1+sT}$, it is another form of the previous case.
   Derivation: $\int_0^\infty e^{-t/T}e^{-st}\mathrm{d}t = \int_0^\infty e^{-(s+1/T)t}\mathrm{d}t \left[\frac{e^{-(s+1/T)t}}{-(s+1/T)}\right]_0^\infty = \frac{1}{s+1/T} = \frac{T}{1+sT}$.

   Pole-zero from: $\frac{1}{s+1/T}$

   Time-constant form: $\frac{T}{1+sT}$

6. $\mathfrak{L}\{1 - e^{-t/T}, s\} = \frac{1}{s(1+sT)}$ (time-constant form)

7. $\mathfrak{L}\left\{\frac{1}{T_1 - T_2}(e^{-t/T_1} - e^{-t/T_2}), s\right\} = \frac{1}{(1+sT_1)(1+sT_2)}$ (time-constant form)

8. $\mathfrak{L}\{e^{at}\sin(bt), s\} = \dfrac{b}{(s-a)^2 + b^2}$

9. $\mathfrak{L}\{e^{at}\cos(bt), s\} = \dfrac{s-a}{(s-a)^2 + b^2}$

2.4. Inverse Laplace transform

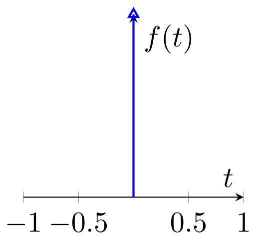
Definition 4. The inverse Laplace transformation of the complex signal $F(s)$ is given by the following complex line integral:

$$f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{c-jT}^{c+jT} F(s)e^{st}\mathrm{d}s \tag{13}$$

where $c \in \mathbb{R}$ is greater than the real parts of F(s)'s singularities.

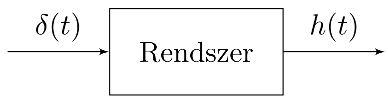
2.5. Input, system response

1. **Dirac impulse**



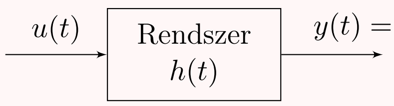
$$f_\tau(t) = \begin{cases} \frac{1}{\tau} & \text{ha } 0 \le t < \tau \\ 0 & \text{egyébként} \end{cases} \qquad \delta(t) = \lim_{\tau \to 0+} f_\tau(t)$$

2. **the output (response) of the system to the Dirac impulse (impulse response)**: $h(t)$
   E.g.. if I strike on a trapdoor $(\delta(t))$, then it will dampedly oscillate$(h(t))$.



   Convolutional time-invariance: $\delta(t-\tau)$, $h(t-\tau)$.

3. **The system response to $u(t)$ (transfer function)**: Causal convolution



$$y(t) = (u * h)(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

Example 1.

Let us compute the convolution of $f(t) = t$ and $g(t) = t^2$:

$$(f * g)(t) = \int_0^t (t-\tau)\tau^2 d\tau = \int_0^t t\tau^2 - \tau^3 d\tau = \left[\frac{t\tau^3}{3} - \frac{\tau^4}{4}\right]_0^t = \frac{t^4}{12} \tag{14}$$

3. applying laplace transform to solve initial value problems

Example 2. Constant coefficient second order linear differential equation

Solve the following initial value problem:

$$\ddot{y} - 2\dot{y} + 5y = -8e^{-t} \quad y(0) = 2 \quad \dot{y}(0) = 12$$

One can compute the Laplace transform as follows (elementwise).

$$\mathfrak{L}\{\ddot{y}\} - 2\mathfrak{L}\{\dot{y}\} + 5\mathfrak{L}\{y\} = -\frac{8}{s+1} \tag{15}$$

Laplace transform in the case of derivated function: $\mathfrak{L}\{\dot{y}\} = sY(s) - y(0) = sY(s) - 2$. and the second derivative: $\mathfrak{L}\{\ddot{y}\} = s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - 2s - 12$. such a way the equation (15) has the following form:

$$(s^2Y(s) - 2s - 12) - 2(sY(s) - 2) + 5Y(s) = -\frac{8}{s+1} \tag{16}$$

expressing $Y(s)$ we get:

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} \xrightarrow{\mathfrak{L}^{-1}} y(t) =? \tag{17}$$

Example 3. Partial fraction decomposition

Tel us solve the following initial value problem:

$$\dddot{y} + 7\ddot{y} + 14\dot{y} + 8y = 0 \quad y(0) = 0 \quad \dot{y}(0) = 0 \quad \ddot{y}(0) = 2$$

The physical interpretation of initial value: nyugalomban lévő testre hat egy gyorsulásvektor (pl. gravitációs gyorsulás). Az előző feladathoz hasonlóan ha vesszük az egyenlet mindkét oldalának The Laplace transform is the following:

$$Y(s) = \frac{2}{(s+1)(s+2)(s+4)} = \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{s+4} \quad \xrightarrow{\mathfrak{L}^{-1}} \quad y(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{-4t}$$

I fall the roots of the denominator have single multipülicity, then the following formula can be applied:

$$C_i = \lim_{s\to\alpha_i}(s-\alpha_i)Y(s), \quad \text{ahol } \alpha_i \text{ az } \frac{C_i}{s+\alpha_i} \text{ gyöke}$$

$$C_1 = \lim_{s\to-1}(s+1)Y(s) = \frac{2}{(s+2)(s+4)}\,|_{s=-1} = \frac{2}{3}$$

$$C_2 = \lim_{s\to-2}(s+2)Y(s) = \frac{2}{(s+1)(s+4)}\,|_{s=-2} = -1$$

$$C_3 = \lim_{s\to-4}(s+4)Y(s) = \frac{2}{(s+1)(s+2)}\,|_{s=-4} = \frac{1}{3}$$

$$Y(s) = \frac{\frac{2}{3}}{s+1} + \frac{-1}{s+2} + \frac{\frac{1}{3}}{s+4} \tag{18}$$

Tehát a megoldás:

$$y(t) = \frac{2}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t} \tag{19}$$

Matlab 1. Inverse Laplace transform `partfrac,ilaplace,residue,poly2sym,sym2poly`

3. Continuation of the example:

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} = \frac{3s+5}{s^2 - 2s + 5} - \frac{1}{s+1} \quad \Rightarrow \quad y(t) = 3\mathrm{e}^t\left(\cos(2t) + \frac{4\sin(2t)}{3}\right) - \mathrm{e}^{-t}$$

By means of the symbolic toolbox:

```
>> syms s
>> Y = partfrac( (2*s^2 + 10*s) / ((s+1) * (s^2 - 2*s + 5)) )
Y =
(3*s + 5)/(s^2 - 2*s + 5) - 1/(s + 1)
```

```
>> ilaplace(Y)
ans =
3*exp(t)*(cos(2*t) + (4*sin(2*t))/3) - exp(-t)
```

by means of numerical computations:

```
>> Y = expand((s+1) * (s^2 - 2*s + 5))
Y =
s^3 - s^2 + 3*s + 5
>> B = [2 10 0];
>> A = sym2poly(Y)
A =
     1    -1     3     5
>> [r,p,k] = residue(B,A)
r =
   1.5 - 2i
   1.5 + 2i
  -1   + 0i
p =
   1   + 2i
   1   - 2i
  -1   + 0i
k =
     []
```

$$Y(s) = \frac{B(s)}{A(s)} = \sum_i \frac{r_i}{s - p_i} + K(s) = -\frac{1}{s+1} + \frac{1.5 - 2j}{s - 1 - 2j} + \frac{1.5 + 2j}{s - 1 + 2j} \tag{20}$$

```
>> Y = sum(r ./ (s - p)) + poly2sym(k)
Y =
- 1/(s + 1) + (3/2 - 2i)/(s - 1 - 2i) + (3/2 + 2i)/(s - 1 + 2i)
>> latex(Y)
ans =
 - \frac{1}{s + 1} + \frac{\frac{3}{2} - 2\, \mathrm{i}}{s - 1 - 2\, \mathrm{i}} + \frac{\frac{3}{2} + 2\, [...]
>> ilaplace(Y)
ans =
- exp(-t) + exp(t*(1 - 2i))*(3/2 + 2i) + exp(t*(1 + 2i))*(3/2 - 2i)
>> Y = simplify(Y)
ans =
(2*s*(s + 5))/(s^3 - s^2 + 3*s + 5)
>> ilaplace(Y)
ans =
3*exp(t)*(cos(2*t) + (4*sin(2*t))/3) - exp(-t)
```

$$y(t) = -\mathrm{e}^{-t} + \mathrm{e}^{t\,(1-2j)} \left(\frac{3}{2} + 2j\right) e^{t\,(1+2j)} \left(\frac{3}{2} - 2j\right) = 3\mathrm{e}^{t}\left(\cos(2t) + \frac{4\sin(2t)}{3}\right) - \mathrm{e}^{-t} \tag{21}$$

Example 4.

Constant coefficient linear differential equation system

$$\begin{aligned}\dot{x}_1 &= 2x_1 + 3x_2 \\ \dot{x}_2 &= 2x_1 + x_2\end{aligned} \quad \rightarrow \quad \dot{x} = Ax \quad A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution: $x(t) = e^{At}x_0$, $\boxed{e^{At} = Se^{Dt}S^{-1} = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}.}$

from the eigenvalue-eigenvector decomposition of the first equation (previous practice). moreover in 1 dimension:

$$e^{at} = \mathfrak{L}^{-1}\{(s-a)^{-1}\} = \mathfrak{L}^{-1}\left\{\frac{1}{s-a}\right\} \tag{22}$$

Both of the expressions can be used. In this case the second:

$$\det(sI - A) = \begin{vmatrix} s-2 & -3 \\ -2 & s-1 \end{vmatrix} = (s-2)(s-1) - 6 = s^2 - 3s - 4 = (s-4)(s+1) \tag{23}$$

$$(sI - A)^{-1} = \frac{1}{(s-4)(s+1)} \begin{pmatrix} s-1 & 3 \\ 2 & s-2 \end{pmatrix}$$

According to the linearity of the Laplace transform:

$$e^{At}x_0 = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \dfrac{3}{(s-4)(s+1)} \\ \dfrac{s-2}{(s-4)(s+1)} \end{pmatrix}\right\}$$

Partial fraction decomposition:

$$\frac{3}{(s-4)(s+1)} = \frac{3}{5}\frac{(s+1)-(s-4)}{(s-4)(s+1)} = \frac{0.6}{s-4} - \frac{0.6}{s+1} \tag{24}$$

Using a simpler method:

$$\frac{s-2}{s^2-3s-4} = \frac{C_3}{s+1} + \frac{C_4}{s-4} \rightarrow C_3 = 0.6 \quad C_4 = 0.4 \tag{25}$$

Finally:

$$x(t) = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \dfrac{-0.6}{s+1} + \dfrac{0.6}{s-4} \\ \dfrac{0.6}{s+1} + \dfrac{0.4}{s-4} \end{pmatrix}\right\} = \begin{pmatrix} -0.6e^{-t} + 0.6e^{4t} \\ 0.6e^{-t} + 0.4e^{4t} \end{pmatrix} \tag{26}$$

Applying the second formula: $e^{At} = Se^{Dt}S^{-1}$, the decomposition is not required, but the eigenvalues and eigenvectors are necessray.

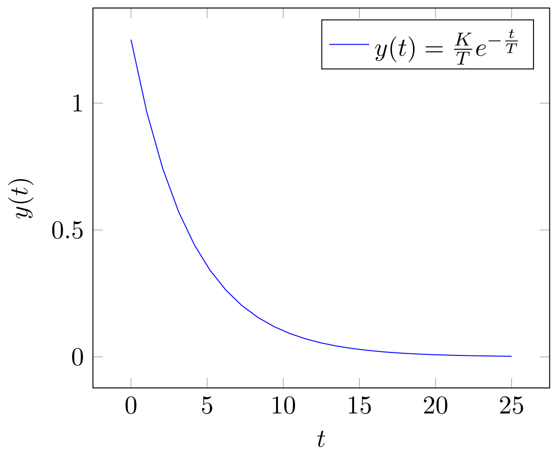
Matlab 2. $\dot{x} = Ax,\ x(0) = x_0$ solution with symbolic toolbox `eig,syms,expand,pretty,diag`

$$\dot{x} = Ax\ ,\quad x(0) = x_0 \text{ megoldása } x(t) = e^{At}x_0\ ,\quad e^{At} = Se^{Dt}S^{-1} \text{ képlettel} \tag{27}$$

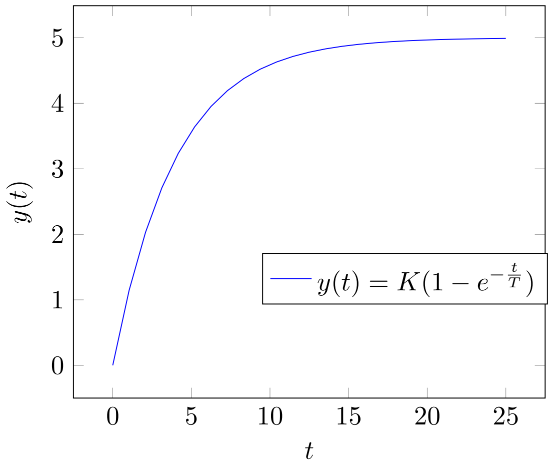
```
syms t real

A = [2 3 ; 2 1];
x0 = [0;1];

[S,D] = eig(A);

SDS_A_iszero = S * D / S - A

exp_Dt = diag(exp(diag(D)*t));
fprintf('\nexp(Dt) = \n\n')
pretty(exp_Dt)

exp_At = expand(S * exp_Dt / S);
fprintf('\n[Matlabbal szamolt sajatvektorok] \nexp(At) = \n\n'), pretty(exp_At)

xt = exp_At * x0;
fprintf('\nA differencialegyenlet megoldasa: x(t) = \n\n')
pretty(expand(xt))
```

Eredmény:

```
exp(Dt) =

/ exp(4 t),    0    \
|                   |
\     0,    exp(-t) /


[Eigenvectors computed by Matlab]
exp(At) =

/ 2 exp(-t)   exp(4 t) 3  exp(4 t) 3   3 exp(-t) \
| --------- + ----------, ---------- - --------- |
|     5           5           5            5     |
|                                                |
| exp(4 t) 2   2 exp(-t)  3 exp(-t)   exp(4 t) 2 |
| ---------- - ---------, --------- + ---------- |
\      5           5          5            5     /

A differencialegyenlet megoldasa: x(t) =

/ exp(4 t) 3   3 exp(-t) \
| ---------- - --------- |
|      5           5     |
|                        |
| 3 exp(-t)   exp(4 t) 2 |
| --------- + ---------- |
\     5           5      /
```



(a) Response to the Dirac impulse



(b) Response to the unit step impulse

Example 5. Applying Laplace transform

The differential equation system describing the system: $T\dot{y} + y = Ku(t) \quad y(0) = 0$
Let us determine the system's response in the followingh cases:

1. $u(t) = \delta(t)$
2. $u(t) = 1(t)$

The Laplace transform of the system: $TsY(s) + Y(s) = KU(s)$, where $T \in \mathbb{R}$ and $K \in \mathbb{R}$ parameters depending on the system. Impulse response function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K}{1 + Ts} \tag{28}$$

System's response: $Y(s) = \frac{K}{1+Ts}U(s)$

1. impulse response

$$u(t) = \delta(t) \xrightarrow{\mathfrak{L}} U(s) = 1$$

$$Y(s) = K\frac{1}{1 + Ts} \quad \xrightarrow{\mathfrak{L}^{-1}} \quad y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \frac{K}{T}\mathfrak{L}^{-1}\left\{\frac{1}{s + \frac{1}{T}}\right\} = \frac{K}{T}e^{-t/T}$$

2. transfer function (response to the unit step function)

$$u(t) = 1(t) \xrightarrow{\mathfrak{L}} U(s) = \frac{1}{s}$$

$$Y(s) = K\frac{1}{s(1 + Ts)} \quad \xrightarrow{\mathfrak{L}^{-1}}$$

$$y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \frac{K}{T} \cdot \mathfrak{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s + \frac{1}{T}}\right\} = \frac{K}{T} \cdot (1(t) * e^{\frac{-t}{T}}) = K(1 - e^{-t/T})$$

for the values $K = 5$, $T = 4$ the solution is depicted in the above picture.

4. Állapotegyenlet megoldása

- only excitation ($x_0 = 0, u(t) \neq 0$) $\rightarrow$: Laplace transform
- only initial value ($x_0 \neq 0, u(t) = 0$) $\rightarrow e^{At}x_0$, state trajectories
- both excitation and initial values

Example 6. SSM solution – unit step input

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad u(t) = 1(t)$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{29}$$

Applying Laplace transfromation:

$$sX(s) = AX(s) + BU(s) \rightarrow sX(s) - AX(s) = BU(s)$$

$$(sI - A)X(s) = BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s)$$

$$H(s) = Y(s)/U(s) = C(sI - A)^{-1}B = \frac{s}{s^2 - 3s - 4} = \frac{s}{(s+1)(s-4)}$$

$$Y(s) = H(s)U(s) = \frac{s}{(s+1)(s-4)} \cdot \frac{1}{s} = \frac{1}{(s+1)(s-4)} = \frac{0.2}{s-4} - \frac{0.2}{s+1}$$

$$y(t) = 0.2e^{4t} - 0.2e^{-t}$$

Example 7. SSM solution – autonomous system

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u(t) = 0$$

Applying Laplace transfromation:

$$sX(s) - x_0 = AX(s) \quad \rightarrow \quad X(s) = (sI - A)^{-1}x_0 \quad \rightarrow \quad x(t) = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}x_0 = e^{At}x_0$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s-4)} \cdot \begin{pmatrix} s-1 & 3 \\ 2 & s-2 \end{pmatrix}$$

Output: $y(t) = Cx(t) = C \cdot \mathfrak{L}^{-1}\left\{(sI - A)^{-1}\right\} \cdot x_0 = \mathfrak{L}^{-1}\left\{C(sI - A)^{-1}x_0\right\} = \mathfrak{L}^{-1}\left\{\frac{s-2}{(s+1)(s-4)}\right\} = 0.6e^{-t} + 0.4e^{4t}$

Example 8. SSM solution – unit steo velocity

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u(t) = t$$

Applying Laplace transformation:

$$sX(s) - x_0 = X(s) + BU(s) \quad \rightarrow \quad X(s) = (sI - A)^{-1}(x_0 + BU(s)) \quad \rightarrow$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

If $t_0 = 0$, then $e^{A(t-t_0)} = e^{At}$

$$e^{At} = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \frac{s-1}{s^2-3s-4} & \frac{3}{s^2-3s-4} \\ \frac{2}{s^2-3s-4} & \frac{s-2}{s^2-3s-4} \end{pmatrix}\right\} = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \frac{0.6}{s-4} + \frac{0.4}{s+1} & \frac{0.6}{s-4} - \frac{0.6}{s+1} \\ \frac{0.4}{s-4} - \frac{0.4}{s+1} & \frac{0.4}{s-4} + \frac{0.6}{s+1} \end{pmatrix}\right\}$$

$$e^{At} = \begin{pmatrix} 0.6e^{4t} + 0.4e^{-t} & 0.6e^{4t} - 0.6e^{-t} \\ 0.4e^{4t} - 0.4e^{-t} & 0.4e^{4t} + 0.6e^{-t} \end{pmatrix}$$

$$e^{A(t-\tau)} = \begin{pmatrix} 0.6e^{4(t-\tau)} + 0.4e^{-(t-\tau)} & 0.6e^{4(t-\tau)} - 0.6e^{-(t-\tau)} \\ 0.4e^{4(t-\tau)} - 0.4e^{-(t-\tau)} & 0.4e^{4(t-\tau)} + 0.6e^{-(t-\tau)} \end{pmatrix}$$

$$e^{A(t-\tau)}B = \begin{pmatrix} 1.2e^{4(t-\tau)} - 0.2e^{-(t-\tau)} \\ 0.8e^{4(t-\tau)} + 0.2e^{-(t-\tau)} \end{pmatrix} \rightarrow e^{A(t-\tau)}Bu(\tau) = \begin{pmatrix} 1.2e^{4(t-\tau)}\tau - 0.2e^{-(t-\tau)}\tau \\ 0.8e^{4(t-\tau)}\tau + 0.2e^{-(t-\tau)}\tau \end{pmatrix}$$

Elementwise integral:
$\int_0^t c_1 e^{c_2(t-\tau)}\tau d\tau = c_1 e^{c_2 t}\int_0^t e^{-c_2\tau}\tau d\tau = \frac{c_1}{c_2^2}(e^{c_2 t} - c_2 t - 1)$ (Partial integration)

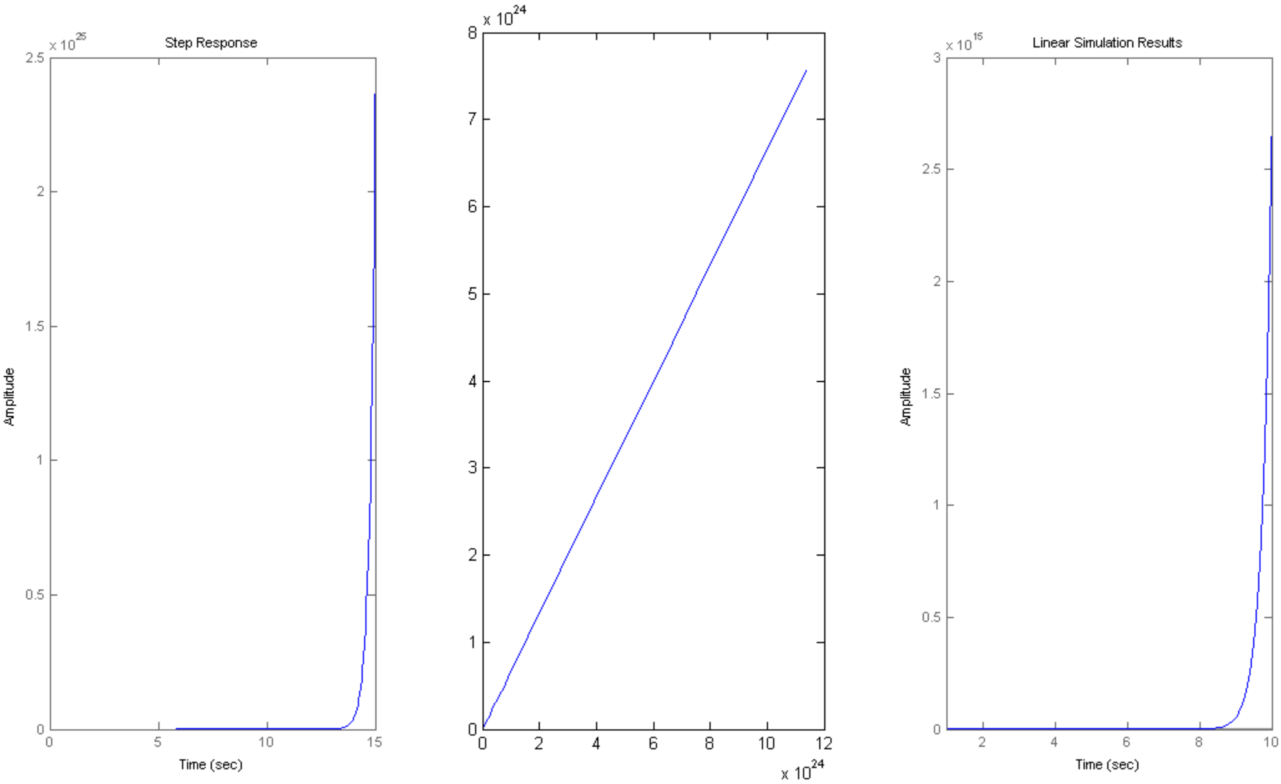
$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{pmatrix} 0.075e^{4t} - 0.2e^{-t} - 0.5t + 0.125 \\ 0.05e^{4t} + 0.2e^{-t} - 0.25 \end{pmatrix}$

$e^{At}x_0$ the same value as in the case of 2. example

$$x(t) = \begin{pmatrix} 0.675e^{4t} - 0.8e^{-t} - 0.5t + 0.125 \\ 0.45e^{4t} + 0.8e^{-t} - 0.25 \end{pmatrix}$$

$$y(t) = Cx(t) = 0.45e^{4t} + 0.8e^{-t} - 0.25$$

A 2. we can see the solution of the three example in order.


x 10^25
Step Response
Amplitude
Time (sec)
x 10^24
x 10^24
x 10^15
Linear Simulation Results
Amplitude
Time (sec)

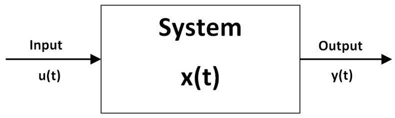
2. ábra

Computer controlled systems

Lecture 3

version: 2018.10.02. – 11:53:38

1 System representations



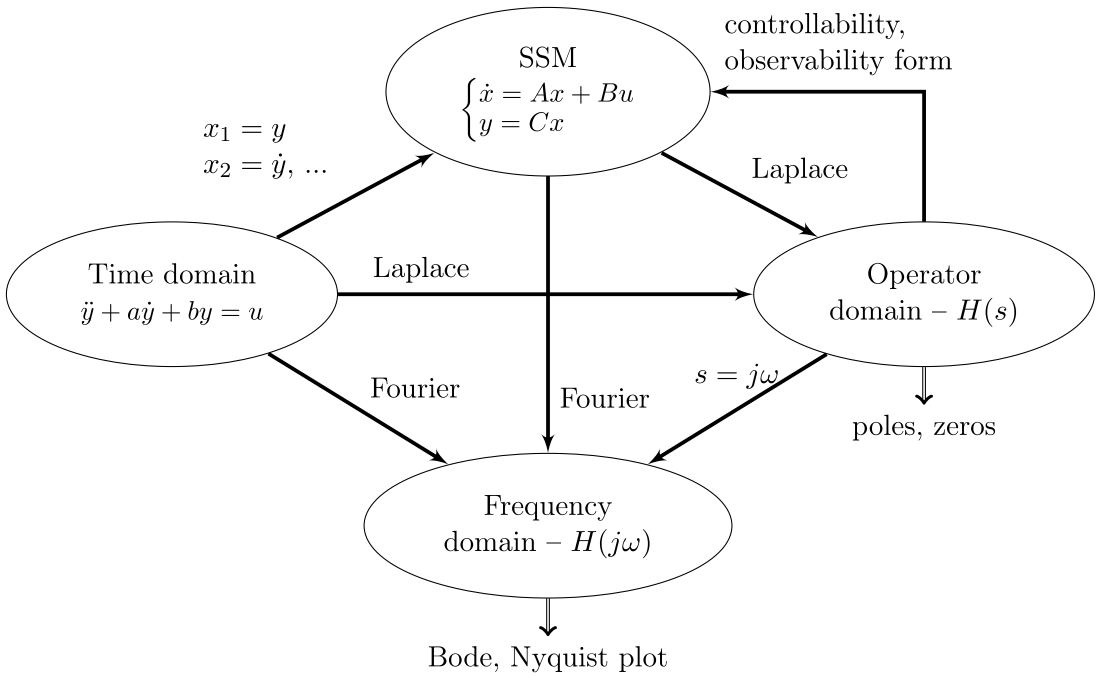


Figure 1. System representations, SSM (State Space Model) – ÁTM (Állapottér model)

1.1 Time domain → Operator domain

Example 1. The system's differential equation:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\ddot{u}(t) - 3\dot{u}(t) + u(t)$$

Assume that the initial conditions are all zero. The Laplace transform of the above equation is:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 2s^2U(s) - 3sU(s) + U(s)$$

$$(s^2 + 3s + 2)Y(s) = (2s^2 - 3s + 1)U(s)$$

From this, we obtain the system's transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 - 3s + 1}{s^2 + 3s + 2} = \underbrace{\frac{-9s - 3}{s^2 + 3s + 2}}_{C(sI-A)^{-1}B} + \underbrace{2}_{D} \quad \Rightarrow$$

Controller, observer form (in advance)

$$\begin{cases} \dot{x} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y = \begin{pmatrix} -9 & -3 \end{pmatrix} x + 2u \end{cases} \quad \text{ctr. form}$$

$$\begin{cases} \dot{x} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} x + \begin{pmatrix} -9 \\ -3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + 2u \end{cases} \quad \text{obs. form}$$

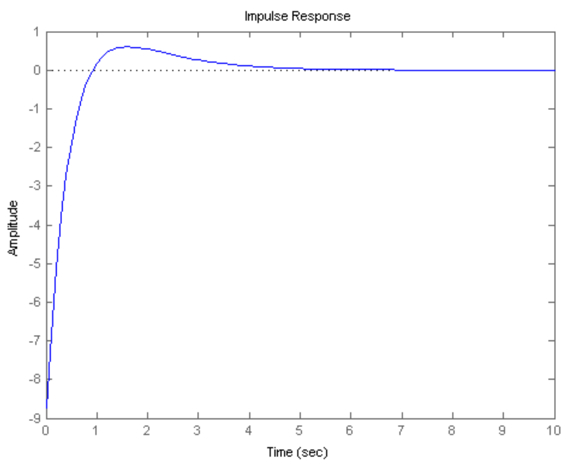
From $H(s)$, we can determine the system's impulse response function (using inverse Laplace transformation):

$$H(s) = 2 + \frac{-9s - 3}{(s+1)(s+2)} = 2 + \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

$$C_i = \lim_{s \to \alpha_i} (s - \alpha_i)H(s)$$

$$C_1 = 6 \quad C_2 = -15$$

$$h(t) = \mathfrak{L}^{-1}\{H(s)\} = \mathfrak{L}^{-1}\{2 + \frac{6}{s+1} + \frac{-15}{s+2}\} = 2\delta(t) + 6e^{-t} - 15e^{-2t}$$



In a strictly proper system the input does not affect the output *directly*. In the operator domain, this means that the degree of the transfer function's numerator is less then the degree of the denominator. In the state space model, matrix D is zero if the system is strictly proper.

Figure 2. Impulse response $h(t)$ of the system.

Matlab 1. conv,deconv — Polynomial multiplication and division

Polynome multiplication

$(s^2 + 3s + 2) \cdot (s + 4) = s^3 + 7s^2 + 14s + 8$

```
>> C = conv([1 3 2],[1 4])
C = 1      7     14      8
```

Polynome division

$\frac{2s^2-3s+1}{s^2+3s+2} = \frac{-9s-3}{s^2+3s+2} + 2$

```
>> [Q,R] = deconv([2 -3 1],[1 3 2])
Q = 2
R = 0     -9     -3
```

Matlab 2. ss2tf — Compute the transfer function of a SSM

```
>> [num,den] = ss2tf([-3 -2 ; 1 0],[1 0]',[-9 -3],2)
num = 2     -3      1
den = 1      3      2
```

1.2 Time domain → SSM

State space model: *system* of first order linear differential equations (elsőrendű differenciálegyenletekből álló egyenletrendszer)

$$\dot{x} = f(x) + g(x)u, \quad \text{where } f : \mathbb{R}^n \to \mathbb{R}^n, \quad g : \mathbb{R}^n \to \mathbb{R}^{n\times r}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r. \tag{1}$$

Linear case:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2}$$

Example 2. It is given the following second order linear scalar differential equation

$$\ddot{y} = -y$$

We introduce the following notation:

$$x_1 = y \ , \quad x_2 = \dot{y} = \dot{x}_1$$

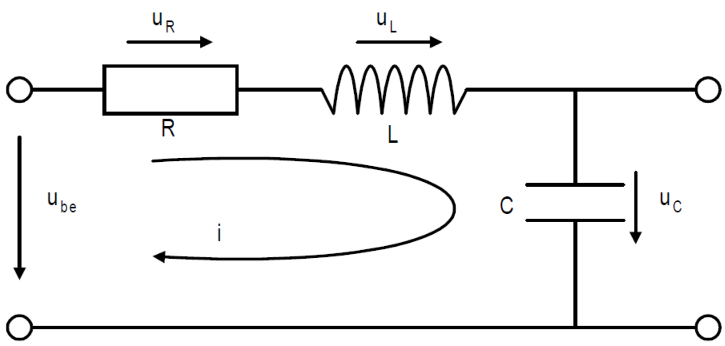
SSM:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{x}_1 = \ddot{y} = -y = -x_1 \end{aligned}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Usually, the procedure for (time domain → SSM) is the following:
Linear higher order scalar differential equation with constant coefficient → compute its transfer function $H(s)$ → SSM (Controller or Observer form).

Example 3.



The system's differential equation (u_{be} stands for u_{in} – is the input voltage):

$$u_{be} = u_R + u_L + u_C$$

$$i = C \cdot \frac{\mathrm{d}u_C}{\mathrm{d}t}, \quad u_L = L \cdot \frac{\mathrm{d}i}{\mathrm{d}t}, \quad u_R = R \cdot i$$

State equations (let the state variables be: i, u_C):

$$\begin{cases} u_{be} = R \cdot i + L \cdot \dfrac{\mathrm{d}i}{\mathrm{d}t} + u_C \\ i = C \cdot \dfrac{\mathrm{d}u_C}{\mathrm{d}t} \end{cases} \Rightarrow \begin{cases} \dfrac{\mathrm{d}i}{\mathrm{d}t} = -\dfrac{R}{L}i - \dfrac{1}{L}u_C + \dfrac{1}{L}u_{be} \\ \dfrac{\mathrm{d}u_C}{\mathrm{d}t} = \dfrac{1}{C}i \end{cases} \tag{3}$$

In matrix form:

$$\begin{pmatrix} \frac{\mathrm{d}i}{\mathrm{d}t} \\ \frac{\mathrm{d}u_C}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ u_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u_{be}$$

Hence, the state vector is $x(t) = \begin{pmatrix} i \\ u_C \end{pmatrix}$, $u(t) = u_{be}$

We must define the output of the system: $y(t) = u_C = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ u_C \end{pmatrix}$

Now we consider a concrete numerical example. Let $R = 1.5\Omega$, $L = 0.25H$, $C = 0.5F$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

In Figure 3, you can see the impulse and step response of the system. Furthermore, Figure 4 illustrates the system's output in case of a sinusoidal input function $u(t) = 2\sin(3t)$ (szinuszos gerjesztés).

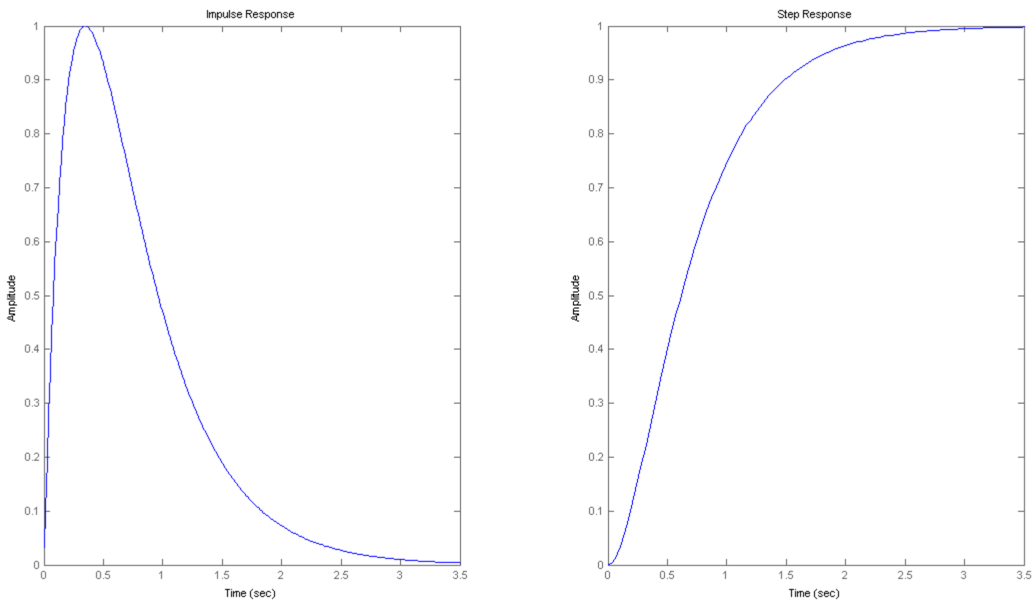


Figure 3. Impulse and step response of the RLC circuit

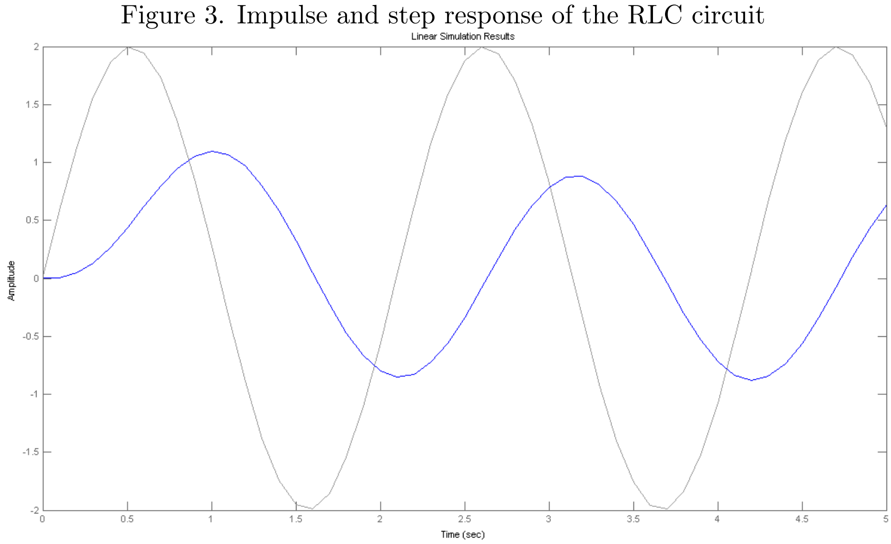


Figure 4. Excitation (gerjesztés) of the RLC circuit with a sinusoidal input function. In case of $f = 3$Hz frequency signal the system's gain is $g = 0.4438$, phase shift is $\phi = -93.1798°$. HU: 3 Hz esetén a rendszer egyenáramú erősítése $g = 0.4438$, fáziseltolása $\phi = -93.1798°$.

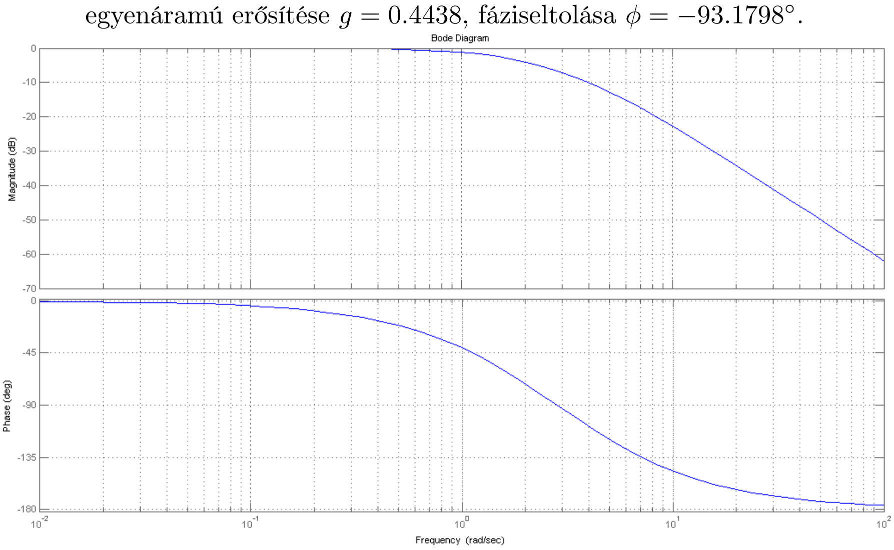


Figure 5. Bode diagram of the system describing the RLC circuit.

.

Example 4. (Diagonal SSM)

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-6 & -4\\ 2 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}4\\ 0\end{pmatrix}u$$

$$y = \begin{pmatrix}0 & 1\end{pmatrix}x$$

We compute the eigenvalue/eigenvector decomposition (alternatively: ***spectral decomposition***) of matrix A

$$\begin{vmatrix}\lambda+6 & 4\\ -2 & \lambda\end{vmatrix} = (\lambda+6)\lambda+8 = (\lambda+2)(\lambda+4)$$

Eigenvalues: $\lambda_1 = -2$, $\lambda_2 = -4$
Eigenvectors in case of $\lambda_1 = -2$:

$$\begin{pmatrix}-6 & -4\\ 2 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = -2\begin{pmatrix}v_1\\ v_2\end{pmatrix} \rightarrow \begin{matrix}-6v_1-4v_2=-2v_1 \rightarrow v_2=-v_1\\ 2v_1=-2v_2\end{matrix}$$

$$v = \begin{pmatrix}1\\ -1\end{pmatrix}\cdot p \qquad p\in\mathbb{R}\setminus\{0\}$$

Eigenvectors in case of $\lambda_2 = -4$:

$$\begin{pmatrix}-6 & -4\\ 2 & 0\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} = -4\begin{pmatrix}w_1\\ w_2\end{pmatrix} \rightarrow \begin{matrix}-6w_1-4w_2=-4w_1 \rightarrow w_1=-2w_2\\ 2w_1=-4w_2\end{matrix}$$

$$w = \begin{pmatrix}-2\\ 1\end{pmatrix}\cdot q \qquad q\in\mathbb{R}\setminus\{0\}$$

Transformation matrix:

$$T = S^{-1} \quad T^{-1} = S$$

$$T^{-1} = \begin{pmatrix}v & w\end{pmatrix} = \begin{pmatrix}1 & -2\\ -1 & 1\end{pmatrix} \rightarrow T = \begin{pmatrix}-1 & -2\\ -1 & -1\end{pmatrix}$$

State space transformation:

$$\bar{A} = TAT^{-1} = \begin{pmatrix}-1 & -2\\ -1 & -1\end{pmatrix}\begin{pmatrix}-6 & -4\\ 2 & 0\end{pmatrix}\begin{pmatrix}1 & -2\\ -1 & 1\end{pmatrix} = \begin{pmatrix}-2 & 0\\ 0 & -4\end{pmatrix}$$

$$\bar{B} = TB = \begin{pmatrix}-1 & -2\\ -1 & -1\end{pmatrix}\begin{pmatrix}4\\ 0\end{pmatrix} = \begin{pmatrix}-4\\ -4\end{pmatrix}$$

$$\bar{C} = CT^{-1} = \begin{pmatrix}-1 & 1\end{pmatrix}$$

1.3 SSM → Operator domain

$$\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = 0 \end{matrix} \rightarrow \begin{matrix} sIX(s) = AX(s) + BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s) \\ Y(s) = CX(s) + DU(s) = (C(sI - A)^{-1}B + D)U(s) \end{matrix}$$

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Example 5.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

$$H(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+6 & 4 \\ -2 & s \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{(s+6)s+8} \begin{pmatrix} s & -4 \\ 2 & s+6 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \frac{1}{s^2+6s+8} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 4s \\ 8 \end{pmatrix} = \frac{8}{(s+2)(s+4)}$$

Proposition 1. (special case – if the SSM is diagonal, i.e. matrix A is diagonal)

$$H(s) = C(sI - A)^{-1}B = \begin{pmatrix} c_1 \cdots c_n \end{pmatrix} \begin{pmatrix} s-\lambda_1 & 0 & \cdots & 0 \\ 0 & s-\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s-\lambda_n \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} =$$

$$= \begin{pmatrix} c_1 \cdots c_n \end{pmatrix} \begin{pmatrix} \frac{1}{s-\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s-\lambda_n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \frac{c_1 b_1}{s-\lambda_1} + \frac{c_2 b_2}{s-\lambda_2} + \cdots + \frac{c_n b_n}{s-\lambda_n} = \sum_{i=1}^{n} \frac{c_i b_i}{s-\lambda_i}$$

Example 6.

$$\dot{x} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ -10 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 6 & 8 & 2 & -1 \end{pmatrix} x$$

$$H(s) = \frac{0 \cdot 6}{s-4} + \frac{1 \cdot 8}{s+3} + \frac{(-10) \cdot 2}{s+2} + \frac{2 \cdot (-1)}{s+6} = \frac{8}{s+3} - \frac{20}{s+2} - \frac{2}{s+6}$$

1.4 Operator domain → SSM : Controller form

Theorem 2. (Controller, observer form – only SISO)

$$H(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{1s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D = \frac{b(s)}{a(s)} + D\ , \qquad D \in \mathbb{R}$$

The denominator must always be monic (with leading coefficient 1).
A nevezőben szereplő polinom vezéregyütthatója mindig 1 kell, hogy legyen!

$$\text{controller form:} \quad \begin{cases} \dot{x} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\ y = \begin{pmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{pmatrix} x + Du \end{cases} \tag{4}$$

$$\text{observer form:} \quad \begin{cases} \dot{x} = \begin{pmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 1 & 0 \\ -a_0 & 0 & \cdots & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} x + Du \end{cases} \tag{5}$$

Example 7.

$$H(s) = \frac{4s+38}{(s+1)(s+2)(2s+6)} = \frac{4s+38}{2s^3+12s^2+22s+12} = \frac{2s+19}{s^3+6s^2+11s+6}$$

$$\dot{x} = \begin{pmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 2 & 19 \end{pmatrix} x$$

Remark. The controller form produces a controllable SSM!
A controller form mindig irányítható ÁTM-et eredményez! (Lásd később)

Check whether it leads, indeed, to the initial transfer function:

$$H(s) = C(sI-A)^{-1}B = \begin{pmatrix} 0 & 2 & 19 \end{pmatrix} \begin{pmatrix} s+6 & 11 & 6 \\ -1 & s & 0 \\ 0 & -1 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 2 & 19 \end{pmatrix} \frac{1}{s^2(s+6)+6+11s} \begin{pmatrix} s^2 & -11s-6 & 6s \\ s & s(s+6) & -6 \\ 1 & s+6 & s(s+6)+11 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\frac{1}{s^3+17s+6} \begin{pmatrix} 0 & 2 & 19 \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \frac{2s+19}{s^3+6s^2+11s+6}$$

! **Only SISO case.** The controller and observer forms are very similar:

$$A_c = A_o^T, \quad B_c = C_o^T, \quad C_c = B_o^T \tag{6}$$

SISO $\Rightarrow C(sI-A)^{-1}B \in \mathbb{R}$, thus

$$C(sI-A)^{-1}B = \left(C(sI-A)^{-1}B\right)^T = B^T(sI-A)^{-T}C^T = B^T(sI-A^T)^{-1}C^T \tag{7}$$

Computer controlled systems

Lecture 4

version: 2018.10.09. – 15:13:40

1 State space transformation

As we shall already know, the state space model is not unique. For the given example, define a new SSM using a state space transformation.

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Let the linear transformation of the state vector be the following:

$$\bar{x}_1 = x_1 + x_2$$
$$\bar{x}_2 = 3x_1 - 2x_2$$

In matrix form:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\bar{x} = Tx, \; x = T^{-1}\bar{x} \quad \rightarrow \quad$ state state space equation can be written for the new state vector $\bar{x}$ as well

$$\dot{x} = Ax + Bu \quad \rightarrow \quad T^{-1}\dot{\bar{x}} = AT^{-1}\bar{x} + Bu$$

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu \quad \rightarrow \quad \bar{A} = TAT^{-1} \quad \bar{B} = TB$$

$$y = Cx = CT^{-1}\bar{x} \quad \rightarrow \quad \bar{C} = CT^{-1}$$

Returning to the example:

$$T = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \quad T^{-1} = -\frac{1}{5}\begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{pmatrix} -4 & 0 \\ -16 & -2 \end{pmatrix} \quad \bar{B} = TB = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad \bar{C} = CT^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

If the original and the transformed SSM are (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, respectively, determine the transformation matrix T, which connects them.

$$A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{1}$$

$$\bar{A} = \begin{pmatrix} 1.8 & 1.6 \\ -4.4 & 2.2 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \bar{C} = \begin{pmatrix} 0.4 & -0.2 \end{pmatrix} \tag{2}$$

Solution. $\bar{B} = TB, \; \bar{A}\bar{B} = TAB \quad \rightarrow \quad T \cdot [B|AB] = [\bar{B}|\bar{A}\bar{B}] \quad \rightarrow \quad \boxed{T = \bar{\mathcal{C}}_2 \cdot \mathcal{C}_2^{-1}}$, where $\mathcal{C}_2 = [B|AB]$ and $\bar{\mathcal{C}}_2 = [\bar{B}|\bar{A}\bar{B}]$ are the controllability matrices of (1) and (2), respectively.
Remark. B and AB are (2×1) matrices.

$$\mathcal{C}_2 = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}, \quad \mathcal{C}_2^{-1} = \frac{1}{-8}\begin{pmatrix} -3 & -5 \\ -1 & 1 \end{pmatrix}, \quad \bar{\mathcal{C}}_2 = \begin{pmatrix} 3 & 7 \\ 1 & -11 \end{pmatrix}$$

$$T = \frac{1}{-8} \cdot \begin{pmatrix} 3 & 7 \\ 1 & -11 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \qquad T^{-1} = \frac{-1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Just as in the previous example, determine the transformation matrix T.

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{3}$$

$$\bar{A} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \bar{C} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{4}$$

Solution. $T = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \quad T^{-1} = \frac{-1}{4} \cdot \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix}$

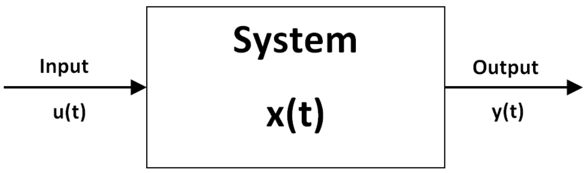
Remark. In case of SISO model, this method can be applied for an even higher dimensional state-space model, but then the controllability matrix will involve further rows. If the state vector is n-dimensional ($A \in \mathbb{R}^{n \times n}$), than $\mathcal{C}_n = [B|AB|A^2B|\ldots|A^{n-1}B]$. To conclude, if the SSM is controllable:

$$\boxed{T = \bar{\mathcal{C}}_n \cdot \mathcal{C}_n^{-1}} \tag{5}$$

Megjegyzés: SISO modell esetén a fenti módszer több állapotváltozó esetén is alkalmazható, de ekkor több oszlopra van szükség. Ha $A \in \mathbb{R}^{n \times n}$, akkor a $[B|AB|A^2B|\ldots|A^{n-1}B]$ alakú mátrixokkal lehet számolni.

2 Controllability, observability

In general Given the following CT-LTI system: The question arouse: In the full knowledge of $y(t)$ and



$u(t)$ can we say something about the unknown state vector $x(t)$? In the other words is $x(t)$ **observable**?

The second question would be the following: is there an input function $u(t)$, with which we can lead the system from the initial state x_0 to state x_1 in a finite time. If we can do so (for every possible initial and final states), we say that the system is **controllable**.

2.1 Observability

Theorem 1. Sufficient and necessary condition for observability

A state space model described by matrices (A, B, C) is observable if and only if (iff) its observability matrix $\mathcal{O}_n$ is full-rank:

$$\mathcal{O}_n = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \qquad \text{rank}\left(\mathcal{O}_n\right) = n$$

Remark. In SISO case $\mathcal{O}_n$ is a square matrix, which is full-rank iff its determinant is nonzero.

Example 1. Is the system (A, B, C) observable?

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The observability matrix is the following

$$CA = \begin{pmatrix} 2 & 1 \end{pmatrix} \to \mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \ , \quad \det(\mathcal{O}_2) = -2 \neq 0 \Rightarrow \mathcal{O}_2 \text{ is full-rank}$$

Hence, $x(t)$ is observable, namely, using $y(t)$ and its time derivative $\dot{y}(t)$, we can compute the actual value of $x(t)$

$$\begin{cases} y(t) = Cx(t) \\ \dot{y}(t) = CAx(t) + CBu(t) \end{cases} \Rightarrow x(t) = \mathcal{O}_2^{-1} \begin{pmatrix} y(t) \\ \dot{y}(t) - CBu(t) \end{pmatrix} \tag{6}$$

Example 2. Unobservable subspace (mathematical background presented in B.1)

Given the state space model:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}, \quad B : \text{arbitrary}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \mathcal{O}_n = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{7}$$

A basis for the kernel of $\mathcal{O}_n$ is $v_1 = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$. This means that

→ if there is a zero input and $x(0) = \lambda v_1 \in \mathcal{O}_2$, than $x(t) \in \text{Ker}(\mathcal{O}_2)$ (Proposition 9) and $y(t) = 0$ for every $t > 0$.

→ for a given input $u(t)$ and with an initial condition $x(0) = x_0 + \lambda v_1 \in x_0 + \text{Ker}(\mathcal{O}_2)$ (where $\lambda \in \mathbb{R}$ is arbitrary) the system will produce *the same ouput* $y(t)$.

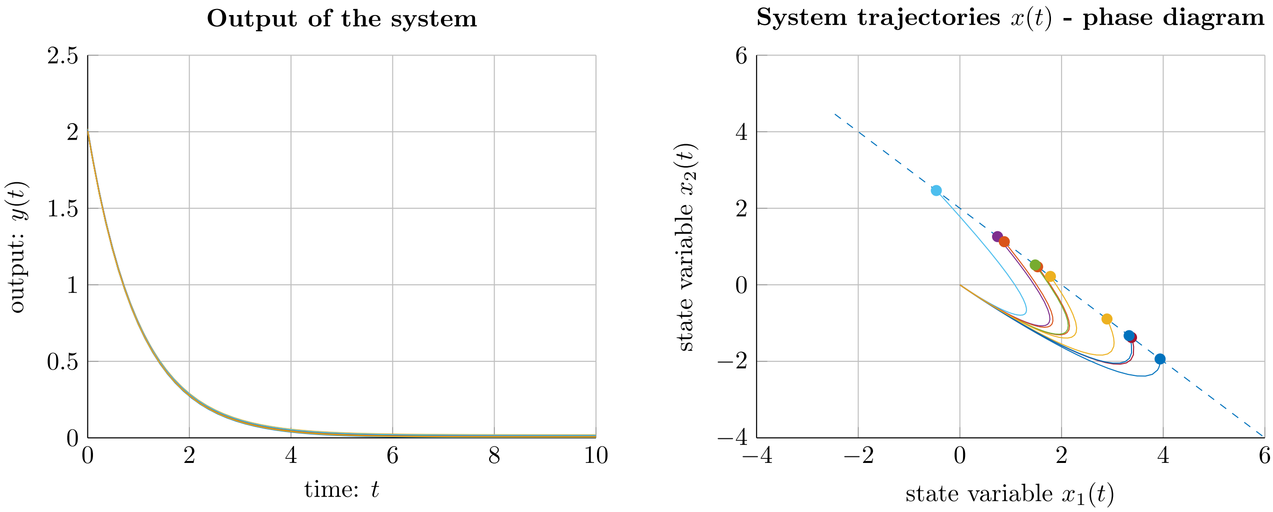


Figure 1. Simulation of system (7) from different initial conditions $x(0) \in x_0 + \text{Ker}(\mathcal{O}_2)$ (denoted by dots) with zero input. As one can observe, the state trajectories are different, however this difference does not appear in the output of the system. In this example $u \equiv 0$ and $x_0 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. The blue dashed line in the right figure illustrates the actual unobservability subspace of the system corresponding to x_0.

2.2 Controllability

Given a strictly proper state space model (A, B, C) with $x(t_0)$ initial and $x(t_1) \neq x(t_0)$ final condition. The question arises, is there any input function $u(t)$, which leads the system from $x(t_0)$ to $x(t_1)$ in a finite time.

Theorem 2. Controllability

A state space model described by matrices (A, B, C) is controllable iff its controllability matrix $\mathcal{C}_n$ is full-rank:

$$\mathcal{C}_n = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}, \quad \operatorname{rank}(\mathcal{C}_n) = n$$

Remark. In SISO case $\mathcal{C}_n$ is a square matrix, which is full-rank iff its determinant is nonzero.

Example 3.

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad \mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}$$

This system is controllable, since the determinant of $\mathcal{C}_2$ is nonzero. In this case the controllability subspace is the whole $\mathbb{R}^2$ itself. If we start the system from zero initial condition, we can lead the system (with an appropriate input) to any other states of the controllability subspace in a finite time.

Example 4. Controllable subspace (mathematical background presented in B.2)

Given the following state space system and its rank-deficient controllability matrix:

$$A = \begin{pmatrix} -1 & 2 & -2 \\ -\frac{2}{3} & -6 & \frac{20}{3} \\ -\frac{1}{2} & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}, \quad \text{eigenvalues of } A\text{: } \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix}, \quad \mathcal{C}_3 = \begin{pmatrix} 0 & 16 & -96 \\ 8 & -48 & 224 \\ 0 & -8 & 48 \end{pmatrix} \tag{8}$$

The basis vectors of $\operatorname{Im}(\mathcal{C}_3)$ are: $v_1 = \begin{pmatrix} 0.3832 \\ -0.9036 \\ -0.1916 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0.8082 \\ 0.4285 \\ -0.4041 \end{pmatrix}$. They span a 2-dimensional subspace in $\mathbb{R}^3$, illustrated by the green plane in the Figure 2. If we start the system from an initial condition which is an element of this subspace $x(0) \in \operatorname{Im}(\mathcal{C}_3)$, the system trajectory will never leave this subspace. If the initial condition is outside of $\operatorname{Im}(\mathcal{C}_3)$ and A is stable, the system trajectory will tend to this subspace.

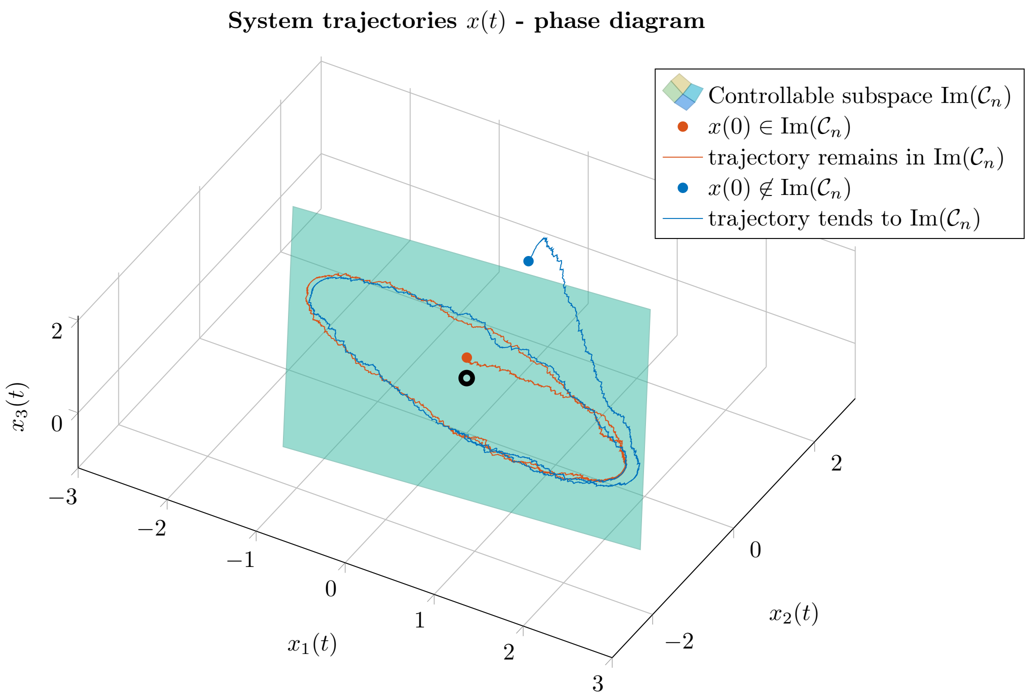


Figure 2. Simulation of system (7) from different initial conditions

Example 5.

Compute the controllable subspace of $\dot{x} = Ax + Bu$, where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

To check your solutions, we give:

$$A^2 = \begin{pmatrix} -1 & 4 & -4 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \ \mathcal{O}_3 = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}. \tag{10}$$

2.3 Controllability and observability in case of a diagonal SSM

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad AB = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \quad CA = \begin{pmatrix} c_1 a_1 & c_2 a_2 \end{pmatrix}$$

$$\mathcal{C}_2 = \begin{pmatrix} b_1 & a_1 b_1 \\ b_2 & a_2 b_2 \end{pmatrix} \qquad \mathcal{O}_2 = \begin{pmatrix} c_1 & c_2 \\ c_1 a_1 & c_2 a_2 \end{pmatrix}$$

SISO rendszer diagonális A mátrix esetén
irányítható $\iff$ a főátlóbeli elemek páronként különbözőek, és $\forall i \ b_i \neq 0$
megfigyelhető $\iff$ a főátlóbeli elemek páronként különbözőek, és $\forall j \ c_j \neq 0$

Theorem 3. The rank of $\mathcal{O}_n$ and $\mathcal{C}_n$ is invariant to the state space transformations.

Proof.

$$\bar{A} = TAT^{-1} \quad \bar{B} = TB \quad \bar{C} = CT^{-1}$$

$$\bar{\mathcal{C}}_n = \begin{pmatrix} TB & TAT^{-1}TB \end{pmatrix} = T\begin{pmatrix} B & AB \end{pmatrix} = T\mathcal{C}_n$$

$$\bar{\mathcal{O}}_n = \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \end{pmatrix} = \begin{pmatrix} C \\ CA \end{pmatrix} T^{-1} = \mathcal{O}_n T^{-1}$$

□

2.4 Markov parameters

$$CA^iB$$

Markov parameters are invariant to the state space transformations.

$$\bar{C}\bar{B} = CT^{-1}TB = CB$$

$$\bar{C}\bar{A}\bar{B} = CT^{-1}TAT^{-1}TB = CAB$$

3 Joint controllability and observability

- Egy $H(s) = \frac{b(s)}{a(s)}$ (SISO) átviteli függvény n-edrendű realizációjának nevezzük az (A, B, C, D) állapottér-modellt, ha $H(s) = C(sI - A)^{-1}B + D$, ahol $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$ (nem egyértelmű!)
- Egy $H(s)$ átviteli függvény n-edrendű realizációját minimálisnak nevezzük, ha nem létezik nála kisebb rendű realizáció.

- Egy n-dimenziós (A, B, C, D) állapottér-modellt együttesen irányíthatónak és megfigyelhetnek nevezünk, ha teljesülnek rá az irányíthatóság és a megfigyelhetőség feltételei (azaz $\mathcal{O}_n$ és $\mathcal{C}_n$ teljes rangú).
- Egy ÁTM minimális $\iff$ egyszerre irányítható és megfigyelhető.

Example 6. Is the state space representation minimal?

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Transfer function: $H(s) = \frac{s}{s^2-3s-4}$. This SSM is minimal, since $H(s)$ is irreducible and the degree of the denominator is equal to the order of the state space realization ($n = 2$).

Example 7. Is the state space representation minimal?

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$H(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2+4s+3} = \frac{s+1}{(s+1)(s+3)}$$

This SSM is not minimal, meaning the one of two properties is broken: the SSM is controllable but its is no observable.

Example 8. Is the state space representation minimal?

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Controllability matrix:

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 4 & -24 \\ 0 & 8 \end{pmatrix}$$

The determinant of matrix $\mathcal{C}_2$ is nonzero, therefore, it is controllable.
Observability matrix:

$$\mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

The determinant of matrix $\mathcal{O}_2$ is nonzero, therefore, it is observable. Consequently, the SSM is minimal.

Example 9. (MIMO case) Is the state space representation minimal?

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 9 & 16 & 1 \\ 2 & -2 & -2 \end{pmatrix} \quad CA = \begin{pmatrix} -3 & 8 \\ -14 & 14 \end{pmatrix}$$

$$\mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 7 \\ -3 & 8 \\ -14 & 14 \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 & 9 & 16 & 1 \\ 2 & 3 & 0 & 2 & -2 & -2 \end{pmatrix}$$

Matrix $\mathcal{O}_2$ is full-column-rank, and $\mathcal{C}_2$ is full row rank, meaning that the system is jointly controllable and observable and (A, B, C) is minimal.

Example 10. Is the SSM minimal? If not give a minimal representation.

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 0 & 4 \end{pmatrix}$$

$$H(s) = \sum_{i=1}^{n} \frac{c_i b_i}{s-\lambda_i} = \frac{3\cdot 1}{s+3} + \frac{0\cdot 2}{s-4} + \frac{6\cdot 4}{s-6} = \frac{3(s-6)+24(s+3)}{(s+3)(s-6)}$$
$$H(s) = \frac{27s+54}{s^2-3s-18}$$

The SSM is not minimal, because the transfer function can be reduced.

$$A^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 36 \end{pmatrix} \qquad \mathcal{O}_n = \begin{pmatrix} 3 & 0 & 4 \\ -9 & 0 & 24 \\ 27 & 0 & 144 \end{pmatrix} \qquad \mathcal{C}_n = \begin{pmatrix} 1 & -3 & 9 \\ 2 & 8 & 32 \\ 6 & 36 & 216 \end{pmatrix}$$

A minimal SSM can be given by skipping the single degenerated state variable:

$$A = \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 4 \end{pmatrix}$$

A minimal realization can also be given using the controller form:

$$A = \begin{pmatrix} 3 & 18 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 27 & 54 \end{pmatrix}$$

Example 11. It is given a SSM in the controller form. Is the SSM jointly controllable and observable?

$$A = \begin{pmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 3 & 9 \end{pmatrix}$$

Transfer function:

$$H(s) = \frac{3s+9}{s^3-7s+6}$$

The realization is most be controllable, since it is given in controller form:

$$A^2 = \begin{pmatrix} 7 & -6 & 0 \\ 0 & 7 & -6 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathcal{O}_n = \begin{pmatrix} 0 & 3 & 9 \\ 3 & 9 & 0 \\ 9 & 21 & -18 \end{pmatrix} \qquad \mathcal{C}_n = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} \mathrm{rank}(\mathcal{C}_n) = 3 \\ \mathrm{rank}(\mathcal{O}_n) = 2 \end{matrix}$$

However the SSM is not observable, because it is not minimal: $H(s)$ is reducible by $s+3$. Using the controller form (on the irreducible form of $H(s)$), we can obtain a jointly controllable and observable realization Tehát nem együttesen megfigyelhető és irányítható a rendszer. The a unobservable subspace

$$\mathrm{Ker}(\mathcal{O}_n) = \left\{ \alpha \begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

Felhasználás: Állapotmegfigyelők tervezése
Bizonyos mennyiségeket (pl. szögsebesség) nem tudunk mérni, csak becsülni. Ld.: 3. ábra

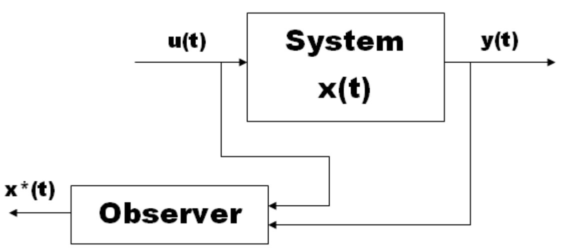


Figure 3. State observer design

References

[1] Alexey Grigorev. The Fundamental Theorem of Linear Algebra. Technische Universität Berlin.

[2] Lantos Béla. *Irányítási rendszerek elmélete és tervezése I.* Akadémiai Kiadó Budapest, 2001.

[3] A. D. Lewis. *A Mathematical Approach to Classical Control*. 2003.

A Supplementary material in linear algebra (not needed for the exam)

Theorem 4. The fundamental theorem of linear algebra

Let $\mathcal{A}:\mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{m\times n}$. Then the followings are true

$$\operatorname{Im}(A) = \operatorname{Ker}\left(A^T\right)^{\perp} \subset \mathbb{R}^m \tag{11a}$$

$$\operatorname{Im}\left(A^T\right) = \operatorname{Ker}(A)^{\perp} \subset \mathbb{R}^n \tag{11b}$$

Furthermore

$$\operatorname{Im}(A) \otimes \operatorname{Ker}\left(A^T\right) = \mathbb{R}^m \tag{12a}$$

$$\operatorname{Im}\left(A^T\right) \otimes \operatorname{Ker}(A) = \mathbb{R}^n \tag{12b}$$

Remark. If $r = \operatorname{rank}(A)$, than

$$\dim \operatorname{Im}(A) = r, \qquad \dim \operatorname{Ker}\left(A^T\right) = m - r \tag{13a}$$

$$\dim \operatorname{Im}\left(A^T\right) = r, \qquad \dim \operatorname{Ker}(A) = n - r \tag{13b}$$

Proof. Proof of (11a) as presented in [1]. Let

$$A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix} \;\Rightarrow\; A^T = \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix} \tag{14a}$$

$$\boldsymbol{x} \in \operatorname{Ker}\left(A^T\right) \;\Rightarrow\; A^T\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a}_1^T\boldsymbol{x} \\ \boldsymbol{a}_2^T\boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_n^T\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{14b}$$

$$\boldsymbol{y} \in \operatorname{Im}(A) \;\Rightarrow\; \exists \alpha_i \in \mathbb{R} \text{ such that } \boldsymbol{y} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i \tag{14c}$$

Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary vector elements of $\operatorname{Ker}\left(A^T\right)$ and $\operatorname{Im}(A)$, respectively. Then we compute the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i^T \boldsymbol{x} = 0, \tag{15}$$

since $\boldsymbol{a}_i^T\boldsymbol{x} = 0$, $\forall i = \overline{1,n}$. Consequently, $\boldsymbol{x} \perp \boldsymbol{y}$ for all possible $x \in \operatorname{Ker}\left(A^T\right)$ and $y \in \operatorname{Im}(A)$, which means that the two subspaces are the ***orthogonal complement*** for each other:

$$\begin{aligned} \operatorname{Im}(A) &= \operatorname{Ker}\left(A^T\right)^{\perp} \\ \operatorname{Im}(A) \cap \operatorname{Ker}\left(A^T\right) &= \{0\} \end{aligned} \tag{16}$$

Following this idea, we can conclude that the subspaces are linearly independent, therefore,

$$\dim\left(\operatorname{Im}(A) \otimes \operatorname{Ker}\left(A^T\right)\right) = r + (m - r) = m. \tag{17}$$

This can only happend if ***direct product*** of the two spaces is $\mathbb{R}^m$, which completes the proof for (12a). □

Proposition 5. (Self-adjoint operator) Let $\mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix: $A = A^T$. Than, as a consequence of Theorem 4, we have that

$$\operatorname{Im}(A) = \operatorname{Ker}(A)^{\perp} \text{ and } \operatorname{Im}(A) \otimes \operatorname{Ker}(A) = \mathbb{R}^n.$$

For more, see [2, Eq. (10.3)].

Proposition 6. Singular value decomposition (SVD)

If we make the SVD for matrix $A \in \mathbb{R}^{m\times n}$

$$A = U\Sigma V^T, \tag{18}$$

where

$$U \in \mathbb{R}^{m\times m} \text{ is unitary: } U^*U = I_m \tag{19a}$$

$$V \in \mathbb{R}^{n\times n} \text{ is unitary: } V^*V = I_n \tag{19b}$$

$$\Sigma \in \mathbb{R}^{m\times n} \text{ eigenvalues in the diagonal.} \tag{19c}$$

After this decomposition, the basis of the four subspaces (12) can be obtained as presented below.

$$\begin{aligned} \mathrm{Im}(A): &\quad \text{the first } r \text{ colums of } U \\ \mathrm{Ker}\left(A^T\right): &\quad \text{the last } m-r \text{ columns of } U \\ \mathrm{Im}\left(A^T\right): &\quad \text{the first } r \text{ columns of } V \\ \mathrm{Ker}(A): &\quad \text{the last } n-r \text{ columns of } V \end{aligned}$$

In short

$$\text{“ } A = \left[\mathrm{Im}(A) \quad \mathrm{Ker}\left(A^T\right)\right] \Sigma \left[\mathrm{Im}\left(A^T\right) \quad \mathrm{Ker}(A)\right]^T \text{ ”} \tag{20}$$

B Subspaces of the state space

Having a strictly proper ($D = 0$) MIMO LTI system:

$$\begin{aligned} \dot{x} &= Ax + By \\ y &= Cx \end{aligned} \tag{21}$$

The state space could be partitioned as follows:

$$X = X_{co} \otimes X_{c\bar{o}} \otimes X_{\bar{c}o} \otimes X_{\bar{c}\bar{o}} \tag{22}$$

where $X_{..}$ are pairwise orthogonal subspaces of the state space, in other words:

$$\begin{aligned} X_{co} \perp X_{c\bar{o}}, \quad X_{co} \perp X_{\bar{c}o}, \quad X_{co} \perp X_{\bar{c}\bar{o}}, \\ X_{c\bar{o}} \perp X_{\bar{c}o}, \quad X_{c\bar{o}} \perp X_{\bar{c}\bar{o}}, \quad X_{\bar{c}o} \perp X_{\bar{c}\bar{o}}. \end{aligned} \tag{23}$$

B.1 Unobservable subspace $X_{\bar{o}} = \mathrm{Ker}(\mathcal{O}_n)$. Observable subspace $X_o = X_{\bar{o}}^{\perp} = \mathrm{Im}\left(\mathcal{O}_n^T\right)$.

Lemma 7. Linear independence of the first k rows of $\mathcal{O}_n$

If $\mathrm{rank}\left(\mathcal{O}_n\right) = k \le n$, then the first k rows of $\mathcal{O}_n$ are linearly independent, and any further rows of it can be expressed as the linear combination of the first k rows.

Formally: $\forall i \in \mathbb{N}\ \exists \alpha \in \mathbb{R}^k$, that $CA^{k+i} = \alpha^T \mathcal{O}_k$, where $\mathcal{O}_k \in \mathbb{R}^{k\times n}$ is defined as $\mathcal{O}_k = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{k-1} \end{pmatrix}$.

Remark. $\mathbb{N} = \{0, 1, 2, ..\}$, $\mathbb{N}_+ := \mathbb{N}\backslash\{0\}$.

Proof. The proof is given in the following three steps:

(i) If $k = n$, the set of row vectors (also called as “covariant vectors”) $C, CA, .., CA^{n-1}$ constitutes a linearly independent (covariant) basis for vector space $\mathbb{R}^n$, which means that any other row vectors in $\mathbb{R}^n$ can be expressed by their linear combinations, the same as $CA^{n+i}, \forall i \in \mathbb{N}$ can be.

(ii) Let k be the first natural number, for which there exists $\alpha \in \mathbb{R}^k$ such that $CA^k = \alpha^T \mathcal{O}_k$. Then

CA^{k+1} can also be expressed by the covariant vectors of $\mathcal{O}_k$:

$$CA^{k+1} = \left(CA^k\right)A = \left(\sum_{j=1}^{k} \alpha_j CA^{j-1}\right)A = \sum_{j=1}^{k-1} \alpha_j CA^j + \alpha_k \sum_{j=1}^{k} \alpha_j CA^{j-1} \tag{24}$$

By induction, we have that for every every $i \in \mathbb{N}$ there exists $\alpha \in \mathbb{R}^k : CA^{k+i} = \alpha\mathcal{O}_k$.

(iii) As a consequence of (ii), we can state that if $\operatorname{rank}(\mathcal{O}_n) = k < n$, that the first k rows of $\mathcal{O}_n$ are linearly independent (i.e. $\operatorname{rank}(\mathcal{O}_k) = k$). □

Lemma 8. For every $v \in \operatorname{Im}(\mathcal{O}_n^T)$, we have that $A^T v \in \operatorname{Im}(\mathcal{O}_n^T)$. In this sense, the observable subspace $X_o = \operatorname{Im}(\mathcal{O}_n^T) = \operatorname{Ker}(\mathcal{O}_n)^\perp \subseteq \mathbb{R}^n$ of the state space $X = \mathbb{R}^n$ is invariant with respect to the linear transformation $\mathcal{A}'(v) = A^T v$, i.e. $\mathcal{A}'(X_o) = X_o$.

Proof. Let $a(s) = \det(sI - A) = a_0 + a_1 s + ... + a_n s^n$. Due to Cayley-Hamilton theorem, we have that

$$a(A) = 0 \;\Rightarrow\; A^n = \frac{1}{a_n}\left(a_0 I + a_1 A + ... + a_{n-1}A^{n-1}\right) \tag{25}$$

□

Proposition 9. $x(0) \in \operatorname{Ker}(\mathcal{O}_n)$ and $u(t) = 0 \;\Rightarrow\; y(t) = 0$

Let $\operatorname{rank}(\mathcal{O}_n) = k < n$. If $x_0 \in \operatorname{Ker}(\mathcal{O}_n)$ and $u \equiv 0$, than $y(t) = 0$ for every $t > 0$, i.e

$$x(t) = e^{At}x_0 \in \operatorname{Ker}(\mathcal{O}_n)$$

In other words, if there is no input signal ($u(t) = 0$) and the initial condition x_0 belongs to the unobservable subspace $\operatorname{Ker}(\mathcal{O}_n)$, than the state response of the system $x(t) = e^{At}x_0$ will remain in this subspace.

Proof. As a consequence of Proposition 7, we have that if $CA^k x_0 = 0$ for $k = \overline{0, n-1}$, than $CA^k x_0 = 0$ holds for every $k \in \mathbb{N}$. If we consider the Taylor expansion of matrix exponent e^{At}, we have:

$$CA^k e^{At} x_0 = \sum_{j=0}^{\infty} \frac{t^k}{k!} \cdot \underbrace{CA^{k+j}x_0}_{0} = 0 \;\; \forall k = \overline{0, n-1} \;\;\Rightarrow\;\; \mathcal{O}_n e^{At} x_0 = 0 \Leftrightarrow e^{At}x_0 \in \operatorname{Ker}(\mathcal{O}_n) \tag{26}$$

Consequently, for a given unobservable state space model (A,B,C,D) if we start the system from the unobservable subspace $x(0) \in \operatorname{Ker}(\mathcal{O}_n)$ and having a zero input ($u \equiv 0$) the output will be zero $y(t) = 0$, for every $t > 0$. □

Proposition 10. Same output for all initial state of an unobservable class

Let us denote $v_1, .., v_{n-k} \in \mathbb{R}^n$, $k < n$ the basis vectors of the null space of $\mathcal{O}_n$:

$$\operatorname{Ker}(\mathcal{O}_n) = \left\{ \alpha_1 v_1 + .. + \alpha_{n-k} v_{n-k} = \alpha^T N \;\middle|\; \alpha \in \mathbb{R}^{n-k} \right\}, \quad \text{where } N := \begin{pmatrix} v_1 & .. & v_{n-k} \end{pmatrix} \in \mathbb{R}^{n \times (n-k)}$$

Matrix N is called an *annihilator* of $\mathcal{O}_n$, since $\mathcal{O}_n N = 0_{n \times (n-k)}$. Now we introduce the following notations:

$$x_0 + \operatorname{Ker}(\mathcal{O}_n) := \left\{ x_0 + \alpha^T N \;\middle|\; \alpha \in \mathbb{R}^{n-k} \right\} \tag{27}$$

From any initial condition $x(0) \in x_0 + \operatorname{Ker}(\mathcal{O}_n)$ and for a given input $u(t)$, the system will produce the same output $y(t)$.

Proof. The explicit solution of the state space model is

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau \tag{28}$$

Considering an initial condition $x(0) = x_0 + \alpha^T N \in x_0 + \operatorname{Ker}(\mathcal{O}_n)$ with an arbitrary $\alpha \in \mathbb{R}^{n-k}$, and keeping in mind, that $\alpha^T N \in \mathrm{Ker}(\mathcal{O}_n)$ (i.e. $CA^i\alpha^T N = 0$ for all $i \in \mathbb{N}$) we obtain:

$$y(t) = Ce^{At}\big(x_0 + \alpha^T N\big) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau \tag{29}$$

Finally, we can observe that the expression for $y(t)$ does not depend on α. It depends only on the input $u(t)$ and on x_0, furthermore, for each x_0 we obtain different outputs, x_0 defines the unobservability class, that the system is actually in. If we can find a particular solution $x(t)$ for the (under-determined) linear equation system

$$\mathcal{Y}(t) = \mathcal{O}_n x(t) + \mathcal{T}\mathcal{U}(t) \qquad [\texttt{lec\_03.pdf}\text{, pg. } 10/31] \tag{30}$$

we can determine the actual unobservability class of the system, but we have no further informations about the state vector itself. □

Remark. Set $x_0 + \mathrm{Ker}(\mathcal{O}_n)$ is not a subspace of $\mathbb{R}^n$, since many properties of the vector space broke (eg. does not have a unity element), however, it is a k dimensional manifold (sokaság) in vector space $\mathbb{R}^n$.

B.2 Controllable subspace $X_c = \mathrm{Im}(\mathcal{C}_n)$. Uncontrollable subspace $X_{\bar{c}} = X_c^{\perp} = \mathrm{Ker}\big(\mathcal{C}_n^T\big)$.

Lemma 11. If (A, B, C) is not controllable $\mathrm{rank}(\mathcal{C}_n) = k < n$, the first k columns of $\mathcal{C}_n$ are linearly independent.

Proof. Same as Lemma 7. □

Lemma 12. For every $v \in \mathrm{Im}(\mathcal{C}_n)$, vector $Av \in \mathrm{Im}(\mathcal{C}_n)$. In this sense, the controllable subspace $X_c = \mathrm{Im}(\mathcal{C}_n) \subseteq \mathbb{R}^n$ of the state space $X = \mathbb{R}^n$ is invariant with respect to the linear transformation $\mathcal{A}(v) = Av$, i.e. $\mathcal{A}(X_c) = X_c$.

Proof. Let $v \in X_c = \mathrm{span}\left\langle B, AB, ..., A^{n-1}B\right\rangle$, therefore, there exist real values $\alpha_1, ..., \alpha_n \in \mathbb{R}$, such that

$$v = \sum_{i=1}^{n} \alpha_i A^{i-1}B \;\Rightarrow\; Av = \sum_{i=1}^{n} \alpha_i A^i B. \tag{31}$$

It is obvious that $A^iB \in X_c$ for all $i = \overline{1, n-1}$, furthermore, due to Lemma 11, A^nB can be expressed as the linear combination of vectors A^{i-1}, B, $i = \overline{1, n}$. Finally, we have that $Av \in X_c$. □

Proposition 13. $x_0 \in \mathrm{Im}\left(\mathcal{C}_n\right) \Rightarrow x(t) \in \mathrm{Im}\left(\mathcal{C}_n\right)$

If the initial condition $x(0) = x_0$ belongs to the controllable subspace of the state space, than the solution $x(t)$ will also belong to it. Formally:

$$x_0 \in \mathrm{Im}(\mathcal{C}_n) \Rightarrow x(t) \in \mathrm{Im}(\mathcal{C}_n)\, \forall t \geq 0. \tag{32}$$

If the initial condition is not an element of $\mathrm{Im}(\mathcal{C}_n)$, but the system is stable, than the trajectory will tend exponentially to the controllable subspace of the state space, i.e.

$$A \prec 0 \Rightarrow x(t) \to \mathrm{Im}(\mathcal{C}_n) \tag{33}$$

Proof. If $x_0 \in \mathrm{Im}(\mathcal{C}_n) = X_c$, than

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau = \sum_{k=0}^{\infty} \frac{t^k}{k!}\underbrace{A^k x_0}_{\in X_c} + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!}\underbrace{A^k B}_{\in X_c} u(\tau)\mathrm{d}\tau \in X_c. \tag{34}$$

If $x_0 \notin X_c$ but $A \prec 0$ (is negative definite), than

$$x(t) = \underbrace{e^{At}x_0}_{\to 0} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{\in X_c} \to X_c. \tag{35}$$

So, the solution tends to the controllable subspace. □

Theorem 14. (Control the system to a given state) If the system is controllable, there exists an input

$$u(t) = -B^T e^{A^T(t_1-t)} P^{-1}(t_1)\big(e^{At_1}x_0 - x_1\big), \text{ where } P(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}\mathrm{d}\tau, \quad t \in [0, t_1], \tag{36}$$

which leads the system from $x(0)$ to $x(t_1) = x_1$ in a finite time $t_1 < \infty$.

Proof. A proof for it can be found in [3, Theorem 2.21]. □

B.3 Controllability staircase form

Proposition 15. Controllability staircase form

We construct the following transformation matrix $T^{-1} = S = \big(v_1, ..., v_k, w_{k+1}, ..., w_n\big)$, where $[v] = [v_1, ..., v_k]$ is the orthonormal (ON) basis of $X_c = \mathrm{Im}(\mathcal{C}_n)$ and $[w] = [w_{k+1}, ..., w_n]$ is the ON basis of $X_{\bar{c}} = \mathrm{Im}(\mathcal{C}_n)^{\perp} = \mathrm{Ker}\big(\mathcal{C}_n^T\big)$. Then the transformed matrices will have the form:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0_{(n-k)\times k} & \bar{A}_{22} \end{pmatrix} \tag{37a}$$

$$\bar{B} = TB = \begin{pmatrix} \bar{B}_1 \\ 0_{(n-k)\times 1} \end{pmatrix} \tag{37b}$$

Using SVD: $\mathcal{C}_n = U_c\Sigma_cV_c^T$, $S := U_c$

Proof. (For simplicity, only for SISO) Since X_c and $X_{\bar{c}}$ are orthogonal complement of each other (i.e. $X_c \otimes X_{\bar{c}} = \mathbb{R}^n$), $[v, w]$ is an ON basis of $\mathbb{R}^n$. In other words: S is an orthogonal matrix with the well-known properties:

$$S^TS = I_n \quad \Rightarrow \quad S^{-1} = S^T = \begin{pmatrix} V^T \\ W^T \end{pmatrix}, \quad \text{where } V = \big(v_1, ..., v_k\big) \text{ and } W = \big(w_{k+1}, ..., w_n\big) \tag{38}$$

Furthermore, $V^TW = 0_{k\times(n-k)}$ and $W^TV = 0_{(n-k)\times k}$ (39). Then the transformed matrix $\bar{A}$ will be:

$$\bar{A} = TAT^{-1} = S^TAS = \begin{pmatrix} V^T \\ W^T \end{pmatrix} A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V^TAV & V^TAW \\ W^TAV & W^TAW \end{pmatrix}. \tag{40}$$

The columns of V are elements of X_c, therefore, the columns of AV are also elements of X_c. The columns of W are the basis vectors of $X_{\bar{c}} = X_c^{\perp}$, therefore, $W^TAV = 0_{(n-k)\times k}$. The transformed matrix $\bar{B}$ will be:

$$\bar{B} = TB = S^TB = \begin{pmatrix} V^T \\ W^T \end{pmatrix} B = \begin{pmatrix} V^TB \\ W^TB \end{pmatrix}. \tag{41}$$

Since $B \in X_c$, $w_j \in X_c^{\perp}$, $W^TB = 0_{(n-k)\times 1}$, $j = \overline{k+1, n}$. □

B.4 Observability staircase form

Proposition 16. Observability staircase form

We construct the following transformation matrix $T^{-1} = S = \big(v_1, ..., v_k, w_{k+1}, ..., w_n\big)$, where $[v] = [v_1, ..., v_k]$ is the orthonormal (ON) basis of $X_o = \mathrm{Ker}(\mathcal{O}_n)^{\perp} = \mathrm{Im}\big(\mathcal{O}_n^T\big)$ and $[w] = [w_{k+1}, ..., w_n]$ is the ON basis of $X_{\bar{o}} = \mathrm{Ker}(\mathcal{O}_n)$. Then the transformed matrices will have the form:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} \bar{A}_{11} & 0_{k\times(n-k)} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \tag{42a}$$

$$\bar{C} = CT^{-1} = \begin{pmatrix} \bar{C}_1 & 0_{1\times(n-k)} \end{pmatrix} \tag{42b}$$

Using SVD: $\mathcal{O}_n = U_o\Sigma_oV_o^T$, $S := V_o$

Proof. (For simplicity, only for SISO) Since X_o and $X_{\bar{o}}$ are orthogonal complement of each other (i.e. $X_o \otimes X_{\bar{o}} = \mathbb{R}^n$), $[v, w]$ is an ON basis of $\mathbb{R}^n$. In other words: S is an orthogonal matrix with the well-known properties:

$$S^T S = I_n \quad \Rightarrow \quad S^{-1} = S^T = \begin{pmatrix} V^T \\ W^T \end{pmatrix}, \quad \text{where } V = \left(v_1, ..., v_k\right) \text{ and } W = \left(w_{k+1}, ..., w_n\right) \tag{43}$$

Furthermore, $V^T W = 0_{k\times(n-k)}$ and $W^T V = 0_{(n-k)\times k}$ (44). The transformed matrix $\bar{A}$ will be:

$$\bar{A} = TAT^{-1} = S^T AS = \begin{pmatrix} V^T \\ W^T \end{pmatrix} A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V^T AV & V^T AW \\ W^T AV & W^T AW \end{pmatrix}. \tag{45}$$

The columns of V are elements of X_o, therefore, the columns of $A^T V$ are also elements of X_o. The columns of W are the basis vectors of $X_{\bar{c}} = X_c^\perp$, therefore, $(A^T V)^T W = V^T AW = 0_{k\times(n-k)}$. The transformed matrix $\bar{C}$ will be:

$$\bar{C} = CT^{-1} = CS = C \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} CV & CW \end{pmatrix}. \tag{46}$$

Since $C^T \in X_o$, $w_j \in X_o^\perp$, $CW^T = 0_{1\times(n-k)}$, $j = \overline{k+1, n}$. □

Proposition 17. If (A, C) has unobservable mode (i.e. is unobservable), there exists $x \in \mathbb{R}^n$, such that $Ax = \lambda x$ and $Cx = 0$. Consequently, λ is a "decoupling zero" of (A, B, C, D), since

$$M = \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \text{ is singular,} \tag{47}$$

namely there exists $\xi = \begin{pmatrix} x \\ 0 \end{pmatrix} \neq 0$ such that $M\xi = 0$. Or in other words, the kernel space of M is not empty, meaning that M is singular.

Proposition 18. The input decoupling zeros are equal to the eigenvalues of the uncontrollable subsystem.

Proof. We assume that (A, B) is uncontrollable:

$$\mathcal{C}_n = \begin{pmatrix} B & AB & ... & A^{n-1}B \end{pmatrix} \in \mathbb{R}^{n\times mn} \tag{48}$$

is rank deficient, that implies a nonempty kernel space $\text{Ker}\left(\mathcal{C}_n^T\right) \subset \mathbb{R}^n$, namely, there exists $x \in \mathbb{R}^n$ such that $\mathcal{C}_n^T x = 0$. Alternatively, we have that

$$\begin{cases} B^T x = 0 \\ B^T A^T x = 0 \\ \cdots \\ B^T \left(A^T\right)^{n-1} x = 0 \end{cases} \tag{49}$$

□

B.5 Kalman decomposition

We produce a controllability staircase form decomposition on the system, than on both subsystems (controllable and uncontrollable) we produce an observability staircase form decomposition.

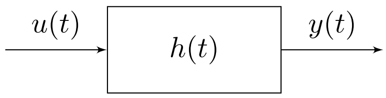
Computer controlled systems

Lecture 5

version: 2018.10.16. – 15:48:55

1 Stability

1.1 Input-output stability



BIBO-stability: Bounded input → bounded output.

$$|u(t)| \leq M_1 < \infty \ \forall t \Rightarrow \exists M_2 : |y(t)| \leq M_2 < \infty \ \forall t$$

Theorem 1. A SISO LTI system is BISO stable iff

$$\int_0^\infty |h(t)| dt \leq M < \infty$$

where $h(t)$ is the impulse response of the system and $M \in \mathbb{R}^+$.

1.2 Equilibrium points

Given the following autonomous system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \text{ (state vector) }, \qquad f : \mathbb{R}^n \to \mathbb{R}^n \text{ (state transition function)} \tag{1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field. This system has an equilibrium point in state $x^* \in \mathbb{R}^n$ if the values of the state variables in x^* do not change, i.e. their derivative are zero. In other words, $x^* \in \mathbb{R}^n$ is an equilibrium point if $f(x^*) = 0$. In order to find all equilibrium point, we need to solve the (possibly nonlinear) system of equations:

$$f(x) = 0 \tag{2}$$

An autonomous system $\dot{x} = f(x)$ may have several equilibrium points, depending on the number of solution of equation (2). An important property of an equilibrium point is its stability, namely how the system reacts when we move the state of the system from the equilibrium equilibrium point to a near point in the state space. Does it converge to the equilibrium point again, or goes away? Consider the hill-valley problem illustrated in Figure (1). It is obvious that the lower equilibrium point (in the valley) is stable, since if we moves the ball away a bit, it returns to the same equilibrium point. However, the upper equilibrium point is unstable, because if we hit the ball, it will fall down.

We aim to find a mathematical apparatus, which can point out whether an equilibrium point is stable or unstable. This apparatus will be the theory of Lyapunov stability.

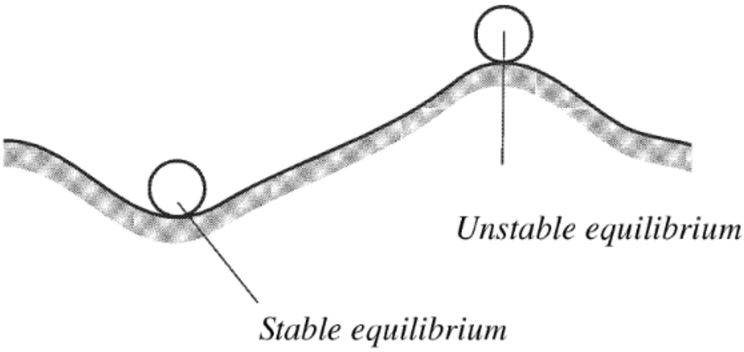


Figure 1. This system has two equilibrium points. One of it is stable, the other one is unstable.

1.3 Lyapunov stability

The Lyapunov theory gives the sufficient conditions for stability of an equilibrium point. Informally, the Lyapunov stability means the following: "If I start the system enough close to the equilibrium point, the system will remain quite closed to it."

Definition 2. (Lyapunov stability) Given the autonomous system $\dot{x} = f(x)$, and its equilibrium point $x^* \in \mathbb{R}^n$, i.e. $f(x^*) = 0$. We say that x^* is ***stable in the sense of Lyapunov***, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad \|x^*(0) - x(0)\| < \delta \quad \text{than} \quad \|x^*(t) - x(t)\| < \varepsilon \quad \text{for all } t > 0 \tag{3}$$

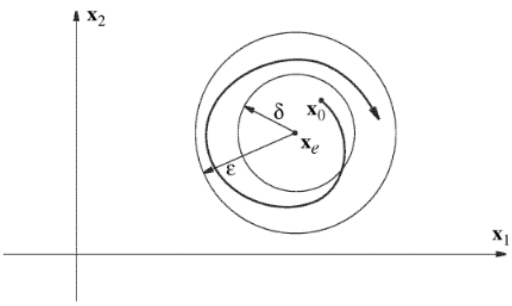


Figure 2. If the equilibrium point $x^* = x_e$ is stable in the sense of Lyapunov, than for all possible positive ε we can find a positive value δ such that if we start the system within a sphere of radius δ of the equilibrium point, the trajectory will not move further then ε.
Ha rendszer Ljapunov értelemben stabil, akkor minden ε-hoz tudok, találni egy olyan δ értéket, amire a rendszer trajektóriája ε határon belül marad, ha a rendszer kezdeti állapota az egyensúlyi pont δ környezetén belül található.

Considering this definition, we can conclude that in the case of the upper equilibrium point of the hill-valley example (Figure (1)) there exist a $\varepsilon > 0$ such that the trajectory of the ball will leave that interval for every little δ perturbation of the equilibrium point (i.e. the ball will fall down).

Ha fenti definíció birtokában nézzük meg a 1. ábrát, akkor láthatjuk, hogy a domb tetjén lévő labdához nem tudunk bármilyen ε-hoz, δ-t találni, hisz már a legkisebb δ elmozdulásra, a labda legurul a dombról és átlépi a tetszőleges választott ε értéket.

1.4 Asymptotic stability

Definition 3. If the state vector $x(t)$ of the system not only approaches the equilibrium point x^* but also tends to it i.e.

$$\lim_{t\to\infty} x(t) = x^* \tag{4}$$

than we say that x^* is ***asymptotically stable***.
We say that the system is ***globally asymptotically stable if for all*** $x_0 \in \mathbb{R}^n$ ***the trajectory of the system will tend to*** x^*

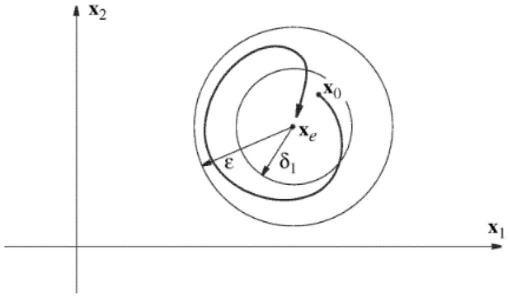


Figure 3. If the system is asymptotically stable that the trajectory of the system will tend do the equilibrium point. Ha a rendszer aszimptotikusan stabil, akkor nem csak ε határon belül marad a trajektóriája, de ez a trajektória vissza is tér az egyensúlyi pontba.

1.4.1 Stability of an LTI system

Theorem 4. An LTI system (A, B, C) is asymptotically stable if and only if the real part of the eigenvalues of matrix A are strictly negative. $\mathrm{Re}\,(\lambda_i(A)) < 0$.

Megj: Ez pontosan azt jelenti, hogy sajátvektorok bázisában felírt A mátrixhoz tartozó e^{At} mátrix minden elemében a kitevő negatív. Azaz minden exponencális értéke nullához tart.

1.5 Lyapunov's direct method

Theorem 5. Consider a system $\dot{x}(t) = f(x(t))$ with the equilibrium point x^*, namely $f(x^*) = 0$. Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a *scalar valued differentiable* function of the state vector $x \in \mathbb{R}^n$.

If $\boxed{V(x) > 0}$ for all $x \neq x^*$, $V(x^*) = 0$, and $\dot{V}(x) := \left[\frac{\partial V(x)}{\partial t}\right]_{\dot{x}=f(x)} = \boxed{\langle \mathrm{grad}\, V, f(x)\rangle \leq 0}$
than x^* is ***stable in the sense of Lyapunov***.

If $\boxed{V(x) > 0}$ for all $x \neq x^*$, $V(x^*) = 0$, and $\dot{V}(x) := \left[\frac{\partial V(x)}{\partial t}\right]_{\dot{x}=f(x)} = \boxed{\langle \mathrm{grad}\, V, f(x)\rangle < 0}$
than x^* is ***asymptotically stable***.

1.6 Examples

Example 1.

Given the following LTI system $\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} x$, with $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ being an equilibrium point. Is $V(x) = x_1^2 + 2x_2^2$ an appropriate Lyapunov function?
Solution:

$$V(x) > 0 \quad \checkmark$$

$$V(x^*) = 0 \quad \checkmark$$

$$\dot{V}(x) := \left[\frac{\partial V(x)}{\partial t}\right]_{\dot{x}=f(x)} = \langle \operatorname{grad} V, f(x) \rangle =$$

$$\begin{pmatrix} 2x_1 & 4x_2 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ -2x_1 + x_2 \end{pmatrix} = 2x_1x_2 - 8x_1x_2 + 4x_2^2 = 4x_2^2 - 6x_1x_2 \not< 0 \text{ for all } x \neq x^*$$

Let $x_1 = 0$ and $x_2 = 1$. Since $4 \cdot 1 - 6 \cdot 0 \cdot 1 = 4$, which is grater than 0, therefore, $V(x)$ is not an appropriate Lyapunov function.
But this does not mean that the x^* is NOT stable. Since this is an LTI system, we have other possibilities to analyse its stability (e.g. the eigenvalues of matrix A):

$$\lambda I - A = \begin{pmatrix} \lambda & -1 \\ 2 & \lambda - 1 \end{pmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda - 1) + 2 = \lambda^2 - \lambda + 2$$

$$\frac{1 + \sqrt{1-8}}{2} \to 0.5 + i\frac{\sqrt{7}}{2}$$

$$\frac{1 - \sqrt{1-8}}{2} \to 0.5 - i\frac{\sqrt{7}}{2}$$

Since the real part of the eigenvalues are positive, the equilibrium point is unstable.

Mivel mindkét esetben a gyök valós része nagyobb mint nulla, ezért nem teljesül az a feltétel, hogy minden sajátértéke valós része negatív, ebből következően ez a rendszer nem stabil. Tehát nem csak hogy a megadott függvény nem volt Ljapunov függvény, hanem a rendszerhez nem is lehet ilyet megadni.

Example 2. Given a system with its transfer function $H(s) = \frac{2}{(s+3)(s+2)}$. Is it stable?
Solution: Stable, since the real parts of its poles $\lambda_1 = -3$ and $\lambda_2 = -2$ are negative.

Example 3. Determine c such that the system $\dot{x} = Ax$ is stable, $A = \begin{pmatrix} 1 & c \\ -2 & -3 \end{pmatrix}$.
Solution.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -c \\ 2 & \lambda + 3 \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 2c$$

$$\lambda^2 + 2\lambda - 3 + 2c = 0$$

$$\lambda_{1,2} = \frac{-2 + -\sqrt{4 - 4(-3 + 2c)}}{2}$$

If $\sqrt{4 - 4(2c - 3)} < 2$ both eigenvalues will have a negative real part.

$$-(2c - 3) < 0$$

$$-2c < 3$$

$$c > \frac{3}{2}$$

Example 4. Given the following system:

$$\dot{x} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix} x$$

1. Is it asymptotically stable?
   Obviously no, since its first eigenvalue is positive.

2. Is it BIBO stable?
   We shall check whether the integral of the impulse response is finite or not.

   $$H(s) = \frac{1}{s+10}$$

   $$h(t) = e^{-10t}$$

   $$\int_0^\infty e^{-10t} dt = \left[\frac{e^{-10t}}{-10}\right]_0^\infty = \frac{1}{10}$$

   Therefore, the system is BIBO stable. It is important to note, that BIBO stability does not imply asymptotic stability. But asymptotic stability implies BIBO stability.

   Tehát a rendszer BIBO stabil. Ez egy fontos példa az előadáson bizonyított tételre, hogy az aszimptotikus stabilitásból következik a BIBO stabilitás, de ez visszafelé nem feltétlenül igaz.

1.6.1 Total energy as a Lyapunov function in case of a mechanical system

We are searching for an appropriate Lyapunov function for the mass-spring-damper system. Its dynamics is given by the following equation:

Keressünk Ljapunov függvényt az alábbi rendszerhez! Adott a korábban bevezetett tömeg-rugó-csillapítás rendszer. Az erők egyensúlyát a következő differenciálegyenlet adja meg:

$$m\ddot{y} + Dy + C\dot{y} = F$$

where m is the mass of the body, D is spring coefficient, C is the damping factor. Let $x_1 := y$ be the position of the body, and let $x_2 := \dot{y} = \dot{x}_1$ be the velocity of the body, which together gives the state vector.

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{D}{m}x_1 - \frac{C}{m}x_2 + \frac{u}{m} \end{aligned} \quad \Rightarrow \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{D}{m} & -\frac{C}{m} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{5}$$

The output of the system (the measured quantity) will be the position of the body $y = x_1$. We use the following parameter configuration: $m = 1kg$, $D = 10\frac{N}{m}$, $C = 0.2\frac{Ns}{m}$, $u = 0N$

$$A = \begin{pmatrix} 0 & 1 \\ -10 & 0.2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

In case of mechanical systems it is a good strategy if we chose the total energy as a Lyapunov function. The total energy is the sum of the followings:

$$\begin{aligned} E_m &= \frac{1}{2}mv^2 \text{ (kinetic energy)} \\ E_h &= \frac{1}{2}Dx^2 \text{ (potential energy due to the extension of spring)} \end{aligned} \tag{6}$$

Then

$$V(x) = \frac{1}{2}Dx_1^2 + \frac{1}{2}mx_2^2 > 0 \quad \forall x \neq 0 \qquad \checkmark \tag{7}$$

Its time derivative regarding the system's dynamics is:

$$\text{grad}V = \begin{pmatrix} Dx_1 & mx_2 \end{pmatrix} \tag{8}$$

therefore

$$\dot{V}(x) = \langle \text{grad}V, Ax \rangle = \begin{pmatrix} Dx_1 & mx_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -\frac{D}{m}x_1 - \frac{C}{m}x_2 \end{pmatrix} \tag{9}$$

$$= Dx_1x_2 + mx_2(-\frac{D}{m}x_1 - \frac{C}{m}x_2) \tag{10}$$

$$= -Cx_2^2 = -0.4x_2^2 < 0 \quad \forall x \neq 0 \qquad \checkmark \tag{11}$$

The time derivative of the Lyapunov function is negative, for all $x \neq 0$ state vector. This physically means that the system loses its energy during its operation, and finally it will stop at the equilibrium point $x^* = 0$. The decrease in the system's total energy is owing to the damper (with damping coefficient C).

A derivált értéke minden x_2 érték esetén negatív lesz. Fizikailag ez azt jelenti, hogy a rendszer kezdeti energiáját elveszti a csillapításon keresztül, és végül megáll.

1.6.2 Total energy as a Lyapunov function in case of an electronic system

We consider the well-known LRC circuit (see the previous lecture notes). Its state space model is the following:

$$\begin{pmatrix} \dot{i} \\ \dot{u}_C \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ u_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u_{be}$$

In this physical system there are two energy conserving elements: the capacitor C and the inductance L:.

A rendszerben két energiatároló elem található, az egyik a kondenzátor, a másik tekercs.

Their energy is given by:

$$E_C = \frac{1}{2}Cu^2 \tag{12}$$

$$E_L = \frac{1}{2}Li^2 \tag{13}$$

Therefore, let the Lyapunov function be the following:

$$V(x) = E_C + E_L > 0 \quad \forall x \neq 0 \qquad \checkmark \tag{14}$$

its derivative

$$\dot{V}(x) = \begin{pmatrix} Lx_1 & Cx_2 \end{pmatrix} \begin{pmatrix} \frac{R}{L}x_1 - \frac{1}{L}x_2 \\ \frac{1}{C}x_2 \end{pmatrix} = -Rx_1^2 < 0 \quad \forall x \neq 0 \qquad \checkmark \tag{15}$$

Since the Lyapunov function is negative for all nonzero state vectors, the system is asymptotically stable. Note that, if there is no resistance (R), than the total energy of the LRC circuit is constant during its operation, hence shall we obtain a harmonically oscillating system.

Mivel a derivált mindig negatív, ezért ez egy jó Ljapunov függvény. Az előző példával analóg módon itt is az egyetlen energiaveszteség az ellenálláson keletkezik, és hő formájában távozik a rendszerből. Tehát ha nincs ellenállás a rendszerben, akkor csillapítatlan rezgést végez a rendszer.

1.7 Lyapunov function in case of an LTI system

If there exist matrix $P > 0$ and matrix $Q > 0$, such that $PA + A^T P = -Q$, than $V(x) = x^T P x = \langle Px, x \rangle$ is an appropriate Lyapunov function, thus the system is asymptotically stable. Matrix P can be constructed as follows:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

However, in case of LTI systems, the eigenvalues of matrix A gives a full knowledge about the system's stability.

Habár lineáris rendszereknél az A mátrix sajátértékei valós részének negativitása garantálja a stabilitást, azonban lehet olyan rendszerméret, ahol a sajátértékek kiszámításánál "olcsóbb", a fenti mátrix-egyenlőtlenség megoldása.

Example 5.

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad u(t) = 0$$

1. Construct a matrix P such that $V(x) = x^T P x$ is an appropriate Lyapunov function, i.e. the followings are satisfied:

$$Q = Q^T \quad Q > 0 \tag{16}$$
$$P = P^T \quad P > 0 \tag{17}$$
$$A^T P + PA = -Q. \tag{18}$$

   Solution:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

   where the positive definite matrix Q can be arbitrarily chose. Let it be

$$Q = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}.$$

2. Check, whether conditions (1),(2),(3) are indeed satisfied!

3. Give the time derivative of the Lyapunov function $\dot{V}(x) = \langle \text{grad} V, Ax \rangle$!

1.8 Local stability analysis of a nonlinear system around the operating point using the linear linearized model

Operating point = munkapont.

Local stability analysis of nonlinear system $\dot{x} = f(x)$, step-by-step:

- Determine the equilibrium points x^* (steady states) of the system by solving the nonlinear system of algebraic equations $f(x) = 0$.
- Compute the Jacobian matrix $f(x)$: $J_f(x) = \mathrm{D}f(x)$.
- Compute the value of $J_f(x)$ when $x = x^*$: $A := [J_f(x)]_{x=x^*}$.
- Compute the eigenvalues of matrix A, which gives whether the system is locally asymptotically stable ($\mathrm{Re}\,\lambda_i < 0$) or not ($\mathrm{Re}\,\lambda_i > 0$).

Example 6. Lotka-Volterra model

Let x_1 and x_2 denote the number of prays and predators, respectively. Their population-dynamics can be described by the following system of nonlinear differential equation:

Jelölje x_1 és x_2 a zsákmányállatok illetve a ragadozók számát, a populáció-dinamikát pedig írja le a következő differenciálegyenlet-rendszer:

$$\begin{aligned} \dot{x}_1 &= \quad ax_1 - bx_1x_2 \\ \dot{x}_2 &= -cx_2 + dx_1x_2 \end{aligned} \qquad \text{with } a, b, c, d > 0 \tag{19}$$

where a and c are the growth rate of the two species in the absence of the other species, the seconds terms in the equations represent the interaction between the two species. If a predator consumes a

pray that means a decrease in the number of prays, but also entails an increase in the number of predators. The rate of the mentioned decrease/increase is given by b and d.
Az egyenletben az a (c) koefficiens a zsákmányállatok (regadozók) szaporulata (elhullási aránya zsákmányállatok nélkül) b a ragadozók hatékonysága és d írja le a ragadozók táplálékbevitel melletti szaporulatát.
Equilibrium points

$$x_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x_2^* = \begin{pmatrix} \frac{c}{d} \\ \frac{a}{b} \end{pmatrix}$$

The Jacobian of f is

$$\begin{pmatrix} a - bx_1 & -bx_1 \\ dx_2 & -c + dx_1 \end{pmatrix}$$

The Jacobian of f evaluated in the equilibrium point x_1^* is:

$$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \quad \rightsquigarrow \quad \lambda_1 = a, \quad \lambda_2 = -c$$

This equilibrium point is locally unstable since there exist eigenvalues with negative real part.
The Jacobian matrix in x_2^* is:

$$\begin{pmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix} \quad \rightsquigarrow \quad \lambda^2 + ac = 0 \quad \rightsquigarrow \quad \lambda_1 = -i\sqrt{ac}, \lambda_2 = i\sqrt{ac},$$

This system is locally stable but NOT asymptotically stable. This means that the system will oscillate around the equilibrium point.

2 Further extra material

Definition 6. The $\mathcal{H}_\infty$ norm of a system operator is the peak gain of the system, namely

$$\mathcal{H}_\infty = \max_{\omega \geq 0} |H(j\omega)| \tag{20}$$

Alternatively, $\mathcal{H}_\infty$ norm is the induced $\mathcal{L}_2$ norm of the convolution operator $\mathcal{S}[u]$:

$$y(t) = \mathcal{S}[u(t)] = (h * u)(t) = \int_0^t h(t-\tau)u(\tau)\mathrm{d}\tau \quad \Rightarrow \quad \mathcal{H}_\infty = \|\mathcal{S}\|_2 = \sup_{u \in \mathcal{L}_2} \frac{\|S[u]\|_2}{\|u\|_2} \tag{21}$$

Theorem 7. The $\mathcal{H}_\infty$ norm is always smaller or equal than the absolute integral of the impulse response. Namely:

$$\max_{\omega} |H(j\omega)| \leq \|h\|_1 = \int_0^\infty |h(t)|\mathrm{d}t \tag{22}$$

Proof. $H(j\omega)$ is the Fourier transform of the impulse repsose $h(t)$, thus, we can write:

$$|H(j\omega)| = \left| \int_0^\infty h(t)e^{-j\omega t}\mathrm{d}t \right| \leq \int_0^\infty |h(t)| \cdot \underbrace{\left|e^{-j\omega t}\right|}_{1} \mathrm{d}t = \int_0^\infty |h(t)|\mathrm{d}t \tag{23}$$

This inequality holds for every ω, therefore, $\max_\omega |H(j\omega)| \leq \int_0^\infty |h(t)|\mathrm{d}t$. □

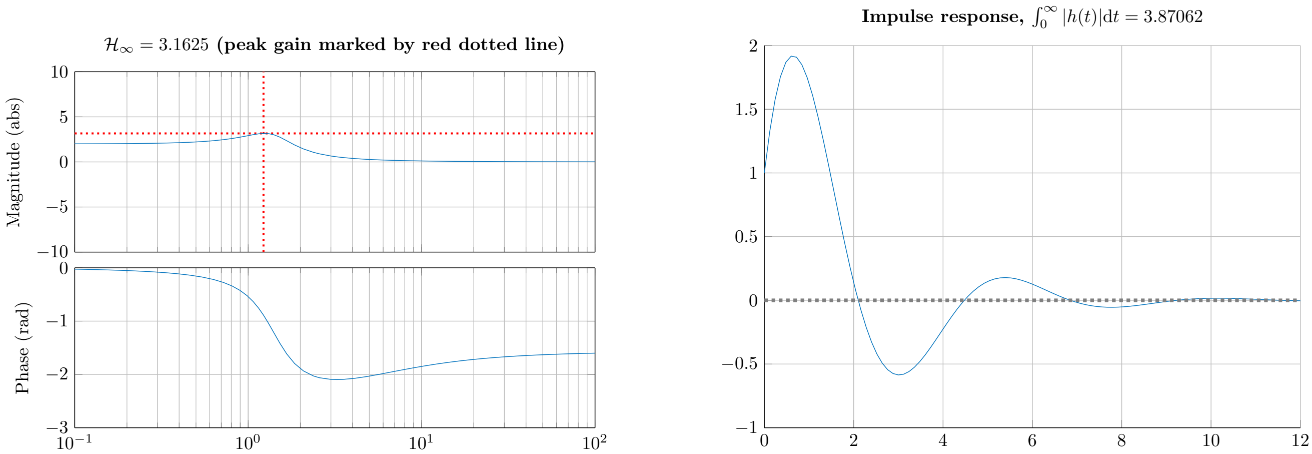


Figure 4. Bode diagram of system $H(s) = \frac{s+4}{s^2+s+2}$

Computer controlled systems

Lecture 6

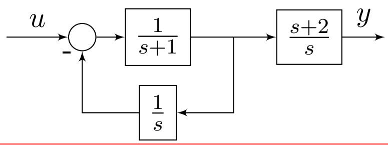
version: 2018.11.08. – 22:20:17

1 Block diagram algebra (Hatásvázlat algebra)

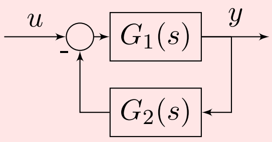
Resultant[1] transfer function computation (Eredő átviteli függvény számolása)

Example 1.

1. What is the resulting transfer function $G(s) = ?$ of $\longrightarrow$
Mi az eredő átviteli függvény: $G(s) = ?$



THE RULE:



$$G_e(s) = \frac{G_1(s)}{1 + G_1(s)\, G_2(s)} \tag{1}$$

$$G(s) = \frac{\frac{1}{s+1} \cdot \frac{s+2}{s}}{1 + \frac{1}{s} \cdot \frac{1}{s+1}} = \frac{\frac{1}{s+1}}{\frac{s^2+s+1}{s(s+1)}} \cdot \frac{s+2}{s} = \frac{1}{s+1} \cdot \frac{s(s+1)}{s^2+s+1} \cdot \frac{s+2}{s} = \frac{s+2}{s^2+s+1}$$

2. Give a possible state space realization for this transfer function!
Controller form:

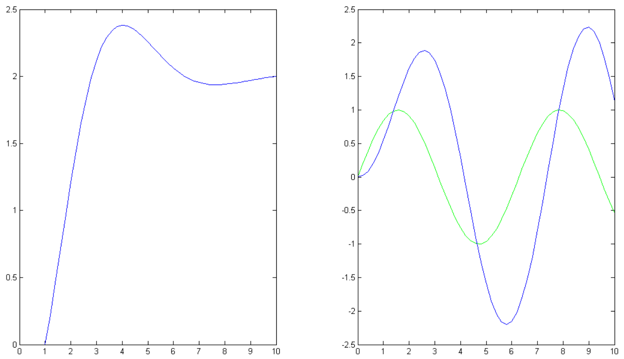
$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \quad \overset{\text{Ctrb N.F.}}{\Rightarrow} \quad \begin{aligned} A_c &= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C_c &= \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \end{aligned} \tag{2}$$

Observer form:

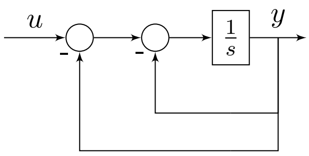
$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \quad \overset{\text{Obsv N.F.}}{\Rightarrow} \quad \begin{aligned} A_o &= \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ C_o &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned} \tag{3}$$

The next figure illustrates the behaviour of the system in case of the unit step function and a sinusoid input function.

[1] ha valakinek van az "eredő" szóra értelmesebb fordítása, kérem írjon: ppolcz@gmail.com



Example 2.



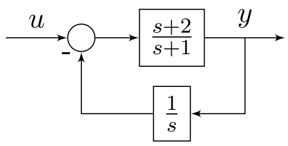
$$H(s) = \frac{1}{s}$$

What is the resulting transfer function $G(s) = ?$

$$G_0(s) = \frac{\frac{1}{s}}{1 + \frac{1}{s}} = \frac{1}{s+1}$$

$$G_1(s) = \frac{\frac{1}{s+1}}{1 + \frac{1}{s+1}} = \frac{1}{s+2}$$

Example 3.



What is the resulting transfer function $G(s) = ?$

$$G(s) = \frac{\frac{s+2}{s+1}}{1 + \frac{1}{s} \cdot \frac{s+2}{s+1}} = \frac{s(s+2)}{s(s+1) + s + 2} = \frac{s^2 + 2s}{s^2 + 2s + 2}$$

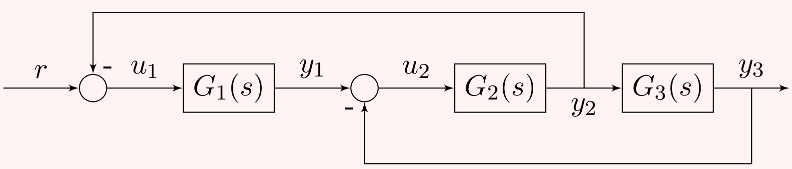
Theoretical questions (minimal computational effort is needed here)

Example 4. Given the following transfer function: $H(s) = \frac{5s^3+2s^2-s+1}{s^4+4s^2-s^2+2s+1}$. Determine whether $H(s)$ is stable or not!

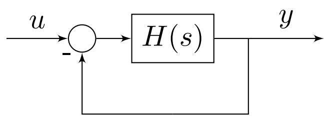
Example 5. Compute the DC-Gain of $H(s) = \frac{s+2}{s^4+3s^2+10s+5}$ in **dB**.

Example 6. (**Computational problem**)

Determine the transfer function $H_{y \to y_3}(s)$ of the following feedback system:



Example 7. Simple negative feedback



Transfer function:

$$\begin{aligned} Y(s) &= H(s)(U(s) - Y(s)) \\ Y(s) + H(s)Y(s) &= H(s)U(s) \\ (1 + H(S))Y(s) &= H(s)U(s) \\ Y(s) &= \frac{H(s)}{1+H(s)}U(s) \quad \Rightarrow \quad G(s) = \frac{H(s)}{1+H(s)} \end{aligned}$$

Using this simple negative feedback, determine whether the system is stabilizable or not, if

1. $H(s) = \frac{1}{s}$

$$G(s) = \frac{\frac{1}{s}}{1+\frac{1}{s}} = \frac{1}{s+1}$$

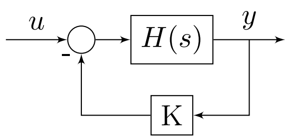
Yes, the system is stabilizable, since the resultant transfer function is stable.

2. $H(s) = \frac{1}{s-2}$

$$G(s) = \frac{\frac{1}{s-2}}{1+\frac{1}{s-2}} = \frac{1}{s-1}$$

No, the system is not stabilizable.

Example 8.



Resulting transfer function:

$$G(s) = \frac{H(s)}{1+KH(s)}$$

$$H(s) = \frac{b(s)}{a(s)} \rightarrow G(s) = \frac{\frac{b(s)}{a(s)}}{1+K\frac{b(s)}{a(s)}} = \frac{b(s)}{a(s)+Kb(s)}$$

Using this negative feedback with gain K, determine whether the system is stabilizable or not.

1. $H(s) = \frac{1}{s-3}$

$$G(s) = \frac{\frac{1}{s-3}}{1+K\frac{1}{s-3}} = \frac{1}{s-3+K}$$

Therefore, if $K > 3$ the closed loop system is stable.

2. $H(s) = \frac{1}{s-10}$

$$G(s) = \frac{\frac{1}{s-10}}{1+K\frac{1}{s-10}} = \frac{1}{s-10+K}$$

Therefore, if $K > 10$, the closed loop system is again stable.

3. $H(s) = \frac{1}{(s-3)(s-2)}$

$$G(s) = \frac{\frac{1}{(s-3)(s-2)}}{1 + K\frac{1}{(s-3)(s-2)}} = \frac{1}{s^2 - 5s + 6 + K}$$

This system is not stabilizable.

2 Control loop

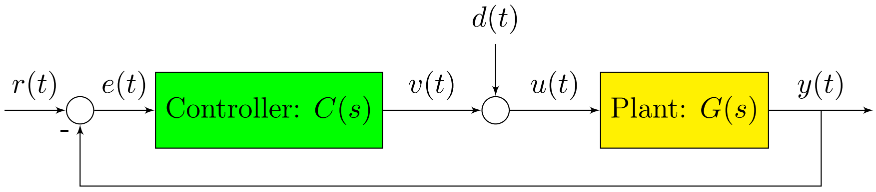


Figure 1

- **Control goal 1. (reference tracking):** to eliminate the error signal $e(t) = r(t) - y(t)$, namely the output signal $y(t)$ converges exponentially to the reference signal $r(t)$. In other words, after a while the output and the reference signal be the same.
- **Control goal 2. (input disturbance reduction):** To lower the transfer between the input disturbance (or actuator fault) $d(t)$ and the output of the error signal $e(t)$, namely: $\left|\frac{E(j\omega)}{D(j\omega)}\right|$ be as smaller as possible.
- Control (or manipulate) signal $v(t)$: the necessary input signal computed by the controller for reference tracking.
- Actuator fault
- The controlled system (Plant) receives the manipulate input $u(t)$ and generates the output signal $y(t)$
- Physical example. Consider a DC motor. Let the input be the current intensity (áramerősség) given to the DC motor, and let the revolution of the motor (fordulatszám) be the output of the DC motor. Then, the error signal will be the difference between the reference revolution and the actual revolution of the DC motor.

Example 9. The control loop presented in Figure 1 can be consider as system with two inputs (reference signal $r(t)$ and input disturbance $d(t)$) and with a single output $y(t)$.

(a) Determine the transfer function $H_{d\to y}(s)$, which is the transfer of $d(t)$ to $y(t)$.

(b) Determine the transfer function $H_{d\to e}(s)$, which is the transfer of $d(t)$ to $e(t)$.

2.1 PID controller

The objective of the PID controller is to eliminate the error signal $e(t) := r(t) - y(t)$, where $r(t)$ is the reference signal, $y(t)$ is the output of the system. In order to do this, the PID controller uses the following signals:

- actual error signal $e(t)$.

- integral of the error signal: $\int_0^t e(\tau)\mathrm{d}\tau$. This constitutes the historical informations of the error signal.
- derivative of the error signal: $\dot{e}(t)$. This gives the actual trend of the error signal.

Therefore, the PID controller *may* contain the following three dynamical components:

- proportional component (P - proportional): $u(t) = K_P \cdot e(t) \qquad H_p(s) = K_P$
- integral component (I - integral): $u(t) = K_I \cdot \int_0^t e(\tau)d\tau \qquad H_I(s) = \frac{K_I}{s}$
- derivative component (D - derivative): $u(t) = K_D \cdot \dot{e}(t) \qquad H_D(s) = s \cdot K_D$

Fontos megjegyezni, hogy a deriváló tag kauzális volta miatt valós rendszerekben a deriváló tagot egy közelítő taggal helyettesítjük.

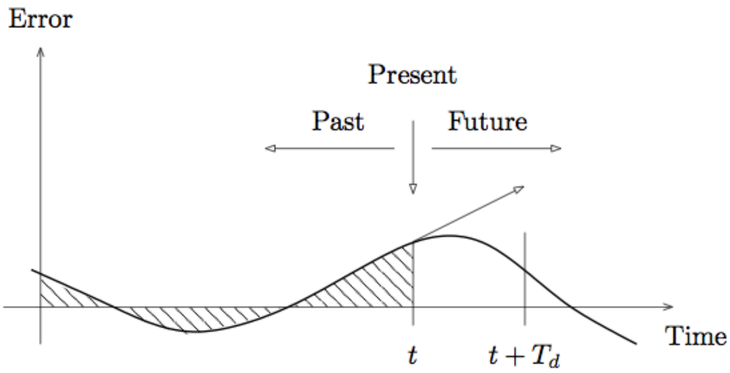


Figure 2

The transfer function of the subsystem (highlighted by the gray dashed box in Figure 3) is the following:

$$H_{PID}(s) = K_p + \frac{K_I}{s} + K_D s = \frac{sK_p + K_I + s^2 K_D}{s}$$

If we use only the P and I components of the PID controller:

$$H_{PI}(s) = K_p + \frac{K_I}{s} = \frac{sK_p + K_I}{s}$$

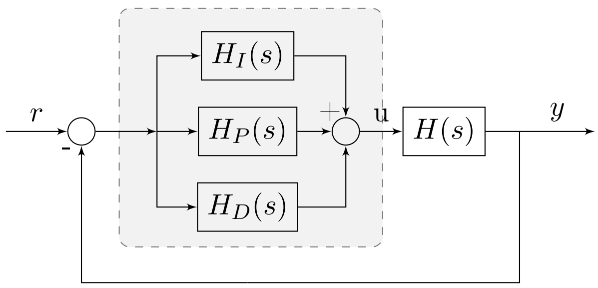


Figure 3

Example 10.

Let us consider the DC motor model, which we mentioned previously:

$$H(s) = \frac{1}{Ms^2 + bs + k} \quad (4)$$

Let $M = 1$, $b = 10$ és $k = 20$

Analyse the response of the system for the unit step function (sse Figure 5). We can see, that the limit at $t \to \infty$ of the output $y(t)$ is much less than the reference signal. This error is called *static error*.

We put into the control loop a proportional term in order to reduce the static error and to obtain a shorter transient (faster rise-time and settling-time).

Helyezzünk a szabályozási körbe egy arányos tagot, ezzel csökkentve a statikus hibát és csökkentve a felfutási időt.

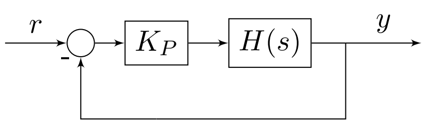


Figure 4. Block diagram of the P controller.

Transfer function of the resulting system:

$$G(s) = \frac{K_p H(S)}{1 + K_P H(s)} = \frac{K_p}{Ms^2 + bs + (k + K_p)} \quad (5)$$

The step response of the system is illustrated in Figure 6.

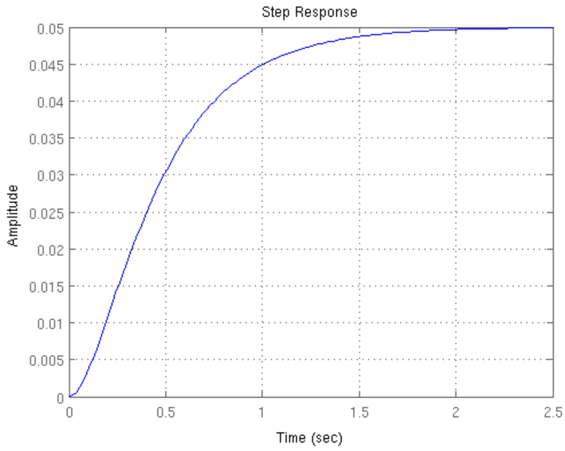


Figure 5. Step response of the uncontrolled system.

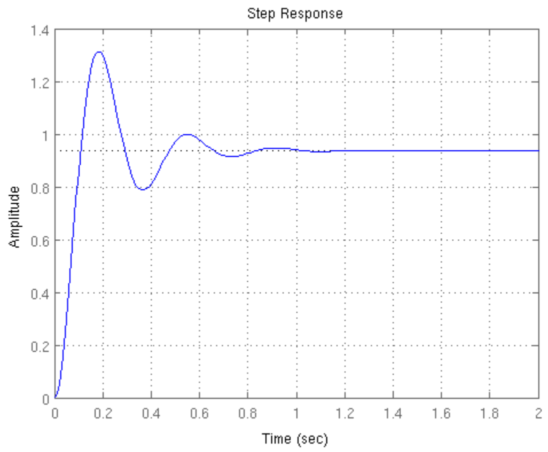


Figure 6. Step response with P controller: $K_p = 300$.

We can see, that the transient time and the static error decreased significantly, however there appears a large overshoot in the step response (the output of the system rises up to 1.3).

Látható, hogy a statikus hiba és a felfutási idő jelentősen csökkent, ugyanakkor jelentős túllövés lett a rendszerválaszban.

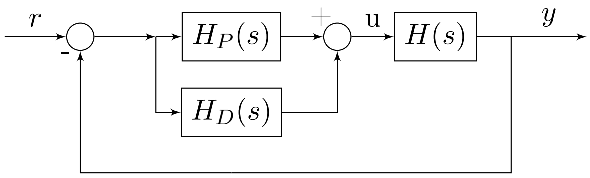


Figure 7. Block diagram of a PD controller.

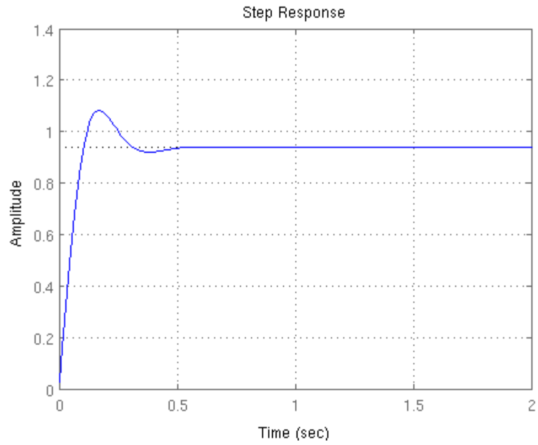


Figure 8. PD controller with $K_p = 30$, $K_i = 70$

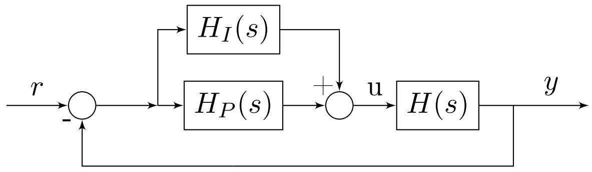


Figure 9. Block diagram of a PI controller.

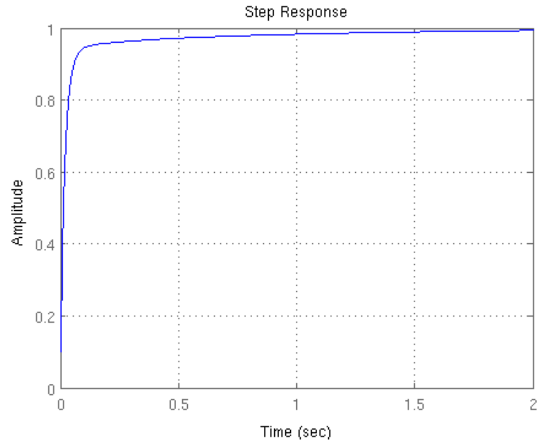
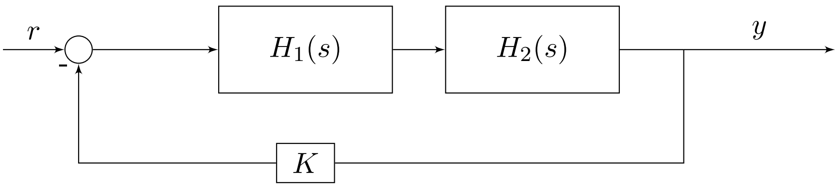


Figure 10. PID controller with $K_p = 350$, $K_d = 50$, $K_i = 300$

Source: http://www.engin.umich.edu/class/ctms/pid/pid.htm

3 További gyakorló feladatok (tipikus ZH feladatok)

1. Adott a következõ hatásvázlat:



   (a) $H_1(s) = \frac{s+2}{s^2+5s+6}$, $H_2(s) = \frac{1}{s+1}$, $K = 1$, adja meg a $G(s)$ eredõ átviteli függvényt! (2p)

   (b) $H_1(s) = \frac{s+1}{s-3}$, $H_2(s) = \frac{s+4}{s^2+3s+2}$, $K = -4$ vagy $K = 2$ értékre lesz az eredõ átviteli függvény stabil? (3p)

   (c) $H_1(s) = \frac{s+2}{s^2+5s+6}$, $H_2(s) =?$, $K = 1$, adja meg $H_2(s)$-t, úgy hogy csak -tetszõleges- instabil pólusai legyenek az eredõ rendszernek! (5p)

2. Tekintsük a következõ átviteli függvényt:

$$H(s) = \frac{s + l_1}{s^3 + l_2 s^2 + s + 3},$$

   ahol l_1 és l_2 valós paraméterek. Létezik-e olyan véges erõsítésû lineáris kimenet-visszacsatolás (azaz $u = -ky$, ahol $|k| < \infty$), amely aszimptotikusan stabilizálja a rendszert, ha $l_1 > 0$ és $l_2 < 0$? Miért? (3p)

3. Mennyi lesz az az erõsítése decibelben az alábbi átviteli függvénynek konstans bemenet esetén? (2p)

$$H(s) = \frac{s + 1}{s^2 + 10s + 10}$$

4. Minimumfázisú-e a következõ átviteli függvény (Miért)? (2p)

$$H(s) = \frac{(s+1)(s+3)}{s^3 - 3s^2 + 2s + 1}$$

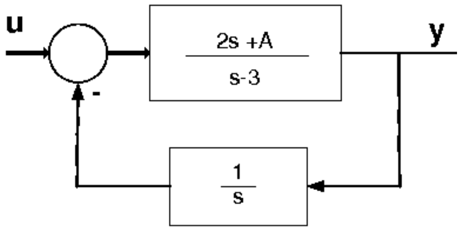
5. Adott a következõ lineáris rendszer:

$$A = \begin{bmatrix} 4 & 3.5 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

   (a) Adja meg a rendszer $H(s)$ átviteli függvényét! (3 pont)

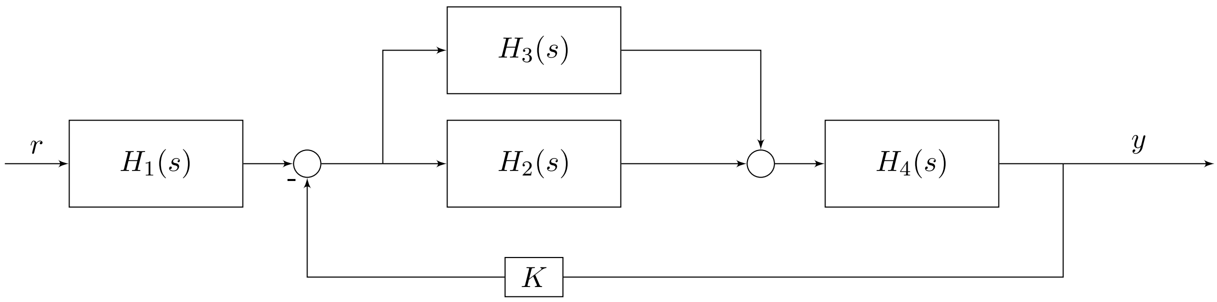
   (b) Adja meg a rendszer pólusait! (1 pont)

   (c) Stabil-e a rendszer? Pontos indoklás! (1 pont)

   (d) 

   Stabil lesz-e a visszacsatolt rendszer $A = 0$ illetve $A = 0.25$ értékek esetén (3p)?

6. Adott a következõ hatásvázlat:



Adja meg a rendszer eredõ átviteli függvényét $G(s)$-t, ha $H_1(s) = \frac{s+2}{s^2-7s+11}$, $H_2(s) = \frac{1}{s}$, $H_3(s) = \frac{s-3}{s+7}$, $H_4(s) = \frac{s+7}{s+1}$ (6 pont)
(Segítség: a gyöktényezõs alak megtartása elõnyös a számolás során, illetve az egyes részrendszerek kiszámítása megkönnyíti a számolást.)

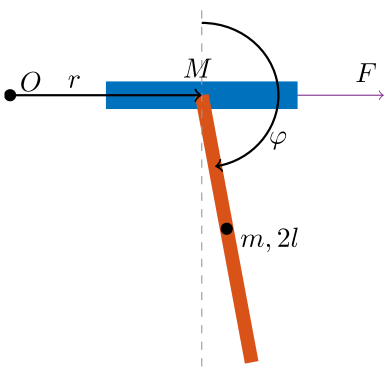
Computer controlled systems

Lecture 7, March 31, 2017

version: 2018.11.27. – 12:39:05

Exercises

We consider a simple pendulum mounted an a cart that can move horizontally:



M	is the mass of the cart
m	is the mass of the pendulum
$2l$	is the length of the pendulum
l	is the distance of the pivot point from the pendulum's center of mass
F	is an external force (input) acting on the cart
b	is the damping factor
r	is the (horizontal) position of the cart
$\dot{r} = v$	is the (horizontal) velocity of the cart
φ	is the angle of the cart (clockwise direction)
$\dot{\varphi} = \omega$	is the angular velocity of the cart (clockwise direction)
$\varphi = 0$	**unstable equilibrium point**: if the pendulum's center of mass is exactly **above** its pivot point (is vertical and pointing towards the sky)
$\varphi = \pi$	**stable equilibrium point**: if the pendulum's center of mass is exactly **below** its pivot point

This system has a nonlinear equation, which can be linearized in a certain operating point[1] (see Appendix). The state vector of the system is the following: $x = \begin{pmatrix} r & v & \varphi & \omega \end{pmatrix}^T$, furthermore, the external force F constitutes the input of the system (u). The nonlinear model of the system is: $\dot{x} = f(x) + g(x)u$, where

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\varphi)\omega^2 - 1.5mg\sin(2\varphi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\varphi)\omega^2 + (M+m)g\sin(\varphi) + b\cos(\varphi)v\Big) \end{pmatrix}, \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\varphi) \end{pmatrix} \tag{1}$$

where $q = 4(M+m) - 3m\cos(\varphi)^2$. For the full derivation see Appendix. For each exercise, you can use your own parameter configuration. Some examples are listed below.

(A) no friction	**(B)** with friction	**(C)** with friction + heavy rod
$M =$ 0.5 [kg]	$M =$ 0.5 [kg]	$M =$ 0.5 [kg]
$m =$ 0.2 [kg]	$m =$ 0.2 [kg]	$m =$ 10 [kg]
$l =$ 1 [m]	$l =$ 1 [m]	$l =$ 1 [m]
$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]
$b =$ 0 [kg/s]	$b =$ 10 [kg/s]	$b =$ 10 [kg/s]

[1] munkapont

1. Linearized model around the *stable* equilibrium point ($\varphi = \pi$)

Linearized model around the operating point $x^* = \begin{pmatrix} 0 & 0 & \pi & 0 \end{pmatrix}^T$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3b}{l(4M+m)} & -\frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)} \begin{pmatrix} 0 \\ 4l \\ 0 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

(`ss,tf`) 1. Determine the system's transfer function:

$$H(s) = \begin{pmatrix} H_{u \to r}(s) \\ H_{u \to \varphi}(s) \end{pmatrix} \tag{3}$$

(`impulse`) 2. Determine the impulse response of the system

(`step`) 3. Determine the step response of the system for both $H_{u\to r}(s)$ and $H_{u\to\varphi}(s)$. Determine the DC gain of the system.

(`eig`) 4. Determine the poles of the system. Is the linearized model locally/globally/asymptotically stable? What can we say about the original nonlinear system's stability? How does the stability properties change if we assume friction?

(`bodeplot`) 5. Determine the Bode plot of the transfer function $H_{u\to\varphi}(s)$. Set the frequency unit to be in Hz. Determine the own (or resonance) frequency (f_r) of the system.

(`nyquist`) 6. Plot the Nyquist diagram of $H_{u\to\varphi}(s)$.

(`lsim`) 7. Plot the output of the system if the input is $u_i(t) = A_i \sin(2\pi f_i t)$, where

(a) $f_2 = f_r\,[Hz],\ A_2 = 1\,[N]$ (b) $f_3 = 4\,[Hz],\ A_3 = 20\,[N]$ (c) $f_1 = 0.1\,[Hz],\ A_1 = 1\,[N]$

Considering the Bode diagram, what is expected to happen in each cases? In certain cases, we shall notice that the system's motion is quite unusual, why?

(`ode45`) 8. Solve the linearized differential equation $\dot{x} = Ax + Bu$ with different initial conditions. The input may be zero first, than you can use the values from the previous example.

(`ctrb`) 9. Is the linearized model controllable?

(`obsv`) 10. Is the linearized model observable? How does this change if we measure only the angle of the rod φ.

(`null`) (a) Compute the kernel (null space) of $\mathcal{O}_4$.

(`orth`) (b) Give the bases of the image space of $\mathcal{O}_4$.

(c) Give the matrix T of the linear state transformation, which produces the observability staircase representation:

$$\begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{pmatrix} B_o \\ B_{\bar{o}} \end{pmatrix} u$$
$$y = \begin{pmatrix} C_o & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

2. Nonlinear system simulation

11. Solve the nonlinear ODE (1) numerically, use the `ode45` solver:

(a) $x_0 = \begin{pmatrix} 0 & 0 & \frac{5\pi}{6} & 0 \end{pmatrix}^T$, $u(t) = 0$

(b) $x_0 = \begin{pmatrix} 0 & 0 & \frac{\pi}{6} & 0 \end{pmatrix}^T$, $u(t) = 0$

(c) $x_0 = 0$, $u(t) = \sin(2\pi f_r t)$

(d) You can play with x_0 and $u(t)$ as you want

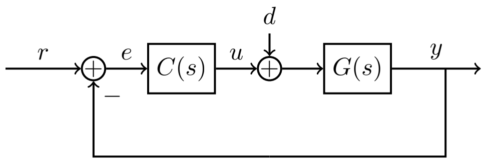
3. PID controller design

12. Consider the following SISO model given by the transfer function:

$$G(s) = \frac{s^2 + 3s + 2}{s^3 + 2s^2 - 6s + 8} \tag{4}$$

(`pzmap`) (a) Determine the poles and the zeros of the system. Is the system minimum-phase?

(`pidTuner`) (b) Design a PID controller $C(s)$ which provides stability and reference tracking.



$$C(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s} \tag{5}$$

Appendix

I. Linearize a nonlinear model around an equilibrium point

We have a nonlinear system in the following form:

$$\dot{x} = F(x, u) = f(x) + g(x)u \tag{6}$$

Let $x^* \in \mathbb{R}^n$ be an equilibrium point of the nonlinear system, which means that $F(x^*, 0) = f(x^*) = 0$. We assume that the system operates around this equilibrium point, and by default there is no input given to the system. Therefore, we say that the system's operating point[2] is $(x^*, u^* = 0)$.

The Jacobian matrix of $F(x, u)$ is

$$\mathrm{D}[F(x,u)] = \left(\frac{\partial F(x,u)}{\partial x} \;\middle|\; \frac{\partial F(x,u)}{\partial u} \right) = \left(\frac{\partial f(x)}{\partial x} + \frac{\partial g(x)}{\partial x} u \;\middle|\; g(x) \right) \tag{7}$$

The value of the Jacobian matrix in this operating point is

$$\mathrm{D}[F(x^*,0)] = \left(\frac{\partial f(x^*)}{\partial x} \;\middle|\; g(x^*) \right) \tag{8}$$

Now we estimate $F(x, u)$ by its first order Taylor polynomial around the operating point:

$$\begin{aligned} F(x,u) &\simeq \underbrace{F(x^*,0)}_{0} + \mathrm{D}[F(x^*,0)] \begin{pmatrix} x - x^* \\ u - 0 \end{pmatrix} \\ F(x,u) &\simeq \frac{\partial f(x^*)}{\partial x}(x - x^*) + g(x^*)u \end{aligned} \tag{9}$$

Hence, the linear model is

$$\dot{x} = A(x - x^*) + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{10}$$

There's only one more thing left, we need to center the system. We introduce the centered state vector $\bar{x} := x - x^*$. Therefore, the time derivative of the transformed state vector will be:

$$\dot{\bar{x}} = \dot{x} = A(x - x^*) + Bu = A\bar{x} + Bu \tag{11}$$

Finally, we obtained the centered linearized model:

$$\dot{\bar{x}} = A\bar{x} + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{12}$$

[2] munkapont

II. Derivation of the inverted pendulum's equation

The equation of the inverted pendulum is the following:

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\varphi}\cos(\varphi) - ml\dot{\varphi}^2\sin(\varphi) &= F \\ ml\ddot{x}\cos(\varphi) + \frac{4}{3}ml^2\ddot{\varphi} - mgl\sin(\varphi) &= 0 \end{aligned} \tag{13}$$

The nonlinear state space equation of the inverted pendulum:

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{q}\big(4ml\sin(\varphi)\omega^2 - 1.5mg\sin(2\varphi) - 4bv\big) + \frac{4}{q}F \\ \dot{\varphi} = \omega \\ \dot{\omega} = \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\varphi)\omega^2 + (M+m)g\sin(\varphi) + b\cos(\varphi)v\Big) - \frac{3\cos(\varphi)}{lq}F \end{cases} \tag{14}$$

where $q = 4(M+m) - 3m\cos(\varphi)^2$. Let the state vector be $x = \begin{pmatrix} x & v & \varphi & \omega \end{pmatrix}^T$.

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\varphi)\omega^2 - 1.5mg\sin(2\varphi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\varphi)\omega^2 + (M+m)g\sin(\varphi) + b\cos(\varphi)v\Big) \end{pmatrix}, \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\varphi) \end{pmatrix} \tag{15}$$

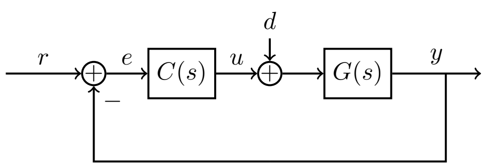
Linearized model around the stable operating point $x^* = \begin{pmatrix} 0 & 0 & \pi & 0 \end{pmatrix}^T$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3b}{l(4M+m)} & -\frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{16}$$

Linearized state space model around the unstable operating point $x^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3b}{l(4M+m)} & \frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$

III. A simple control loop (SISO)



r	reference input
d	input disturbance (eg. wind, noise, fault of the actuator, etc.)
u	control input computed by the controller $C(s)$
y	output of system $G(s)$
e	error: difference between the reference input r and the output y

We derive, how the reference input r and the input disturbance d influence the output of $G(s)$:

$$\begin{aligned} y &= G(s)(u+d) = G(s)(u + C(s)(r-y)) \\ &= G(s)d + G(s)C(s)r - G(s)C(s)y \\ &\boxed{y = \frac{G(s)}{1+G(s)C(s)}d + \frac{G(s)C(s)}{1+G(s)C(s)}r} \end{aligned} \tag{18}$$

In general an actuator[3] has a limited power, and it cannot perform arbitrarily large control input u. Therefore, during the controller design, we need to consider what would be the actual control input (u) determined by the controller $C(s)$. From the closed loop system, we can derive the transfer function

[3] eg. in case of the inverted pendulum the actuator could be the DC motor of cart

describing the influence of r and d on the control input u:

$$\begin{aligned} u &= C(s)(r-y) = C(s)(r - G(s)(d+u)) \\ &= C(s)r - C(s)G(s)d - C(s)G(s)u \\ &\boxed{u = \frac{C(s)}{1+G(s)C(s)}r + \frac{-G(s)C(s)}{1+G(s)C(s)}d} \end{aligned} \tag{19}$$

Computer controlled systems

Lecture 7, March 31, 2017

version: 2018.11.27. – 12:39:16 [β version]

Pole-placement controller

Pole-placement controller based on Bass-Gura formula

$$K = (\underline{\alpha} - \underline{a})T_l^{-1}\mathcal{C}^{-1}$$

where $\underline{\alpha}$ is the expected (prescribed) characteristic polynomial of the closed-loop system, $\underline{a}$ is the characteristic polynomial of the original (uncontrolled) system, $\mathcal{C}$ is the controllability matrix, finally T_l is the following Toeplitz matrix:

$$T_l = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ . & . & . & \cdots & . \end{pmatrix}$$

Example 1. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Solution.

$$\begin{aligned} a(s) &= s^2 - 3s + 2 \\ a_1 &= -3 \\ a_2 &= 2 \end{aligned}$$

The prescribed characteristic polynomial ($\phi_c(s)$):

$$\begin{aligned} \alpha(s) &= s^2 + 3s + 2 \\ \alpha_1 &= 3 \\ \alpha_2 &= 2 \end{aligned}$$

A Toeplitz matrix and the controllability matrix in this case are

$$T_l = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}$$

$$T_l^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix}$$

Than the static state feedback will be the following:

$$K = \begin{pmatrix} 3 - (-3) & 2 - 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 5 \end{pmatrix}$$

Ackermann formula

$$K = [0\,0\,\cdots\,0\,1]\mathcal{C}_n^{-1}\phi_c(A)$$

where $\phi_c(s)$ is the prescribed characteristic polynomial of the closed-loop (controlled) system. In the previous example, it was denoted by $\alpha(s) = \phi_c(s)$.

Example 2. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Solution.

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \to \mathcal{C}_2^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

Legyen $\lambda_1 = -1$ és $\lambda_2 = -2$.

$$\phi_c = (s - \lambda_1)(s - \lambda_2) = s^2 + 3s + 2$$

$$\phi_c(A) = A^2 + 3A + 2I = \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} -4 & 5 \end{pmatrix}$$

Check

$$A - BK = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -4 & 5 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 8 & -9 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Namely, the poles of the obtained closed-loop system are indeed the prescribed values.

Example 3. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Solution.

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \to \mathcal{C}_2^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

Let $\lambda_1 = -1$ and $\lambda_2 = -2$.

$$\phi_c = (s + \lambda_1)(s + \lambda_2) = s^2 + 3s + 2$$

$$\phi_c(A) = A^2 + 3A + 2I = \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \end{pmatrix}$$

Check:

$$A - BK = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Indeed, the poles of the closed loop system are the prescribed values.

Example 4. Given the following CT LTI SISO systems

1. $\begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{cases}$

2. $\begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{cases}$

Design a state feedback controller (if it is possible), that stabilizes the system!

Example 5. Given the following CT LTI SISO system $H(s) = \frac{2s-4}{s^2+s-6}$.

1. Is the system asymptotically stable?
2. If it is possible, design a controller, that shifts the system's poles to $p_1 = -3$ and $p_2 = -5$! Hint: controllability normal form.

Linear state observer design

Goal: computation of the values of the non-measured state variables of the system using the observed output.

The dynamical system

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = F\hat{x} + Ly + Hu$$

is called a full order state observer, if the error dynamics $e = x - \hat{x}$ tends to zero, i.e. $\lim_{t\to\infty} e = 0$

In case of an LTI system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - F\hat{x} - Ly - Hu + Fx - Fx =$$

$$= Ax + Bu - F\hat{x} - LCx - Hu + Fx - Fx =$$

$$= (A - LC - F)x + (B - H)u + F(x - \hat{x}) = (A - LC - F)x + (B - H)u + F(e)$$

Let $F = A - LC$ and $H = B$

Than $\dot{e} = Fe$

We require that the system be asymptotically stable, namely the real part of the roots of the characteristic polynomial $\det(sI - (A - LC))$ be negative.

$$\det(sI - (A - LC)) = \det\big(sI - (A^T - C^T L^T)\big)$$

We can observe that the state observer design can be traced back to a pole placement problem of (A', B'), where $A' = A^T$, $B' = C^T$, and the result (K) of the pole placement should be interpreted as $L = K^T$.

Example 6. Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Solution.
Let the characteristic polynomial of the closed-loop system: $\phi_o(s) = (s+3)(s+3)$
In order to use the Ackermann, formula we should substitute $A' = A^T$ into $\phi_o(s)$:

$$\phi_o(A') = \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix}$$

If $B' = C^T$, the obtained controllability matrix for (A', B') (which is actually the transpose of the observability matrix of (A, C)) is:

$$\mathcal{C}_2' = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

Its inverse will be:

$$(\mathcal{C}_2')^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$$

Finally, we compute the feedback gain K:

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

From this:

$$L = K^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad F = A - LC = \begin{pmatrix} -3 & 0 \\ 2 & -3 \end{pmatrix} \quad H = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 7. Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Example 8. Design a state observer AND a stabilizer state feedback controller for the following CT LTI SISO system.

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Separation principle: the observer gain L and the feedback gain K can be designed separately.

Optimal state feedback controller - LQR controller design

We want to minimize the following functional:

$$J(x,u) = \frac{1}{2}\int_0^T x^T Q x + u^T R u \; dt$$

where Q and R are positive definite symmetric matrices. In case of LTI systems this problem can be traced back to a CARE (continuous-time algebraic Riccati equation):

$$KA + A^T K - KBR^{-1}B^T K + Q = 0$$

The system can be stabilized with the $u = -Gx$ state feedback, where

$$G = R^{-1}B^T K$$

Example 9. Design an optimal LQR controller for the following system: $\dot{x} = 2x + u$, i.e $A = 2, B = 1$.
Solution. We minimize the following functional:

$$J = \frac{1}{2}\int 5x^2 + u^2 dt$$

meaning that in our case $Q = 5$ and $R = 1$. In this case (first order system – only one single state variable) the CARE will have the following form:

$$-K^2 + 4K + 5 = 0$$

The solutions for K are 5 and -1. By definition, we should choose the positive one, otherwise, we obtain a positive feedback.

$$G = 1 \cdot 1 \cdot 5 = 5$$

Finally, the computed state feedback: $u = -5x$.

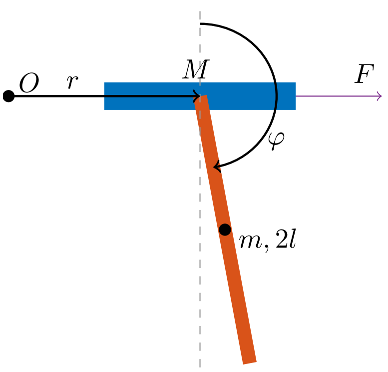
Computer controlled systems

Lecture 7, March 31, 2017

version: 2018.11.27. – 12:39:27

Inverted pendulum model

We consider a simple pendulum mounted an a chart that can move horizontally:



M	is the mass of the chart
m	is the mass of the pendulum
$2l$	is the length of the pendulum
l	is the distance of the pivot point from the pendulum's center of mass
F	is an external force (input) acting on the chart
b	is the damping factor
r	is the (horizontal) position of the chart
$\dot{r} = v$	is the (horizontal) velocity of the chart
ϕ	is the angle of the chart (clockwise direction)
$\dot{\phi} = \omega$	is the angular velocity of the chart (clockwise direction)
$\phi = 0$	**unstable equilibrium point**: if the pendulum's center of mass is exactly **above** its pivot point (is vertical and pointing towards the sky)
$\phi = \pi$	**stable equilibrium point**: if the pendulum's center of mass is exactly **below** its pivot point

This system has a nonlinear equation, which can be linearized in a certain operating point[1] (see Appendix). The state vector of the system is the following: $x = \begin{pmatrix} r & v & \phi & \omega \end{pmatrix}^T$, furthermore, the external force F constitutes the input of the system (u). The nonlinear model of the system is: $\dot{x} = f(x) + g(x)u$, where

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\phi)\omega^2 - 1.5mg\sin(2\phi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\phi)\omega^2 + (M+m)g\sin(\phi) + b\cos(\phi)v\Big) \end{pmatrix}, \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\phi) \end{pmatrix} \tag{1}$$

where $q = 4(M+m) - 3m\cos(\phi)^2$. For the full derivation see Appendix. For each exercise, you can use your own parameter configuration. Some examples are listed below.

(A) no friction	**(B)** with friction	**(C)** with friction + heavy rod
$M =$ 0.5 [kg]	$M =$ 0.5 [kg]	$M =$ 0.5 [kg]
$m =$ 0.2 [kg]	$m =$ 0.2 [kg]	$m =$ 10 [kg]
$l =$ 1 [m]	$l =$ 1 [m]	$l =$ 1 [m]
$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]
$b =$ 0 [kg/s]	$b =$ 10 [kg/s]	$b =$ 10 [kg/s]

[1]munkapont

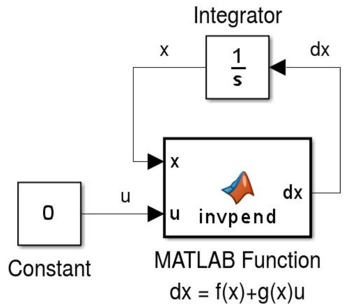
Linearized model around the *unstable* equilibrium point ($\phi = 0$)

Linearized state space model around the unstable operating point $x^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3b}{l(4M+m)} & \frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix} , \quad B = \frac{1}{l(4M+m)} \begin{pmatrix} 0 \\ 4l \\ 0 \\ -3 \end{pmatrix} , \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

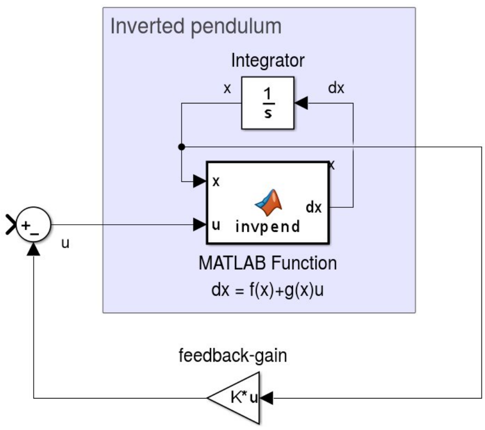
Exercises

1. Simulate the motion of the inverted pendulum in Simulink, use the original **nonlinear** model of the system.
   Instructions.
   - Using the Simulink's "MATLAB function", you can implement the equation $\dot{x} = f(x) + g(x)u$ as a Matlab function $\texttt{dx} = \texttt{invpend}(\texttt{x}, \texttt{u})$ with two input arguments (the state variables $x \in \mathbb{R}^4$ and input $u \in \mathbb{R}$) and a single output argument ($\dot{x} \in \mathbb{R}^4$ the time derivative of x)
   - The time derivative of $\dot{x}$ is fed back through an integrator (see figure below).
   - In order to plot the result, use the "Scope" block diagram.
   - If you want to export the numerical values to the Matlab's global workspace use "To Workspace" block.
   - The initial value of the system can be given as the initial value of the integrator: open the "Block Parameters" dialog of the integrator.

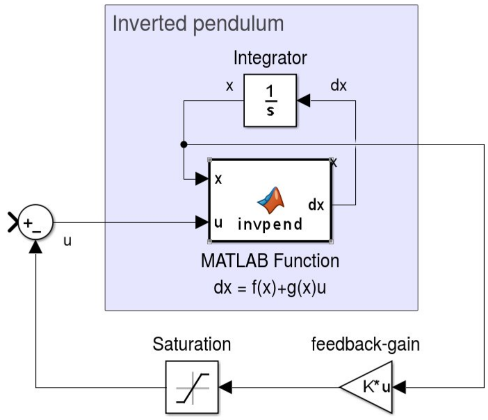


2. Design a state feedback gain in Matlab for the (**linearized**) system, which
   (a) translates the poles into $\{-1, -2, -3, -4\}$ (or into arbitrary stable poles).
   (b) minimizes the functional $J(x, u) = \int_0^\infty x^T Q x + u^T R u \, \mathrm{d}t$, where $Q = I_4$ and $R = 1$ (LQR design).

3. Apply the state feedback gain on the **nonlinear** model, and simulate it in Simulink.
   Instructions.
   - Use the "Gain" block of Simulink, open its "Block Parameters" dialog, and type there the value of the obtained K.
   - Be aware that the multiplication rule is set to be "Matrix(K*u)" (i.e. matrix by matrix multiplication).



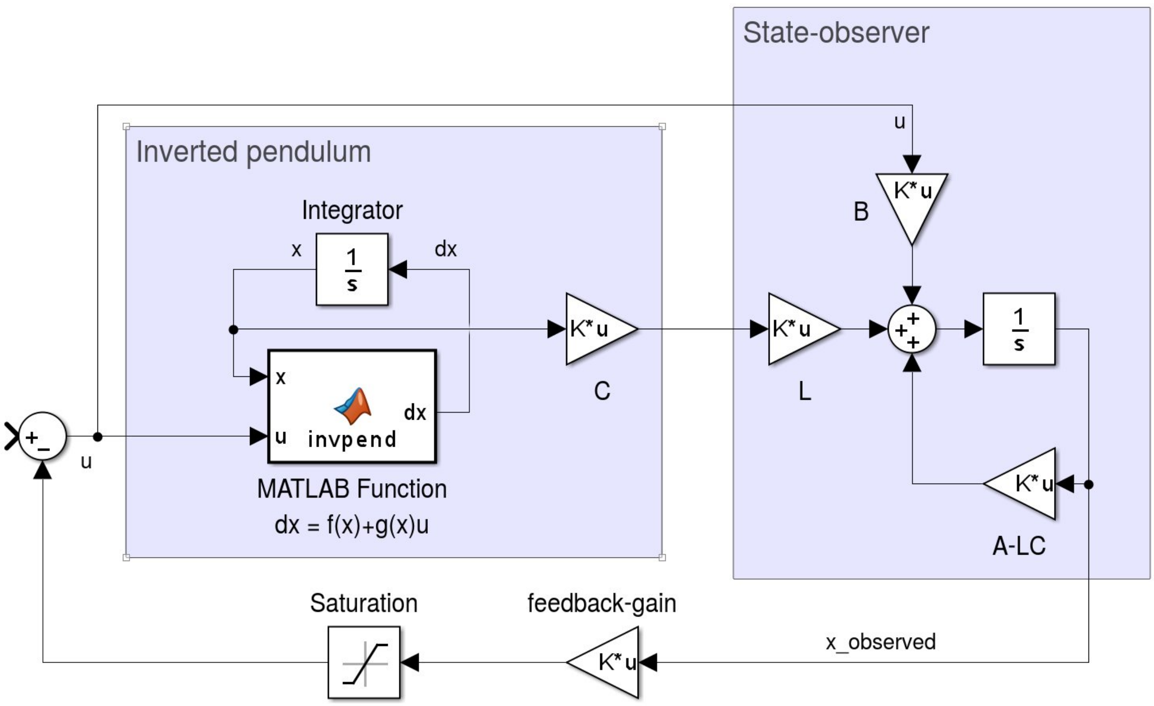
4. In practical applications the actuator has a finite power to act on the system, so it cannot execute arbitrarily large input values. Simulate this saturation effect in Simulink using the "Saturation" block.



5. Design a stable state observer in Matlab for the (**linearized**) system.

6. Simulate the nonlinear system with the existing static feedback of the **observed** state vector $\hat{x}$.

   - Optionally, you can add Gaussian noise to the input (actuator noise) or to the output (sensor noise). Use the "Gaussian Noise Generator" block.



Appendix

I. Linearize a nonlinear model around an equilibrium point

We have a nonlinear system in the following form:

$$\dot{x} = F(x,u) = f(x) + g(x)u \tag{3}$$

Let $x^* \in \mathbb{R}^n$ be an equilibrium point of the nonlinear system, which means that $F(x^*,0) = f(x^*) = 0$. We assume that the system operates around this equilibrium point, and by default there is no input given to the system. Therefore, we say that the system's operating point[2] is $(x^*, u^* = 0)$.

The Jacobian matrix of $F(x,u)$ is

$$\mathrm{D}[F(x,u)] = \left(\tfrac{\partial F(x,u)}{\partial x} \;\middle|\; \tfrac{\partial F(x,u)}{\partial u} \right) = \left(\tfrac{\partial f(x)}{\partial x} + \tfrac{\partial g(x)}{\partial x} u \;\middle|\; g(x) \right) \tag{4}$$

The value of the Jacobian matrix in this operating point is

$$\mathrm{D}[F(x^*,0)] = \left(\tfrac{\partial f(x^*)}{\partial x} \;\middle|\; g(x^*) \right) \tag{5}$$

Now we estimate $F(x,u)$ by its first order Taylor polynomial around the operating point:

$$\begin{aligned} F(x,u) &\simeq \underbrace{F(x^*,0)}_{0} + \mathrm{D}[F(x^*,0)] \begin{pmatrix} x - x^* \\ u - 0 \end{pmatrix} \\ F(x,u) &\simeq \frac{\partial f(x^*)}{\partial x}(x - x^*) + g(x^*)u \end{aligned} \tag{6}$$

Hence, the linear model is

$$\dot{x} = A(x - x^*) + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{7}$$

There's only one more thing left, we need to center the system. We introduce the centered state vector $\bar{x} := x - x^*$. Therefore, the time derivative of the transformed state vector will be:

$$\dot{\bar{x}} = \dot{x} = A(x - x^*) + Bu = A\bar{x} + Bu \tag{8}$$

Finally, we obtained the centered linearized model:

$$\dot{\bar{x}} = A\bar{x} + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{9}$$

II. Derivation of the inverted pendulum's equation

The equation of the inverted pendulum is the following:

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\phi}\cos(\phi) - ml\dot{\phi}^2\sin(\phi) &= F \\ ml\ddot{x}\cos(\phi) + \frac{4}{3}ml^2\ddot{\phi} - mgl\sin(\phi) &= 0 \end{aligned} \tag{10}$$

The nonlinear state space equation of the inverted pendulum:

$$\begin{cases} \dot{x} = v \\ \dot{v} = \dfrac{1}{q}\big(4ml\sin(\phi)\omega^2 - 1.5mg\sin(2\phi) - 4bv\big) + \dfrac{4}{q}F \\ \dot{\phi} = \omega \\ \dot{\omega} = \dfrac{3}{lq}\left(-\dfrac{ml}{2}\sin(2\phi)\omega^2 + (M+m)g\sin(\phi) + b\cos(\phi)v\right) - \dfrac{3\cos(\phi)}{lq}F \end{cases} \tag{11}$$

[2] munkapont

where $q = 4(M+m) - 3m\cos(\phi)^2$. Let the state vector be $x = \begin{pmatrix} x & v & \phi & \omega \end{pmatrix}^T$.

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\phi)\omega^2 - 1.5mg\sin(2\phi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\phi)\omega^2 + (M+m)g\sin(\phi) + b\cos(\phi)v\Big) \end{pmatrix} \quad , \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\phi) \end{pmatrix} \tag{12}$$

Linearized model around the stable operating point $x^* = \begin{pmatrix} 0 & 0 & \pi & 0 \end{pmatrix}^T$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3b}{l(4M+m)} & -\frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix} \quad , \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ 3 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{13}$$

Linearized state space model around the unstable operating point $x^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3b}{l(4M+m)} & \frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix} \quad , \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{14}$$