Computer Controlled Systems (Introduction to systems and control theory) Lecture 1

Szederkényi Gábor

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

e-mail: szederkenyi@itk.ppke.hu

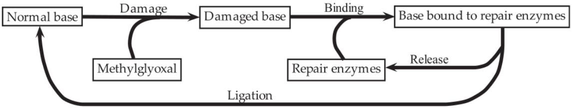
PPKE-ITK, 13 September, 2018

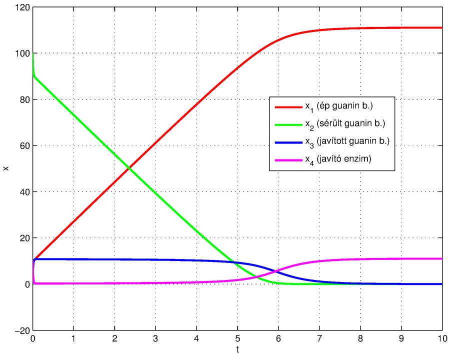
Outline

1. Introduction
2. Brief history
3. Controlled systems in our everyday life and in nature
4. Further examples
5. Basics of signals and systems

Quantitative model of a simple DNA-repair mechanism

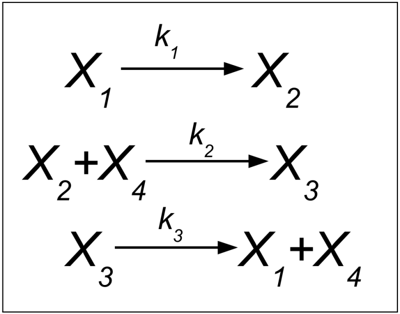
(Karschau et al., Biophysical Journal, 2011)





Introductory example – 2.

Reaction graph:



Kinetic equations:

$$
\begin{aligned}
\dot{x}_1(t) &= k_3 x_3(t) - k_1 x_1(t) \\
\dot{x}_2(t) &= k_1 x_1(t) - k_2 x_2 x_4(t) \\
\dot{x}_3(t) &= k_2 x_2(t) x_4(t) - k_3 x_3(t) \\
\dot{x}_4(t) &= k_3 x_3(t) - k_2 x_2(t) x_4(t),
\end{aligned}
$$

variables:

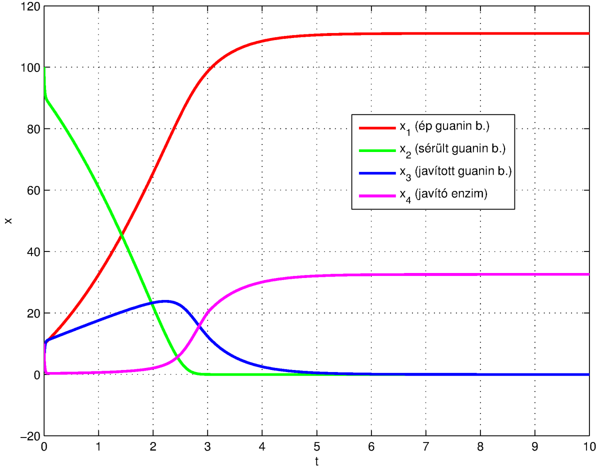
x_1 - no. of undamaged guanine bases

x_2 - no. of damaged guanine bases

x_3 - no. of guanine bases being repaired

x_4 - no. of free repair enzyme molecules

Intervention (to change the operation of the system):
adding more repair enzymes

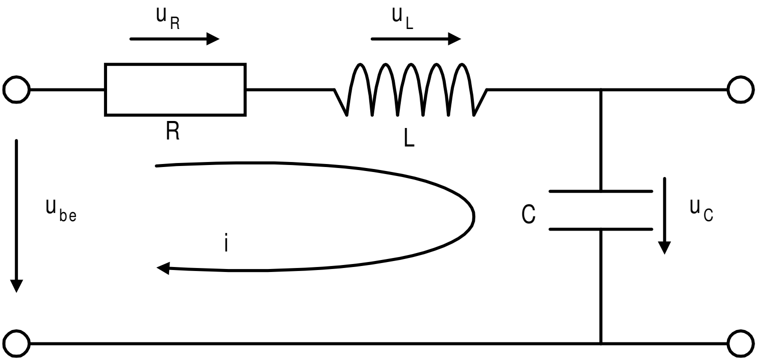


Notion of dynamical models/systems and their application

Dynamical models:

- they are applied to describe [physical] quantities varying in space and/or in time
- they describe the operation of natural or technological processes
- they can be useful to simulate or predict the behaviour of a process
- most often, mathematical models are used to describe dynamics (e.g. ordinary/partial differential equations)
- they can efficiently be solved by computers using various numerical methods
- they are useful to analyse the effect of a given (control) input

Simple RLC circuit



Simple RLC circuit

Kirchhoff's voltage law: $-u_{be} + u_R + u_L + u_C = 0$
Ohm's law: $U_R = R \cdot i$
Operation of the linear capacitor and inductor:

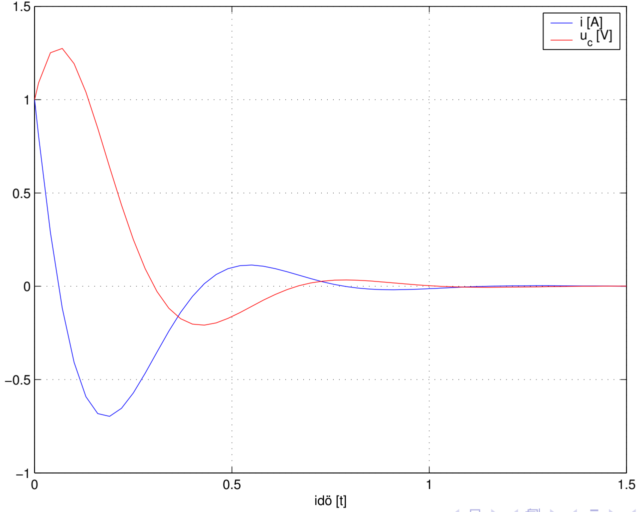
$$u_L = L \cdot \frac{di}{dt}, \quad i = C \cdot \frac{dU_C}{dt}$$

the so-called state equation :

$$\begin{aligned} \frac{di}{dt} &= -\frac{R}{L} \cdot i - \frac{1}{L} u_C + \frac{1}{L} u_{be} \\ \frac{du_C}{dt} &= \frac{1}{C} \cdot i \end{aligned}$$

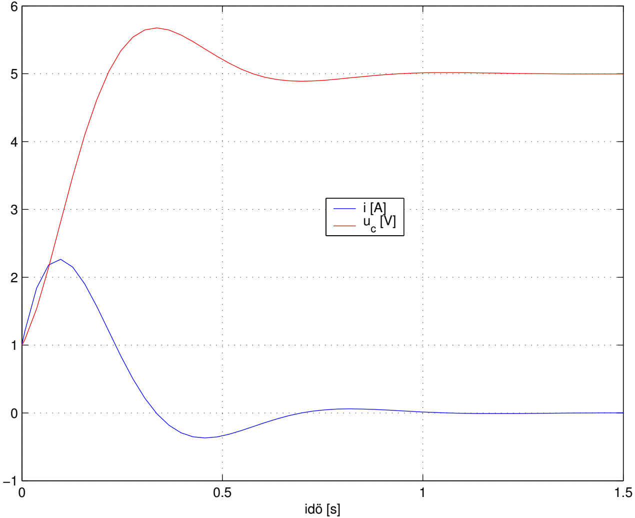
Simple RLC circuit

Parameters: $R = 1\ \Omega,\ L = 10^{-1}H,\ C = 10^{-1}F.$
$u_C(0) = 1\ \text{V},\ i(0) = 1\ \text{A},\ u_{be}(t) = 0\ \text{V}$



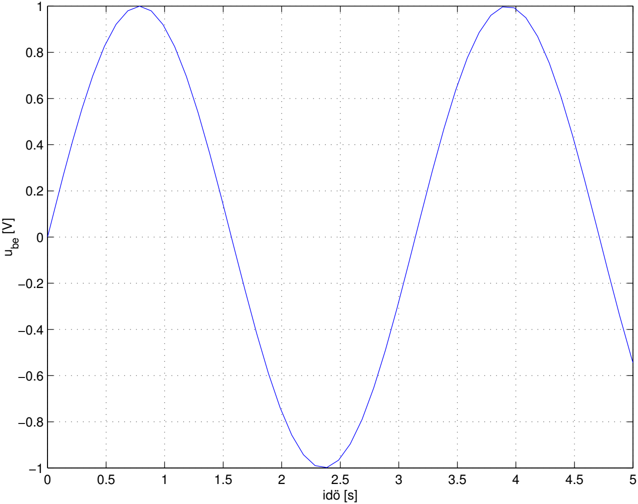
Simple RLC circuit

$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 5$ V



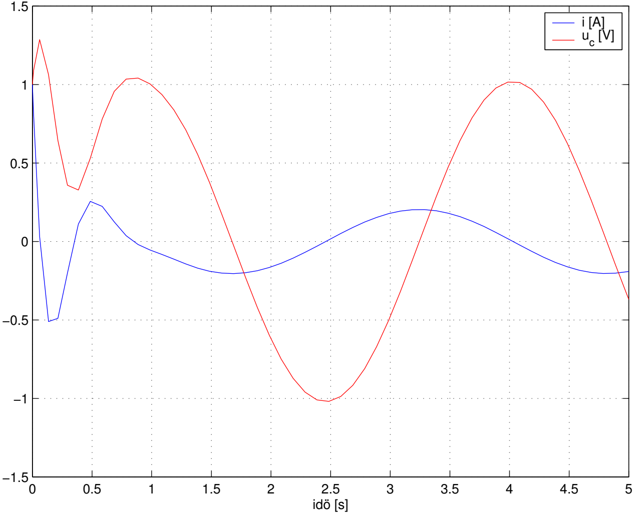
Simple RLC circuit

Periodic input:



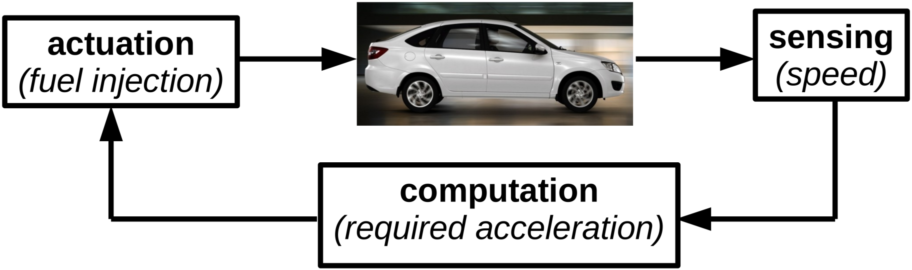
Simple RLC circuit

$u_C(0) = 1$ V, $i(0) = 1$ A



What does control mean? - Example

Control or stabilize the velocity of vehicles (e.g. tempomat)



What does control mean?

To **control** an *object*:

- to **manipulate**
- its **behaviour**
- in order **to reach a goal**.

Manipulation can happen

- through *observing the behaviour* (modeling), then choosing an appropriate control input considering the *desired behaviour*
- through the feedback of the *observed quantities* (measurements) to the input of the system (this can also be model-based)

What does control mean? - Notions

- **System**: What do we want to operate (what are the limits, what are the inputs/outputs)?
- **Control goal**: What kind of behaviour do we want to achieve?
- **System analysis**: Does the problem seem soluble? What can we expect?
- **Sensors**: Detection and monitoring of the the system's behaviour
- **Actuators**: Actual physical intervention (execution)
- **Models**: Mathematical description of the system's operation (over time/space)
- **Control system**: Approach to solve the problem (there can be many solutions based on various principles)
- **Hardware/software**: Controller design and execution of control algorithms

The significance of systems and control theory

- Dynamics : Description of varying quantities in space/time
- Dynamical systems and control systems are present everywhere in our lives: household appliances, vehicles, industrial equipment, communications systems, natural systems (physical, chemical, biological)
- Control becomes mission-critical: if it fails, the whole system may become unusable
- The elements of system theory are (increasingly) utilized by classical sciences
- The principles of control theory has been applied to seemingly distant areas, like economics , biology , drug discovery , etc.

The significance of systems and control theory

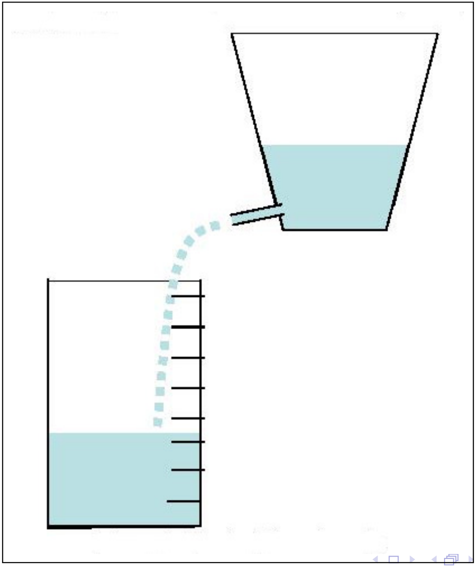
- Systems and control theory is inherently interdisciplinary (construction of mathematical models and analysis; physical components: controlled system, sensors, actuators, communication channels, computers, software)
- Systems theory provides a good environment for the transfer of technology : in general, procedures developed in one area can be useful in other areas, too
- Knowledge and skills obtained in control theory provide a good background for designing and testing complex (technological) systems

Dynamical models (systems) and biology

- dynamics may be essential to understand the operation of important biochemical/biological processes (causes, effects, cross-reactions)
- biology is increasingly available to the traditional engineering approaches (on molecular, cellular and organic levels, too): quantitative modeling, systems theory, computational methods, abstract synthesis methods
- conversely, biological discoveries might serve as a basis for new design methodologies
- a few areas where the dynamics and control have an important role: gene regulation; signal transmission; hormonal, immune and cardiovascular feedbacks; muscle and movement control; active sensing; visual functions; attention; population and disease dynamics

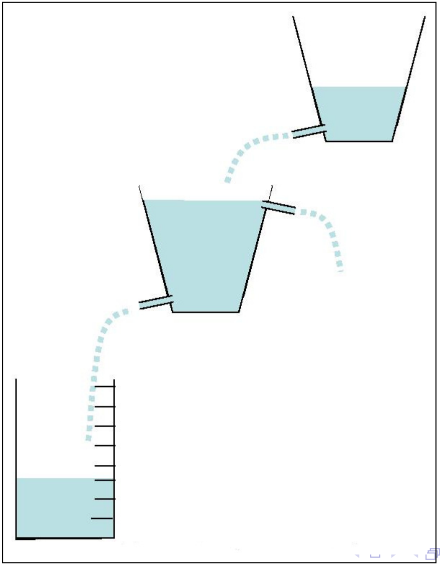
Simple water clock

Before 1000 BC



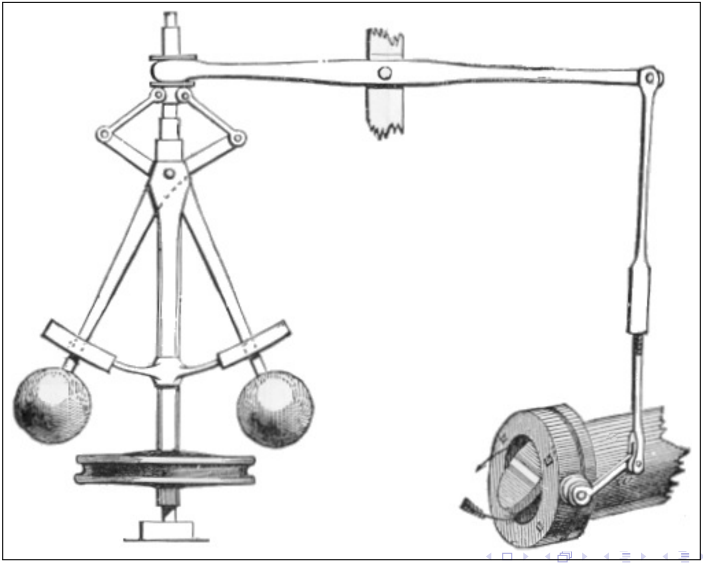
Water clock with water flow rate control

3rd century B.C.



Flyball governor

James Watt, 1788



Birth of systems and control theory as a distinct discipline (approx. 1940-1957)

- 1940-45: Intensive military research (unfortunately); recognizing common principles and representations (radar systems, optimal shooting tables, air defense artillery positioning, autopilot systems, electronic amplifiers, industrial production of uranium etc.).
- Representation of system components using block diagrams
- Analysis and solution of linear differential equations using Laplace transformation, theory of complex functions and frequency domain analysis
- The results of the research in the military were quickly used in other industries as well
- Independent research and teaching of control theory began
- 1957: The International Federation of Automatic Control (IFAC) was founded

The next stage of development (about 1957-1980)

- Motivation: military and industrial application requirements, development of mathematics and computer sciences
- Space Race – space research competition (spacecraft Sputnik, 1957)
- The first computer-controlled oil refinery in 1959
- The use of digital computers for simulation and control systems implementation
- Mathematical precision becomes more important
- The appearance of state-space model based methods

Modern and postmodern control theory (about 1980-)

- Birth of nonlinear systems and control theory based on differential algebra
- The explosive development of numerical optimization methods + computing capacity becomes cheaper
- Handling model uncertainties (robust control)
- Model predictive control (MPC)
- "Soft computing" techniques: fuzzy logics, neural networks etc.
- Energy-based linear and nonlinear control (electrical, mechanical, thermodynamical foundations)
- Control of hybrid systems
- Theory of positive systems
- Control theory and its application to networked systems ("cyber-physical" system)

Controlled technological systems

- thermostat + heating: temperature
- dynamic speed limits on highways: the number of cars passing through during a time unit, exhaust emissions
- power plants' (thermal) power: required electric power
- movement of robotic arms and mobile robots: follow prescribed tracks (guidance)
- aircraft landing/take off: height, speed
- air traffic control: time of landings/take-offs and their order
- re-scheduling of timetables: to minimize all delays
- oxygenation of wastewater treatment plants: speed of bioreactions
- washing machine: weight control, water amount control
- ABS, ESP systems in vehicles: torque, braking force
- CPU clock speed, fan speed: temperature

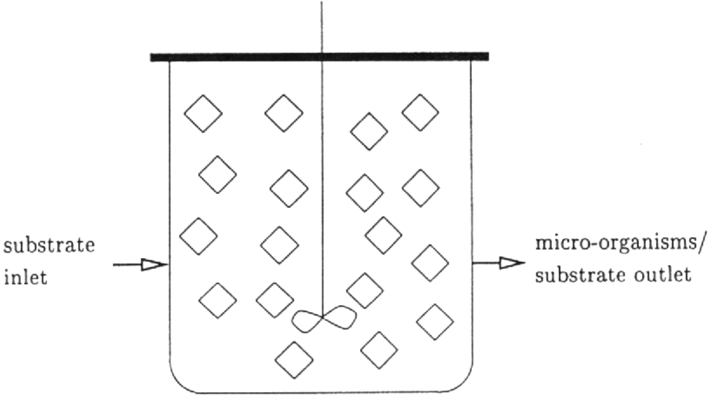
Management of society and economics

- *laws* (including their execution): social life
- *banking systems*: quantity of money in circulation
- *media*: reviews, public taste, agreed standards, overemphasized and concealed informations
- *advertisement*: consumer habits

Control in nature

- control of gene expression (transcription, translation)
- body temperature regulation of warm-blooded animals
- blood glucose control
- hormonal and neural control in organisms/living entities
- swarm of moving animals (birds, insects, fish): speed
- synchronized flashing of light emitting insects
- movement, human walking

Bioreactor–model



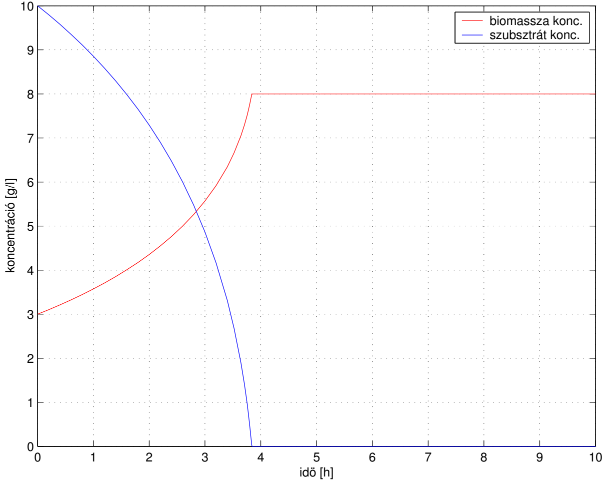
Bioreactor–model

$$\frac{dX}{dt} = \mu(S)X - \frac{XF}{V}$$
$$\frac{dS}{dt} = -\frac{\mu(S)X}{Y} + \frac{(S_F - S)F}{V}$$
$$\text{ahol pl.} \quad \mu(S) = \mu_{max}\frac{S}{K_2S^2 + S + K_1}$$

X	biomass concentration		$[\frac{g}{l}]$	Y	kin.par.	0.5	-
S	substrate concentration		$[\frac{g}{l}]$	μ_{max}	kin.par.	1	$[\frac{1}{h}]$
F	input flow rate		$[\frac{l}{h}]$	K_1	kin.par.	0.03	$[\frac{g}{l}]$
V	volume	4	$[l]$	K_2	kin.par	0.5	$[\frac{l}{g}]$
S_F	substrate feed concentration	10	$[\frac{g}{l}]$				

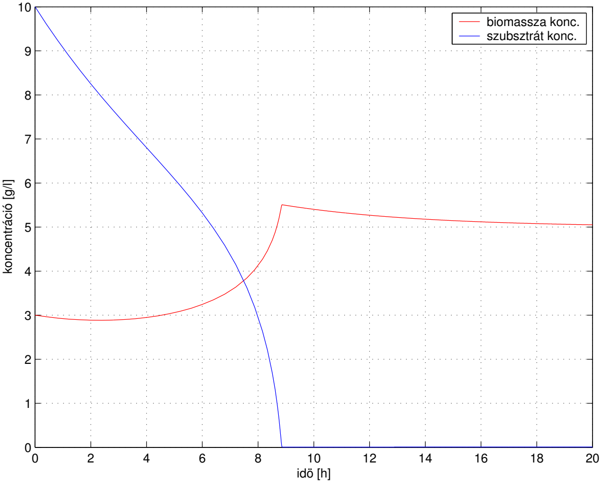
Bioreactor–model

$F = 0 \frac{l}{h}$



Bioreactor–model

$F = 0.8 \frac{l}{h}$



Simple ecological system

$$\frac{dx}{dt} = k \cdot x - a \cdot x \cdot y$$
$$\frac{dy}{dt} = -l \cdot y + b \cdot x \cdot y$$

x – number of preys in a closed area

y – the number of predators in a closed area

k – the natural growth rate of preys in the absence of predators

a – "meeting" rate of predators and preys

l – natural mortality rate of predators in the absence of preys

b – reproduction rate of predators for each consumed pray animal

Parameters:

$k = 2 \; \frac{1}{\text{month}}$

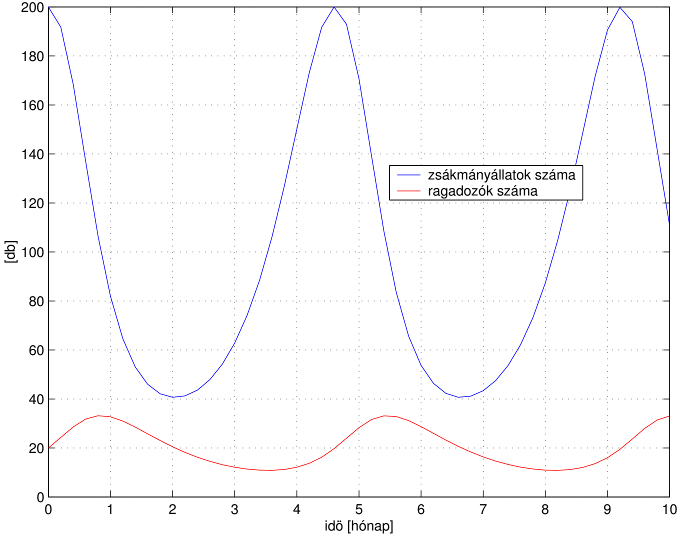
$a = 0.1 \; \frac{1}{\text{pieces} \cdot \text{month}}$

$l = 1 \; \frac{1}{\text{month}}$

$b = 0.01 \; \frac{1}{\text{pieces} \cdot \text{month}}$

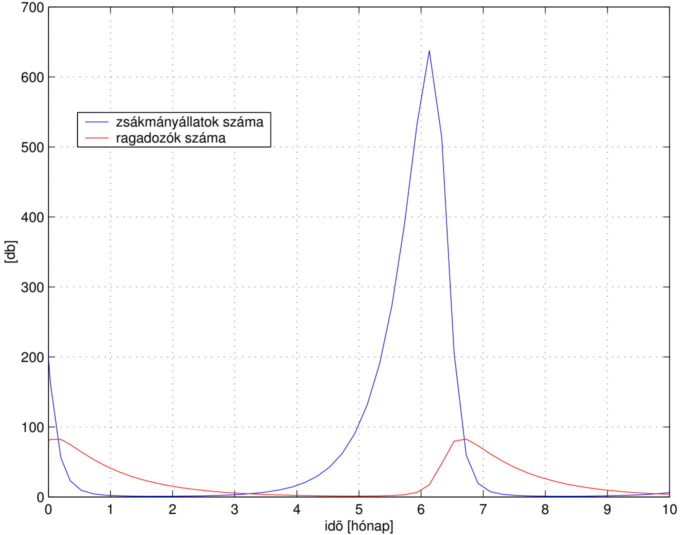
Simple ecological system

$x(0) = 200,\ y(0) = 20$



Simple ecological system

$x(0) = 200, \; y(0) = 80$



SIR disease spreading model

Healing/spreading mechanism:

$$S + I \xrightarrow{b} 2I$$
$$I \xrightarrow{k} R$$

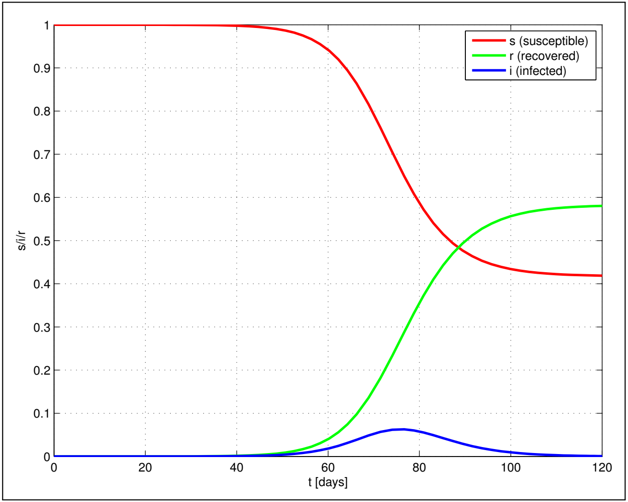
S: susceptible human individuals
I: infected human individuals
R: recovered human individuals
N: number of population
$s = S/N$, $i = I/N$, $r = R/N$
mathematical model:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -b \cdot s(t) \cdot i(t)$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = k \cdot i(t)$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = b \cdot s(t) \cdot i(t) - k \cdot i(t)$$

b, k: constant parameters

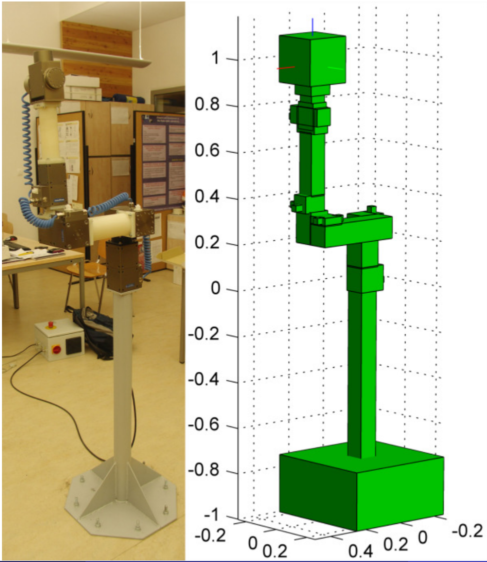
SIR disease spreading model

$N = 10^7,\ S(0) = 9999990,\ I(0) = 10,\ R(0) = 0,\ k = 1/3,\ b = 1/2$



6 degree of freedom robotic arm

(doctoral work of Ferenc Lombai)



6 degree of freedom robotic arm

Planning and execution of a throwing movement

(videos/6dof_dob_1.avi)
(videos/6dof_dob_2.avi)
(videos/6dof_dob_3.avi)

Flexible robotic joint

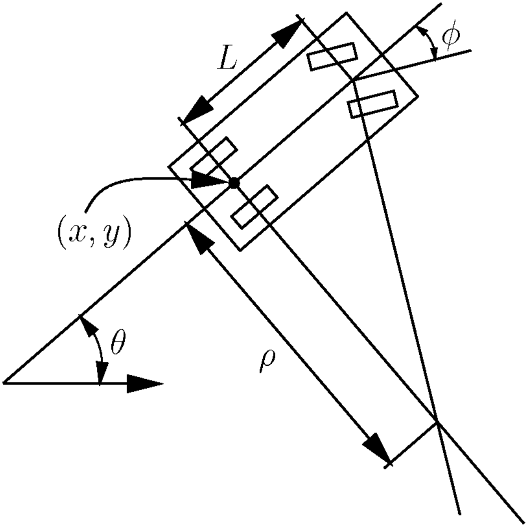
Controlled flexor-extensor mechanism with 2 stepper motor
(doctoral work of József Veres)

http://www.youtube.com/watch?v=qBMs_36gZMg

Task: Active localization of a mobile robot (parallel movement and mapping)
Students' Scientific Conference assignment of
János Rudan and Zoltán Tuza

(videos/SLAM_TDK.mpeg)

Steered car model – 1



Steered car model – 2

Configuration space: $\mathbb{R}^2 \times \mathbb{S}^1$

Configuration: $q = (x, y, \theta)$

Parameters:

S: signed longitudinal direction, speed

ϕ: steering angle

L: distance between front and rear axles

ρ: turning radius for a fixed steering angle ϕ

The dynamical model describes how x, y and θ change in time:

$$\begin{aligned} \dot{x} &= f_1(x, y, \theta, s, \phi) \\ \dot{y} &= f_2(x, y, \theta, s, \phi) \\ \dot{\theta} &= f_3(x, y, \theta, s, \phi) \end{aligned}$$

Steered car model – 3

The most simple control model:

Manipulate input (simplistic assumptions): velocity (u_s), steering angle (u_ϕ), namely $u = (u_s, u_\phi)$

The equations:

$$\begin{aligned}\dot{x} &= u_s \cos\theta \\ \dot{y} &= u_s \sin\theta \\ \dot{\theta} &= \frac{u_s}{L}\tan u_\phi\end{aligned}$$

More accurate (realistic) model using acceleration dynamics:

$$\begin{aligned}\dot{x} &= s \cos\theta \\ \dot{y} &= s \sin\theta \\ \dot{\theta} &= \frac{u_s}{L}\tan u_\phi \\ \dot{s} &= u_t\end{aligned}$$

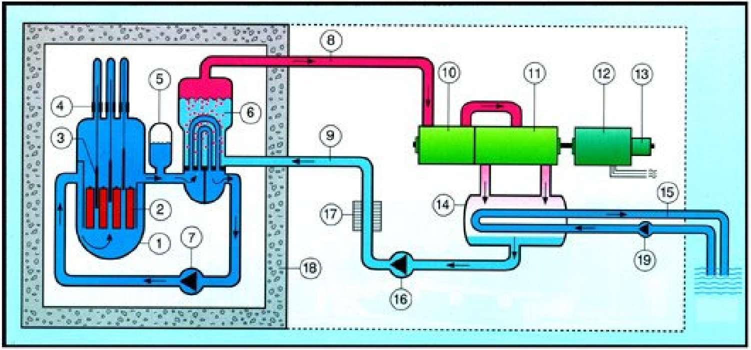
Autonomous and cooperative vehicles

- Following prescribed trajectories (guidance) (videos/car_track.avi)
- Chasing of moving objects, simulations: Gábor Faludi (videos/ref_car.avi)
- (flight) movement in formations (videos/formation.avi)
- Formation change (videos/chg_form.avi)
- Obstacle avoidance (videos/obstacle.avi)

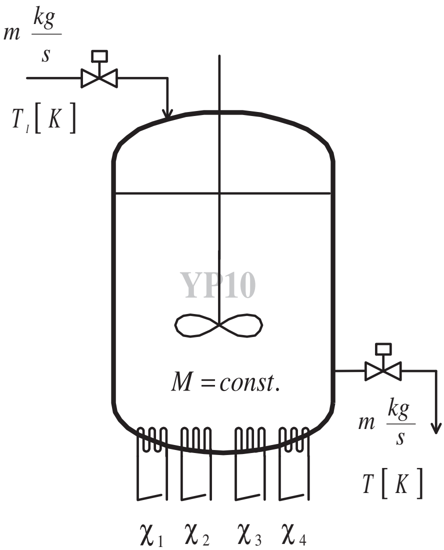
Power system application: primary circuit pressure control

Primary circuit pressure control

Structure of pressurized water reactor unit



Primary circuit pressure control



pressurizer tank

Primary circuit pressure control

Modeling assumptions:

- two perfectly stirred balance volumes: water and the wall of the tank
- constant mass in the two balance volumes
- constant physico-chemical properties
- vapor-liquid equilibrium in the tank

Equations:

water

$$\frac{\mathrm{d}U}{\mathrm{d}t} = c_p m T_I - c_p m T + K_W(T_W - T) + W_{HE} \cdot \chi$$

wall of the tank

$$\frac{\mathrm{d}U_W}{\mathrm{d}t} = K_W(T - T_W) - W_{loss}$$

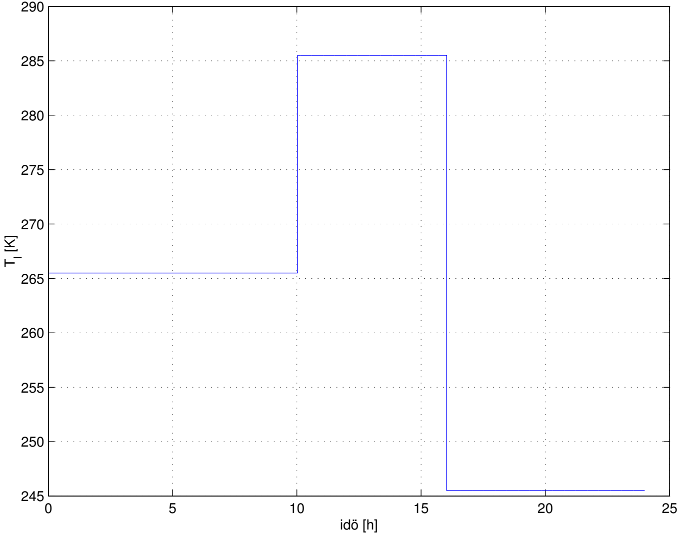
Primary circuit pressure control

Variables and parameters:

T	water temperature	$^\circ C$
T_W	wall temperature	$^\circ C$
c_p	specific heat of water	$\frac{J}{kg^\circ C}$
U	internal energy of water	J
U_W	internal energy of the wall	J
m	water inflow rate	$\frac{kg}{s}$
T_I	temperature of incoming water	$^\circ C$
M	mass of water	kg
C_{pW}	heat capacity of the wall	$\frac{J}{^\circ C}$
W_{HE}	max. power of heaters	W
χ	portion of heaters turned on	-
K_W	heat transfer coefficient of the wall	$\frac{W}{^\circ C}$
W_{loss}	the system's heat loss	W

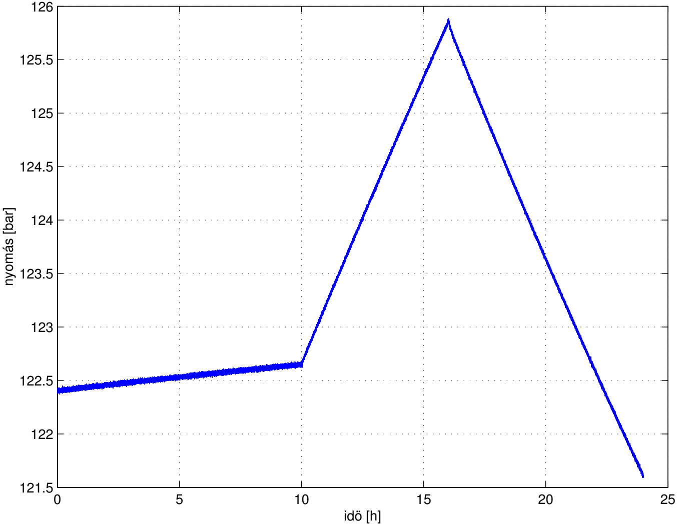
Primary circuit pressure control

Temperature of water inflow



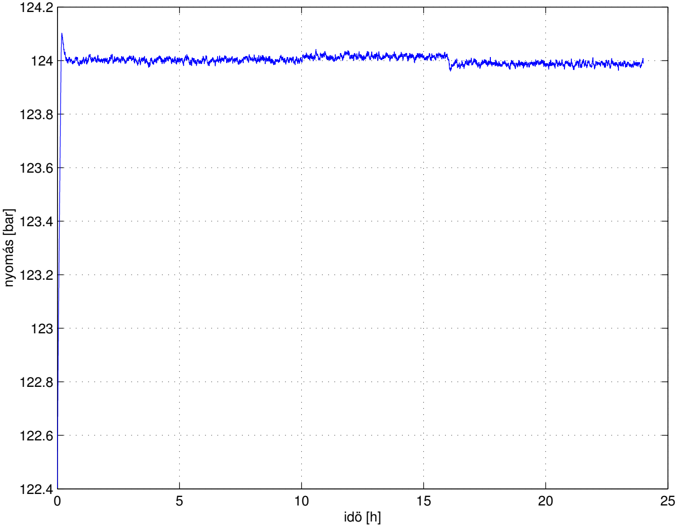
Primary circuit pressure control

Pressure without control

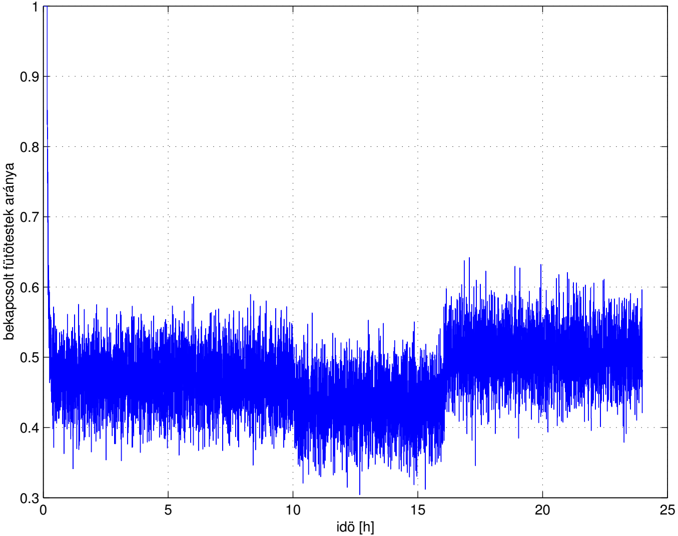


Primary circuit pressure control

Pressure using a control system

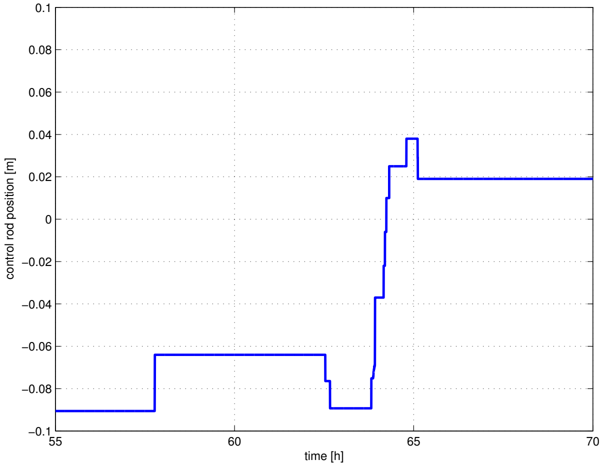


Heating power applied by the the controller



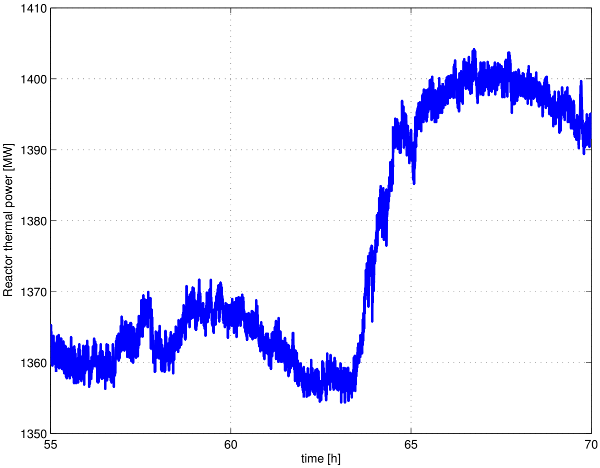
Primary circuit pressure control

Smaller transient: position of control rods



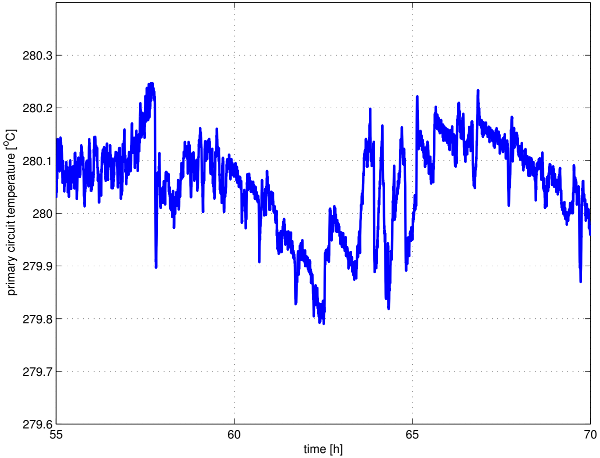
Primary circuit pressure control

Thermal power of the reactor



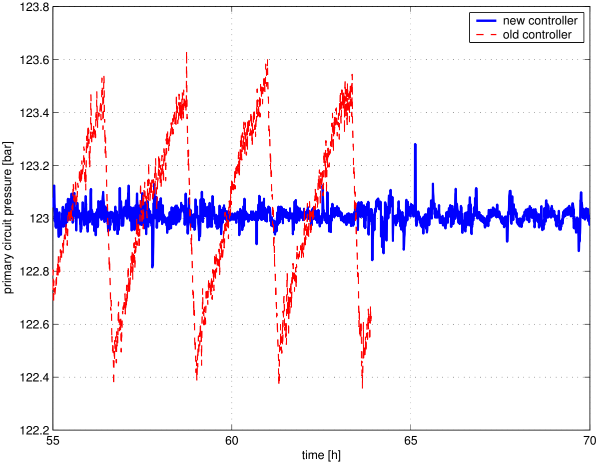
Primary circuit pressure control

Temperature in the primary circuit



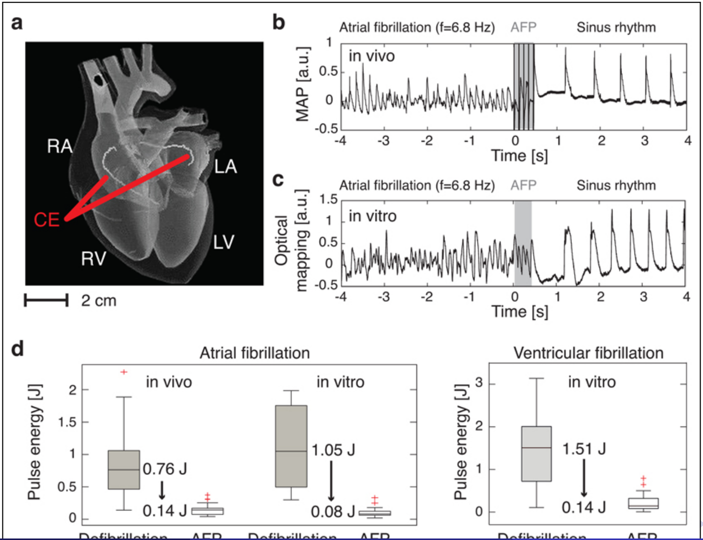
Primary circuit pressure control

primary circuit pressure with the old and the new controller

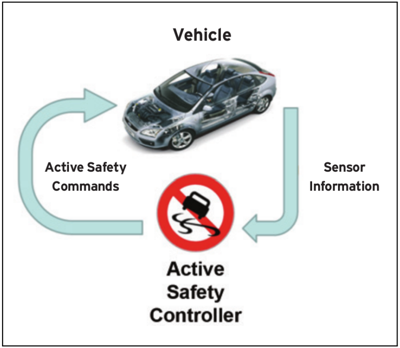


Non-conventional defibrillation

(S. Luther et al. Nature. 2011 Jul 13;475(7355):235-9)
Foundation: a detailed 3D mathematical model of the heart



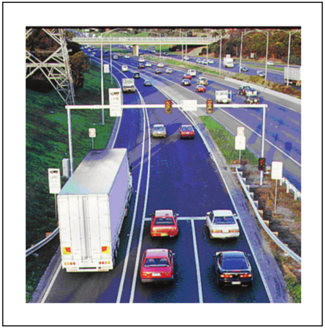
Vehicle safety



- anti blocking system (ABS)
- traction control (TC)
- electronic stability control (ESC)

There is a 4-times payback of the development costs with the avoidance of accidents
Typically, model-based controllers are used

Traffic control on highways



- Australia (Monash Freeway), 2008
- model-based ramp metering control
- problem-free implementation

- traffic jams disappeared
- throughput increased by 4.7 and 8.4% in the morning and afternoon peak period, resp.
- average speed increased by 24.5 and 58.6% in the morning and afternoon, resp.

Why do we study this course?

- primary goal: basic knowledge in **systems theory**
  - ability to observe, analyse and separate systems of the surrounding world
  - ability to determine a system's inputs, outputs, states
- knowledge of basic **system properties** and their analysis (what can we expect?)
- what options of manipulation do we have in order to reach a certain control goal, and how expensive is it (time, energy)?
- establishing an interdisciplinary perspective (electrical, mechanical, chemical, biological, thermodynamic, ecological, economic systems)

Topics

- System classes, basic system properties
- Input/output and state space models of continuous time, linear time invariant (CT-LTI) systems
- BIBO stability and other stability criteria for CT-LTI systems
- Asymptotic stability of CT-LTI systems, Lyapunov's method
- Controllability and observability of CT-LTI systems
- Joint controllability and observability, minimal realization, system decomposition

Topics

- Control design: PI, PID and pole placement controllers
- Optimal (linear quadratic) regulator
- State observer synthesis
- Sampling, discrete time linear time invariant (DT-LTI) models
- Controllability, observability, stability of DT-LTI systems
- Control design for DT-LTI systems

Relations to other subjects

Preliminary studies

- mathematics (linear algebra, calculus, probability theory, stochastic processes)
- physics (determining physical models)
- signal processing (transfer functions, filters, stability)
- electrical networks/circuits theory (linear circuit models)

Further subjects

- robotics (dynamical modeling, regulations and guidance)
- nonlinear dynamical systems (simulation and stability)
- optimization methods, functional analysis (optimal control design, linear system operators)
- computational systems biology (differential equation models, molecular control loops)
- parameter estimation of dynamical systems (construction of dynamical models based on measurements)

Software-tools (possible choices)

- **Commercial**
  - *Matlab/Simulink*: numerical computations, simulations
    http://www.mathworks.com
  - *Mathematica*: symbolic and numerical computations
    http://www.wolfram.com/
  - *Maple*: symbolic, numerical computations, simulations
    http://www.maplesoft.com/
- **Free**
  - *Scilab/Xcos*: numerical computations, simulations
    http://www.scilab.org/
  - *Sage*: symbolic, numerical computations
    http://sagemath.org/

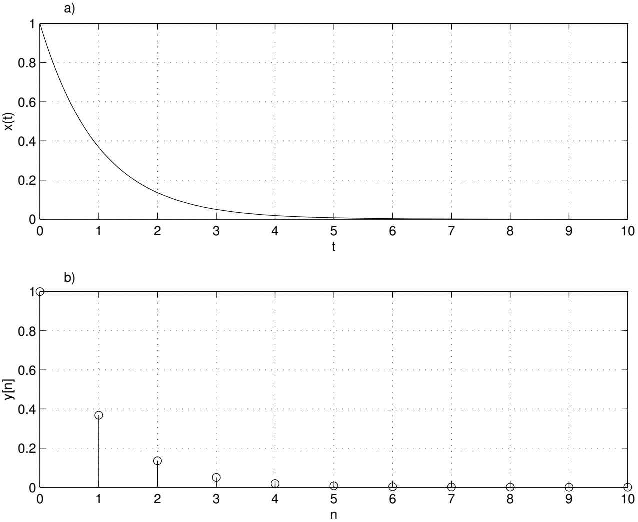
Signals – 1

Signal: A (physical) quantity, which depends on time, space or other independent variables

E.g. (in addition to the introductory examples)

- $x : \mathbb{R}_0^+ \mapsto \mathbb{R}, \quad x(t) = e^{-t}$
- $y : \mathbb{N}_0^+ \mapsto \mathbb{R}, \quad y[n] = e^{-n}$
- $X : \mathbb{C} \mapsto \mathbb{C}, \quad X(s) = \frac{1}{s+1}$

Signals – 2


a)
x(t)
t
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10
b)
y[n]
n
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10

Signals – 3

- room temperature: $T(x, y, z, t)$
  (x, y, z: spatial coordinates, t: time)
- image of a color TV: $I : \mathbb{R}^3 \mapsto \mathbb{R}^3$

$$I(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t), \end{bmatrix}$$

Classification of signals

- dimension of the independent variable
- dimension of the dependent variable (signal)
- real or complex valued
- continuous vs discrete time
- bounded vs not bounded
- periodic vs aperiodic
- even vs odd

Signals with particular significance – 1

Dirac-δ or the *unit impulse* function

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

where $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$ is an arbitrary smooth (infinitely many times continuously differentiable) function.
consequence

$$\int_{-\infty}^{\infty} 1 \cdot \delta(t)dt = 1$$

Signals with particular significance – 1

The physical meaning of the unit impulse:

- current impulse ⇒ charge
- temperature impulse ⇒ energy
- force impulse ⇒ momentum
- pressure impulse ⇒ mass
- density impulse: point mass
- charge impulse: point charge

Heaviside (*unit step*) function

$$\eta(t) = \int_{-\infty}^{t} \delta(\tau) d\tau,$$

in other words:

$$\eta(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

Basic operations on signals – 1

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

- addition:
  $(x + y)(t) = x(t) + y(t), \quad \forall t \in \mathbb{R}_0^+$
- multiplication by a scalar:
  $(\alpha x)(t) = \alpha x(t) \quad \forall t \in \mathbb{R}_0^+, \ \alpha \in \mathbb{R}$
- scalar product:
  $\langle x, y \rangle_\nu (t) = \langle x(t), y(t) \rangle_\nu \quad \forall t \in \mathbb{R}_0^+$

Basic operation on signals – 2

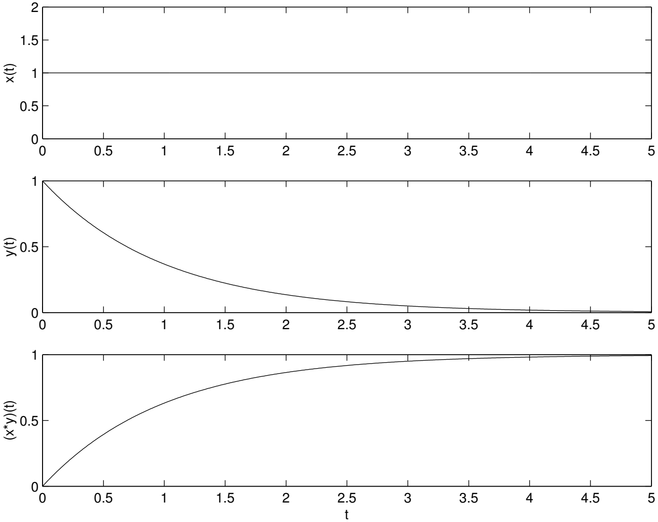
- time shifting:
  $\mathbf{T}_a x(t) = x(t-a) \quad \forall t \in \mathbb{R}_0^+, a \in \mathbb{R}$
- causal time shifting:
  $\mathbf{T}_a^c x(t) = \eta(t-a)x(t-a) \quad \forall t \in \mathbb{R}_0^+, a \in \mathbb{R}$

Convolution – 1

$x, y : \mathbb{R}_0^+ \mapsto \mathbb{R}$

$$(x * y)(t) = \int_0^t x(\tau)y(t-\tau)d\tau, \quad \forall t \geq 0$$

Convolution – 2



Laplace transform

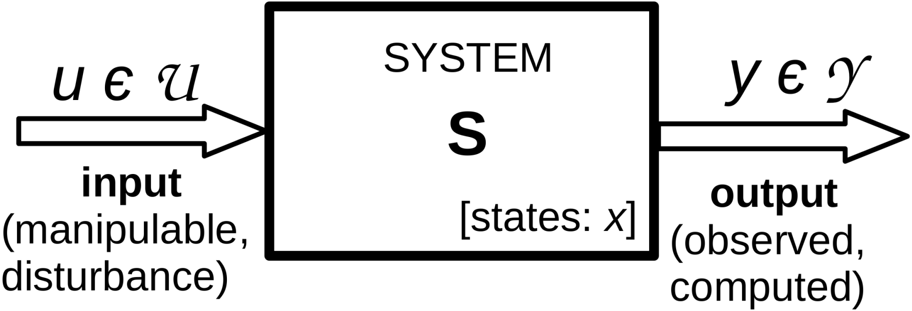
Domain (of interpretation):

$$\Lambda = \{\ f \mid f : \mathbb{R}_0^+ \mapsto \mathbb{C}, f \text{ is integrable on } [0, a]\ \forall a > 0 \text{ and}$$
$$\exists A_f \geq 0, a_f \in \mathbb{R},\ \text{such that } |f(x)| \leq A_f e^{a_f x}\ \forall x \geq 0\ \}$$

Definition:

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt, \quad f \in \Lambda,\ s \in \mathbb{C}$$

System: A physical or logical device that performs operations on signals. (Processes input signals, and generates output signals.)



Summary

- changing (physical) quantities: dynamical models
- mathematical representation: differential equations
- system: operator , input-output mapping
- systems theory is interdisciplinary : describes and treats physical, biological, chemical, technological processes in a common framework
- control is present everywhere and is often mission-critical
- control design and implementation requires knowledge from mathematics, physics, hardware, software and computer science
- control principles can be found in purely natural systems as well
- why to study: to be able to describe, understand and influence (control) dynamical processes

Computer Controlled Systems
(Introduction to systems and control theory)
Lecture 2

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, 27 September, 2018

Contents

1. **Systems**

2. Basic system properties

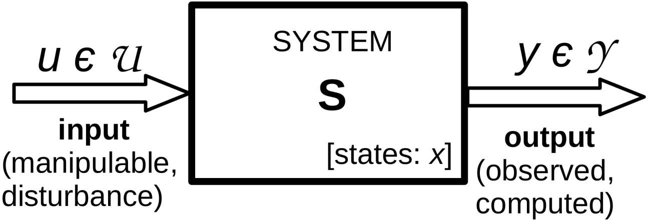
3. Mathematical models of CT-LTI systems
   - Input output models
   - State space systems

Systems

System (**S**): performs operations on signals (abstract operator)

$$y = \mathbf{S}[u]$$

- input signal space: $\mathcal{U}$
- output signal space: $\mathcal{Y}$
- inputs: $u \in \mathcal{U}$
- output: $y \in \mathcal{Y}$



Systems – example

From the previous lecture: systems with possible inputs and outputs

- RLC circuit, eq.
  - input: u_{be}, output: u_C
  - input: u_{be}, output: i
- Primary circuit pressure control tank
  - input: *heating power*, output: *primary circuit pressure*
- steered car model
  - input: (u_ϕ, u_t), output: (x, y, θ)

Basic system properties – 1

Linearity

$$\mathbf{S}[c_1 u_1 + c_2 u_2] = c_1 y_1 + c_2 y_2 \quad (1)$$

$c_1, c_2 \in \mathbb{R}$, $u_1, u_2 \in \mathcal{U}$, $y_1, y_2 \in \mathcal{Y}$, and
$\mathbf{S}[u_1] = y_1 \quad , \quad \mathbf{S}[u_2] = y_2$
i.e. satisfies the principle of *superposition*

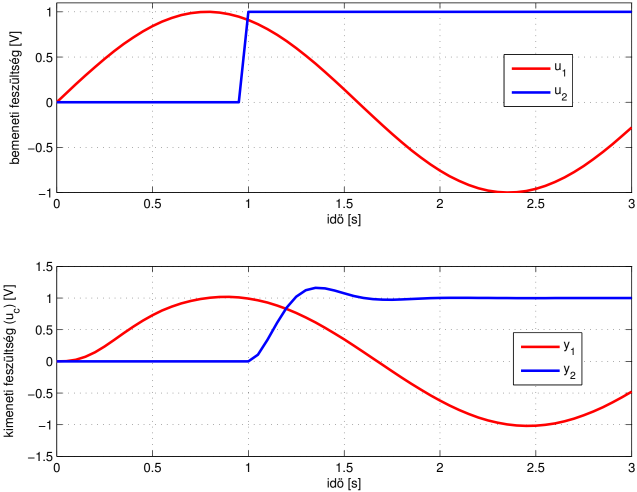
Examples

- the RLC circuit is linear
- the bioreactor model is nonlinear

Checking whether a system is linear or not: by definition (1)

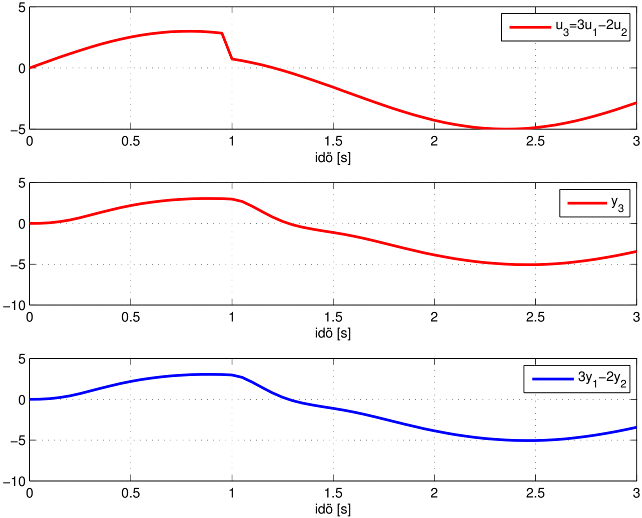
Example: RLC circuit

The system's output for two different inputs:



Example: RLC circuit

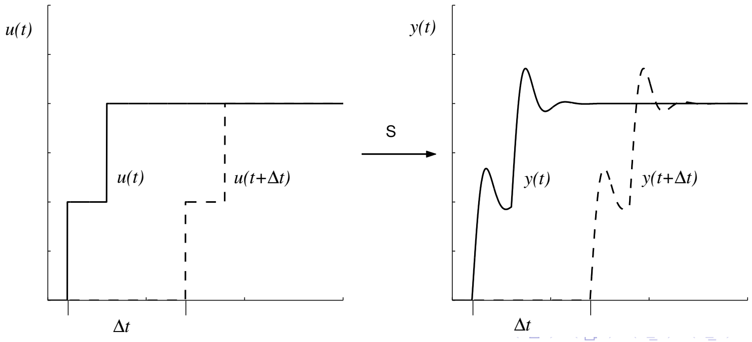
The system's output for a linear combination of the previous two inputs:



time invariance: the shift operator and the system operator commute, i.e.

$$\mathbf{T}_\tau \circ \mathbf{S} = \mathbf{S} \circ \mathbf{T}_\tau$$

where $\mathbf{T}_\tau$ denotes the shift operator (in time), i.e. $\mathbf{T}_\tau\, x(t) = x(t - \tau)$
Checking whether a system is time invariant: **constant (time independent) parameters in the system's ordinary differential equations**



Basic system properties – 3

- *continuous time and discrete time systems*
  continuous time: ($\mathcal{T} \subseteq \mathbb{R}$)
  discrete time: $\mathcal{T} = \{\cdots, t_0, t_1, t_2, \cdots\}$
- *single input – single output (SISO)*
  multiple input – multiple output (MIMO) systems
- *causal/non causal systems*

CT-LTI system models

- **input-output models of SISO systems**
  - time domain (t)
  - operator domain (s - Laplace transform)
  - frequency domain (ω - Fourier transform)
- **State space models**

CT-LTI system models – 1

Time domain

Linear differential equations with constant coefficients

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_0 u + b_1 \frac{du}{dt} + \ldots + b_m \frac{d^m u}{dt^m}$$

with given initial conditions

$$y(0) = y_{00} \; , \; \frac{dy}{dt}(0) = y_{10} \; , \; \ldots \; , \; \frac{d^{n-1}y}{dt^{n-1}}(0) = y_{(n-1)0}$$

CT-LTI system models – 2

Operator domain, SISO systems
Transfer function

$$Y(s) = H(s)U(s)$$

if zero initial conditions assumed (!)

$Y(s)$	Laplace transform of the output signal
$U(s)$	Laplace transform of the input signal
$H(s) = \frac{b(s)}{a(s)}$	*the system's transfer function*
	where $a(s)$ and $b(s)$ are polynomials
	$\deg\ b(s) = m$
	$\deg\ a(s) = n$

Strictly proper transfer function: $m < n$
Proper: $m = n$,
improper: $m > n$

CT-LTI system models – 3

Time domain – *Impulse response function*

$Y(s) = H(s)U(s) \to \mathcal{L}^{-1} \to y(t) = (h * u)(t)$, i.e.

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau$$

using the definition of Dirac-δ, one can obtain:

$$\int_0^\infty \delta(t-\tau)h(\tau)d\tau = \int_0^t \delta(t-\tau)h(\tau)d\tau = h(t)$$

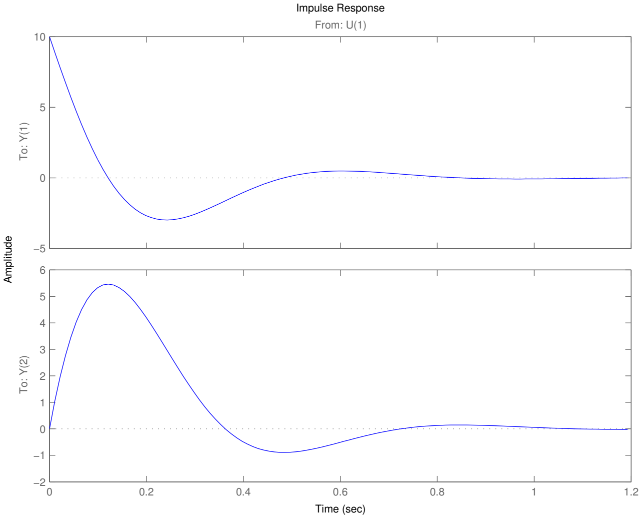
and

$$L(\delta)(s) = \int_0^\infty \delta(t)e^{-st}dt = 1$$

consequently, h is the system's response to a Dirac-δ input

Example

Impulse response functions of the RLC circuit ($u = u_{be}$, $y_1 = i$, $y_2 = u_C$)



CT-LTI I/O models (SISO)

Transfer function – linear differential equation

$$\mathcal{L}\{a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y\} =$$
$$= \mathcal{L}\{b_0 u + b_1 \frac{du}{dt} + \dots + b_m \frac{d^m u}{dt^m}\}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}$$

Transfer function – impulse response

$$H(s) = \mathcal{L}\{h(t)\}$$

CT-LTI I/O models: key points

- the Laplace transform converts (higher order) linear differential equations into algebraic equations
- zero initial conditions are assumed for transfer functions (initial state information is not included!)
- knowing the input, the output can be computed (Laplace transform (and inverse), convolution)
- the whole system operator is represented as a time-domain signal ($h(t)$) and/or its Laplace transform ($H(s)$)
- the model parameters are the coefficients in $b(s)$ and $a(s)$

General form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad &\text{(state equation)} \\ y(t) &= Cx(t) + Du(t) \qquad &\text{(output equation)} \end{aligned}$$

- for a given initial condition $x(t_0) = x(0)$ and $x(t) \in \mathbb{R}^n$,
- $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
- model parameters

$$A \in \mathbb{R}^{n \times n} \, , \; B \in \mathbb{R}^{n \times r} \, , \; C \in \mathbb{R}^{p \times n} \, , \; D \in \mathbb{R}^{p \times r}$$

State transformation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad \dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t)$$
$$y(t) = Cx(t) + Du(t) \quad , \quad y(t) = \overline{C}\overline{x}(t) + \overline{D}u(t)$$

invertible transformation of the states:

$$T \in \mathbb{R}^{n \times n} \quad , \quad det\ T \neq 0 \quad , \quad \overline{x} = Tx \quad \Rightarrow \quad x = T^{-1}\overline{x}$$

$$dim\ \mathcal{X} = dim\ \overline{\mathcal{X}} = n$$

$$T^{-1}\dot{\overline{x}} = AT^{-1}\overline{x} + Bu$$

$$\dot{\overline{x}} = TAT^{-1}\overline{x} + TBu \quad , \quad y = CT^{-1}\overline{x} + Du$$

$$\overline{A} = TAT^{-1} \quad , \quad \overline{B} = TB \quad , \quad \overline{C} = CT^{-1} \quad , \quad \overline{D} = D$$

Transfer function computed from the state space model

Laplace transform of the state space model

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \qquad &\text{(state equation, } x(0) = 0\text{)} \\ Y(s) &= CX(s) + DU(s) \qquad &\text{(output equation)} \end{aligned}$$

$$\begin{aligned} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= \{C(sI - A)^{-1}B + D\}U(s) \end{aligned}$$

The system's transfer function $H(s)$, expressed with the corresponding state space model matrices (A, B, C, D):

$$H(s) = C(sI - A)^{-1}B + D$$

Solution of the state space model

We determine the inverse Laplace transform of

$$X(s) = (sI - A)^{-1} BU(s)$$

by considering the Taylor series of (matrix) expression: $(sI - A)^{-1}$:

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \ldots\right)$$

Thus, the inverse Laplace transform of $(sI - A)^{-1}$ is

$$\mathcal{L}^{-1}\{(sI - A)^{-1}\} = I + At + \frac{1}{2!}A^2 t^2 + \ldots = e^{At} \quad , \quad t \geq 0$$

Finally, we obtain the unique solution $x(t)$ of the state space model for the initial condition $x(0)$:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Markov parameters

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Assuming $x(0) = 0$, $D = 0$ and $u(t) = \delta(t)$, we obtain the impulse response:

$$h(t) = Ce^{At}B = CB + CABt + CA^2B\frac{t^2}{2!} + \ldots$$

Markov parameters

$$CA^iB \quad , \quad i = 0, 1, 2, \ldots$$

are ***invariant*** *for the state transformations.*

state space models: key points

- the Laplace transform converts sets of first order linear differential equations into algebraic equations
- SS models can handle non-zero initial conditions
- knowing the input and the initial condition, the output can be computed (Laplace transform (and inverse), convolution)
- the model parameters are the A, B, C, D matrices ($x(0)$ is also needed for the solution)
- SS models can be easily transformed to I/O models through Laplace transform assuming $x(0) = 0$

Summary

- fundamental system properties: linearity (superposition), time-invariance
- LTI I/O models: higher order linear differential equations containing only the input and the output (and derivatives)
- transfer function, impulse response function: LTI system operators given in the form of signals
- state space models: sets of first order ODEs with state variables, inputs and outputs ; initial conditions not necessarily zero
- SS and I/O models can be converted to each other
- key role of Laplace transform in handling/solving I/O and SS models

Computer Controlled Systems
(Introduction to systems and control theory)
Lecture 3

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, 4 October, 2018

Contents

Needed from mathematics

- matrices: row/column rank, image, left/right kernel, determinant, characteristic polynomial
- matrix polynomials, Cayley-Hamilton theorem
- quadratic forms

Revision from previous lecture

- **system**: operator (input signals $\longrightarrow$ output signals)
- **LTI models**: I/O: $H(s)$, state space: (A, B, C, D)
- **states**: form the state space; knowing the model, input, and initial state the future states and outputs can be computed

Brief problem statements of observability and controllability

General form of SS models – revision

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

- signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
- system parameters: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ (assumption without loss of generality: $D = 0$)

Studied system properties:

- **observability**: determining the initial condition
  (we need state information from the measurements (output) knowing the model)
- **controllability**: setting the initial condition
  we want to influence (change) the state with appropriate input knowing the model

Observability of LTI systems – 1

Problem formulation

Given:

- a state space model (A, B, C) $(D = 0)$
- input u
- measured values y on a finite time horizon

To be computed:

The value of state vector x on a finite time horizon

It is sufficient to compute: $x(t_0) = x_0$

Definition. *The system (A, B, C) (or, equivalently, the pair (A, C)) is observable, if $x(t_0)$ can be determined from a finite measurement of y.*

Observability – example 1

We consider the known RLC circuit. We measure the voltage of the capacitor (u_C). We want to obtain the value of the current (i).

$$x = \begin{bmatrix} i \\ u_C \end{bmatrix}, \quad u = u_{be}, \quad y = x_2$$

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Observability – example 2

Elementary acceleration model (without friction, air resistance etc.):

$$F = m \cdot a = m \cdot \ddot{x}_1$$

in state space form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u,$$

where x_1 is the position, x_2 is the velocity, and $u = \frac{F}{m}$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Problems/tasks:

a) Can we determine the velocity if the position is measured only? (i.e. $C = [1\ 0]$)

b) Can we determine the position if the velocity is measured only? (i.e. $C = [0\ 1]$)

Observability of LTI systems – 2

Necessary and sufficient condition.

A state space model (A, B, C) is *observable* **if and only if** the observability matrix $\mathcal{O}_n$ is full-rank.

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

Observability of LTI systems – 3

Proof: (by construction)

$$
\begin{aligned}
y &= Cx \\
\dot{y} &= C\dot{x} = CAx + CBu \\
\ddot{y} &= C\ddot{x} = CA(Ax + Bu) + CB\dot{u} = CA^2x + CABu + CB\dot{u} \\
&. \\
&. \\
y^{(n-1)} &= Cx^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + ... + CABu^{(n-3)} + CBu^{(n-2)}
\end{aligned}
$$

$$
\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ . \\ . \\ . \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ CB & 0 & . & . & . & 0 \\ CAB & CB & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ CA^{n-2}B & CA^{n-3}B & . & . & CB & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ . \\ . \\ . \\ u^{(n-1)} \end{bmatrix}
$$

Observability of LTI systems – 4

$$\mathcal{Y}(t) = \mathcal{O}_n x(t) + \mathcal{T}\mathcal{U}(t)$$

Expressing $x(t)$

$$x(t) = \mathcal{O}_n^{-1}(\mathcal{Y}(t) - \mathcal{T}\mathcal{U}(t)),$$

where $\mathcal{O}^{-1}$ denotes the (generalized) inverse of $\mathcal{O}_n$

$x(t)$ **can be uniquely determined if and only if** $\textit{rank}\ \mathcal{O}_n(A, C) = n$.

Controllability of LTI systems – 1

Problem formulation
Given:

- a state space mode (A, B, C)
- **initial condition** $x(0)$, and $x(T) \neq x(0)$ desired **final state**

To be computed:
an appropriate u input signal, which drives the system from state $x(t_1)$ to $x(t_2)$ **in finite time**.

Definition. *The system (A, B, C) (or, equivalently, the pair (A, B)) is controllable if, given a finite duration $T > 0$ and two arbitrary points $x_0, x_T \in \mathbb{R}^n$, there exists an appropriate input u such that for initial condition $x(0) = x_0$, the value of the state vector at time T is $x(T) = x_T$.*

Controllability – example

System: RLC circuit $i(0) = 1\text{A}$, $u(0) = 0\text{V}$
Does there exist an input voltage function u_{be}, such that we have $i(t_1) = 5\text{A}$, $u(t_1) = 10\text{V}$, and $t_1 < M < \infty$?

Controllability – example 2

Elementary acceleration model again

$$\dot{x} = Ax + Bu$$

where x_1 is the position, x_2 is the velocity, and $u = \frac{F}{m}$
matrices of the SS model:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Problem/task:
Compute an acceleration command such that the speed is exactly $x_2 = 30m/s$ at distance $x_1 = 200m$?

Controllability of LTI systems – 2

Necessary and sufficient condition

A state space model with matrices (A, B, C)

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

is controllable **if and only if**, the controllability matrix $\mathcal{C}_n$ is of full-rank

$$\mathcal{C}_n = \left[\begin{array}{cccccc} B & AB & A^2B & . & . & A^{n-1}B \end{array} \right]$$

Controllability of LTI systems – 3

Proof: (by construction)
$\int_{-\infty}^{\infty} f(t)\delta'(t)dt = -f'(0)$
$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)dt = (-1)^n f^{(n)}(0)$
$f(\tau) = e^{-A\tau}, \quad f'(\tau) = -Ae^{-A\tau}$
$f^{(n)}(\tau) = (-1)^n A^n e^{-A\tau}$
Input: linear combination of Dirac-δ and its time derivatives.
$u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + \ldots + g_n\delta^{(n-1)}(t)$
According to the principle of superposition:

$$x(0_+) = x(0_-) + g_1 h(0_-) + g_2\dot{h}(0_-) + \ldots + g_n h^{(n-1)}(0_-)$$
$$x(0_+) = x(0_-) + g_1 B + g_2 AB + \ldots + g_n A^{n-1} B$$

Controllability of LTI systems – 4

Assuming that $x(0_-) = 0$ we get:

$$x(0_+) = \left[\begin{array}{cccccc} B & AB & A^2B & . & . & . & A^{n-1}B \end{array} \right] \left[\begin{array}{c} g_1 \\ g_2 \\ . \\ . \\ . \\ g_n \end{array} \right]$$

for an arbitrary final state value $x(0_+)$ there exists a unique weighting vector $[g_1 ... g_n]^T$ if and only if *rank* $\mathcal{C}_n(A, B) = n$.

Diagonal realization

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where

$$\dot{x} = \begin{bmatrix} \lambda_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & . & . & . & c_n \end{bmatrix} x$$

Controllability in case of a diagonal realization

Controllability matrix

$$\mathcal{C}_n = \begin{bmatrix} B & AB & . & . & A^{n-1}B \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & . & . \end{bmatrix} =$$

$$= \begin{bmatrix} b_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & . & . & \lambda_1^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \lambda_n & . & . & \lambda_n^{n-1} \end{bmatrix}$$

This matrix is the so-called *Vandermonde-matrix* , which is nonsingular if $\lambda_i \neq \lambda_j \quad (i \neq j)$.

$$rank\ \mathcal{C}_n = n \quad \Leftrightarrow \quad det\ \mathcal{C}_n \neq 0$$
$$det\ \mathcal{C}_n = \prod_i b_i \prod_{i<j} (\lambda_i - \lambda_j)$$

Transfer function of a diagonal SISO realization

$$H(s) = C(sI - A)^{-1}B = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)}$$

where I is the unit matrix.

If $c_j = 0$ or $b_k = 0$ for a given j or k, then the transfer function can be rewritten by using a smaller number of partial fractions.

$$H(s) = \sum_{i=1}^{\overline{n}} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)} \quad , \quad \overline{n} < n$$

Computing a realizable (smooth) input for a target state

Given: A, B, $x(0)$ (initial state), $x(\bar{t})$ (target state)
To be determined: u, $\bar{t}$ (finite)
Assumption: input is in the form $u(t) = B^T e^{A^T(\bar{t}-t)} z$, where $z \in \mathbb{R}^n$ ($z =$?)

$$x(\bar{t}) = e^{A\bar{t}} x(0) + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B B^T e^{A^T(\bar{t}-\tau)} \cdot z d\tau$$

Let $\xi = \bar{t} - \tau$, then:

$$x(\bar{t}) = e^{A\bar{t}} x(0) + \underbrace{\left[\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi\right]}_{G_C(\bar{t})} \cdot z$$

From this, the input parameters can be expressed as

$$z = G_c^{-1}(\bar{t}) \left(x(\bar{t}) - e^{A\bar{t}} x(0) \right), \quad \text{provided that } G_c^{-1}(\bar{t}) \text{ exists for some } \bar{t}$$

The controllability Gramian

$$G_C(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$$

is the controllability Gramian

The following is true:

The controllability matrix is of full rank if and only if $G_C(t)$ is positive definite (and therefore, invertible) for some $t \geq 0$.

controllability $\Longrightarrow$ a smooth input can be computed to arbitrarily change the state in finite time

Remark on the powers of A

- Q: Why is it that there is no need for higher powers of A than A^{n-1} in the controllability/observability matrices?
- A: It follows from the Cayley-Hamilton theorem that A^n, A^{n+1}, ... can be expressed as a linear combination of $A^0 = I$, A, ..., A^{n-1}
characteristic polynomial of A:

$$p(\lambda) = \det(\lambda I - A) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$$

Then: $a_0 I + a_1 A + a_2 A^2 + \ldots a_n A^n = 0$

$$A^n = \sum_{i=0}^{n-1} \bar{a}_i A^i$$

$$A^{n+1} = A \cdot A^n = \bar{a}_0 A + \bar{a}_1 A^2 + \cdots + \bar{a}_{n-1} A^n =$$

$$= \bar{a}_0 A + \bar{a}_1 A^2 + \cdots + \bar{a}_{n-1} \sum_{i=0}^{n-1} \bar{a}_i A^i$$

and so on $\Longrightarrow$ $A^i B$ and CA^i for $i \geq n$ cannot increase the rank of the controllability and observability matrix, respectively

1. Problem statement
2. Observability
3. Controllability
4. **Geometrical interpretation**

Geometrical interpretation of observability

(A, C) **unobservability subspace of the system**:

set of initial condition values, which cannot be distinguished from each other knowing (measuring) the output signal

namely, starting the system operation from any initial condition from the unobservability subspace, the system will produce the *same* output

Computing the basis of the unobservability subspace $\ker(\mathcal{O}_n)$

Example

Matrices of the state space model:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Observability matrix:

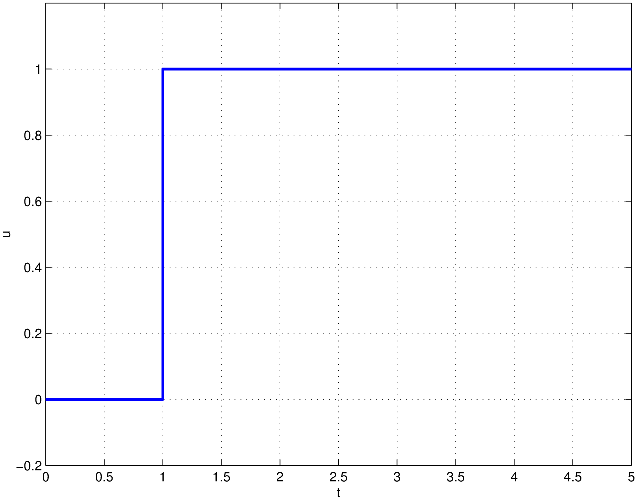
$$\mathcal{O}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Basis of the unobservability subspace:

$$\ker(\mathcal{O}_2) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

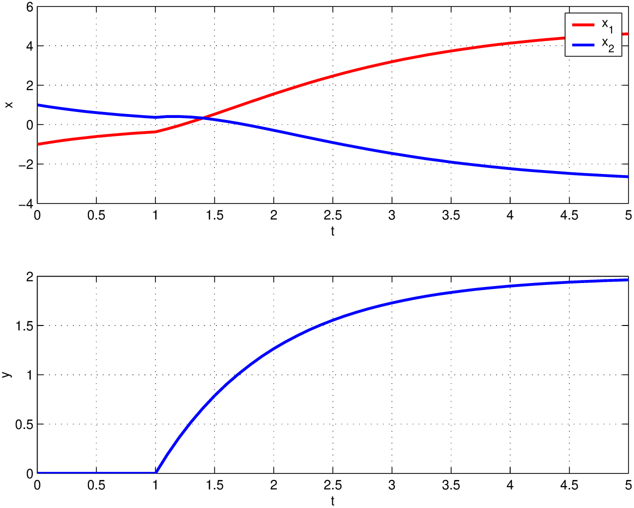
Example

Input given to the system:



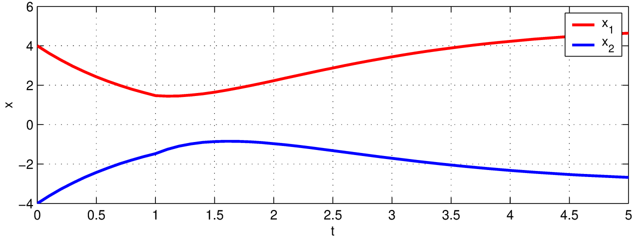
Example

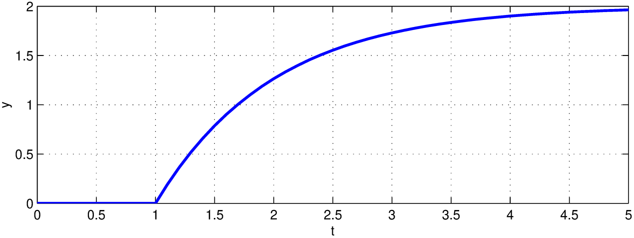
State variables of the system and its output for $x(0) = [-1 \;\; 1]^T$



Example

State variables of the system and its output for $x(0) = [4 \quad -4]^T$





Geometrical interpretation of controllability

(A, B) **controllability subspace of a system**:

set of state vectors $x_1 \in \mathbb{R}^n$, which can be reached in finite time from the origin of the state space ($x(0) = 0$).
$\exists u : [0, T] \to \mathbb{R}^m, \quad T < \infty$ such that $x(T) = x_1$

in other words, there does not exist any input signal $u(t)$ for which the state vector can leave the controllability subspace.

Computing the basis of the controllability subspace $\mathrm{im}(\mathcal{C}_n)$

Example

Matrices of the state space model:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Controllability matrix:

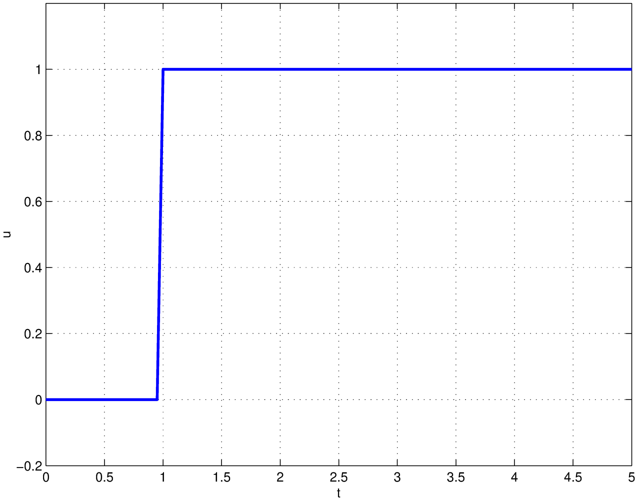
$$\mathcal{C}_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Basis of the controllability subspace:

$$\text{im}(\mathcal{C}_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

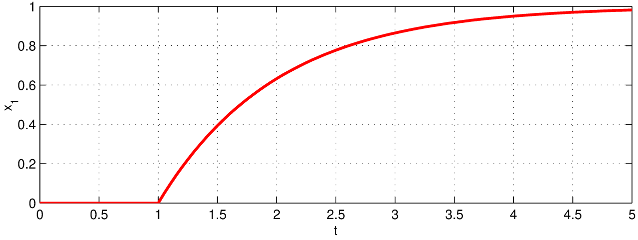
Example

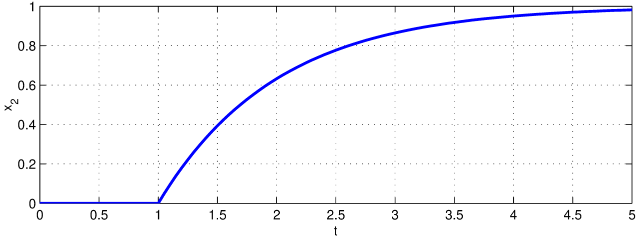
Input given to the system:



Example

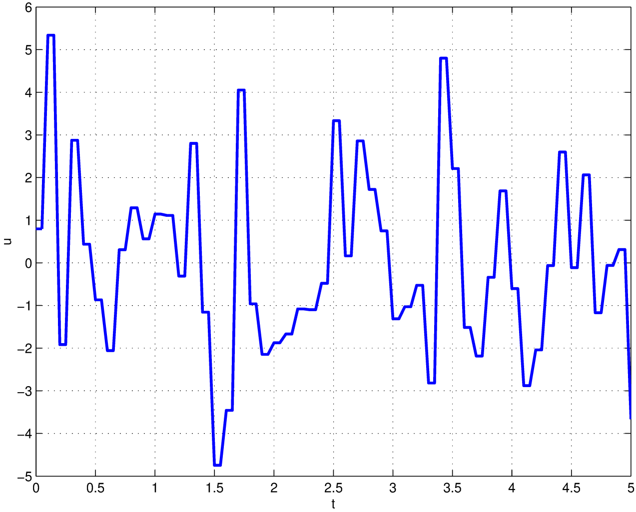
State variables of the system, in case of $x(0) = [0 \quad 0]^T$





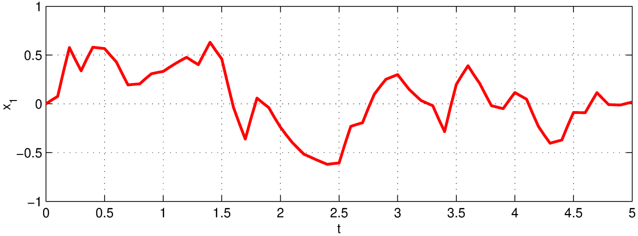
Example

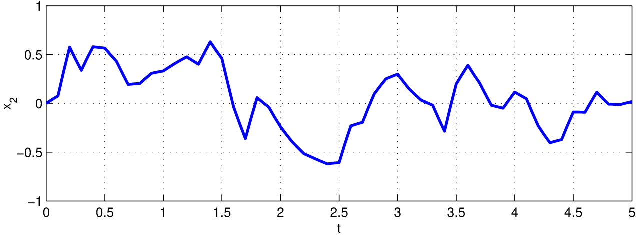
Input given to the system:



Example

State variables of the system, in case of $x(0) = [0 \;\; 0]^T$





Summary

- observability: possibility to compute the state (initial condition) from inputs and outputs knowing the model
- controllability: possibility to reach a given target state (initial condition) with appropriate input knowing the model
- necessary and sufficient condition: full rank of the observability/controllability matrix
- geometry: controllability subspace, unobservability subspace

Computer Controlled Systems
Lecture 4

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, 11 October, 2018

Contents

Introductory example

Consider the following SISO CT-LTI system withe realization (A,B,C)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

The model is *observable* but it is *not controllable*.
Question: Can the model be written in a new coordinates system, such that the new model is both observable and controllable? (and what are the conditions / consequences)
Transfer function:

$$H(s) = \frac{2s^2 + 4s}{s^3 + 2s^2 - s}$$

Introduction – 1

- For a given (SISO) transfer function $H(s) = \frac{b(s)}{a(s)}$, the state space model (A, B, C, D) is called *an nth order realization* if
  $H(s) = C(sI - A)B + D$,
  where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$.
  (The state space repr. for a given transfer function is **not unique**).
- An *n-th order state space realization* (A, B, C, D) of a given transfer function $H(s)$ is called *minimal* , if there exist no other realization with a smaller state space dimension (i.e., with a smaller A matrix)
- An n-th order state *space model* (A, B, C, D) is called *jointly controllable and observable* if both $\mathcal{O}_n$ and $\mathcal{C}_n$ are full-rank matrices.

Assumptions from now on: SISO systems, $D = 0$

Introduction – 2

- The transfer function is invariant for state transformations
- The roots of the transfer function's denominator are the eigenvalues of matrix A ($a(s)$ is the characteristic polynomial of A)
- For a given transfer function $H(s)$, any two arbitrary jointly controllable and observable realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) are connected to each other by the following coordinates transformation

$$T = \mathcal{O}^{-1}(C_1, A_1)\mathcal{O}(C_2, A_2) = \mathcal{C}(A_1, B_1)\mathcal{C}^{-1}(A_2, B_2)$$

(without proof)

Introduction – 3

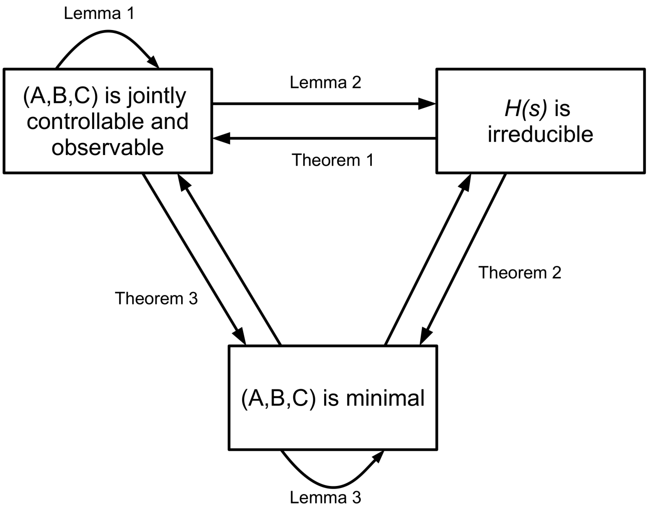
Matrix polynomials:

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0, \quad x \in \mathbb{R}$$
$$p(A) = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I$$

important properties:

- a matrix polynomial commutes with any power of the argument matrix, namely: $A^i P(A) = P(A) A^i$
- eigenvalues: $\lambda_i[P(A)] = P(\lambda_i[A])$
- Cayley-Hamilton theorem: every $n \times n$ matrix is a root of its own characteristic polynomial ($p(x) = \det(A - xI)$)

Overview – 1



equivalent state space and I/O model properties

Overview – 2

Consider **SISO CT-LTI systems** with realization (A, B, C)

- Joint controllability and observability is a **system property**
- Equivalent necessary and sufficient conditions
- Minimality of SSRs
- Irreducibility of the transfer function

Hankel matrices

- A *Hankel matrix* is a block matrix of the following form

$$H[1, n-1] = \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix}$$

- It contains *Markov parameters* CA^iB that are invariant under state transformations.

Lemma 1

Lemma (1)

If we have a system with transfer function $H(s) = \frac{b(s)}{a(s)}$ and there is an n-th order realization (A, B, C) which is jointly controllable and observable, then all other n-th order realizations are jointly controllable and observable.

Proof

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}, \quad \mathcal{C}(A,B) = \begin{bmatrix} B & AB & A^2B & . & . & . & A^{n-1}B \end{bmatrix}$$

$$H[1, n-1] = \mathcal{O}(C,A)\mathcal{C}(A,B)$$

Controller form realization

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix}$$

with the coefficients of the polynomials
$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$ and $b(s) = b_1 s^{n-1} + \ldots + b_{n-1}s + b_n$
that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Observer form realization

$$\dot{x}(t) = A_o x(t) + B_o u(t)$$
$$y(t) = C_o x(t)$$

where

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad D_o = D$$

with the coefficients of the polynomials
$a(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n$ and $b(s) = b_1 s^{n-1} + \dots + b_{n-1}s + b_n$
that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Definitions

Definition (Relative prime polynomials)

Two polynomials $a(s)$ and $b(s)$ are *coprimes* (or relative primes) if $a(s) = \prod(s - \alpha_i)$; $b(s) = \prod(s - \beta_j)$ and $\alpha_i \neq \beta_j$ for all i, j.
In other words: the polynomials have no common roots.

Definition (Irreducible transfer function)

A transfer function $H(s) = \frac{b(s)}{a(s)}$ is called to be *irreducible* if the polynomials $a(s)$ and $b(s)$ are relative primes.

Lemma 2

Lemma (2)

An n-dimensional controller form realization with transfer function $H(s) = \frac{b(s)}{a(s)}$ (where $a(s)$ is an n-th order polynomial) is jointly controllable and observable if and only if $a(s)$ and $b(s)$ are relative primes (i.e., $H(s)$ is irreducible).

Proof

- A controller form realization is controllable and

$$\mathcal{O}_c = \tilde{I}_n b(A_c)$$

$$\tilde{I}_n = \begin{bmatrix} 0 & . & . & 1 \\ 0 & . & 1 & 0 \\ . & . & . & . \\ 1 & 0 & . & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

- Non-singularity of $b(A_c)$

Proof of Lemma 2. – 1

$$\tilde{I}_n = \begin{bmatrix} e_n & e_{n-1} & . & . & e_1 \end{bmatrix} = \begin{bmatrix} e_n^T \\ e_{n-1}^T \\ . \\ . \\ . \\ e_1^T \end{bmatrix} \quad , \quad e_i = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \end{bmatrix} \leftarrow i.$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix} \quad , \quad e_i^T A_c = \begin{cases} [-a_1 \quad -a_2 \quad \dots \quad -a_n] \\ e_{i-1}^T \end{cases}$$

- Computation of the observability matrix $\mathcal{O}_c = \tilde{I}_n b(A_c) \in \mathbb{R}^{n \times n}$
- 1st row:

$$e_n^T b(A_c) = e_n^T b_1 A_c^{n-1} + \ldots + e_n^T b_{n-1} A_c + e_n^T b_n I_n$$

n-th term: $[0 \;\; \ldots \;\; 0 \;\; b_n]$
$(n-1)$-th term: $b_{n-1} e_n^T A_c = b_{n-1} e_{n-1}^T = [0 \;\; \ldots \;\; b_{n-1} \;\; 0]$
...

$$e_n^T b(A_c) = [b_1 \;\; \ldots \;\; b_{n-1} \;\; b_n] = C_c$$

- 2nd row:

$$e_{n-1}^T b(A_c) = e_n^T A_c b(A_c) = e_n^T b(A_c) A_c \quad \Rightarrow \quad e_{n-1}^T b(A_c) = C_c A_c$$

- and so on ...

Proof of Lemma 2. – 3

$\mathcal{O}_c$ is nonsingular

- iff $b(A_c)$ is nonsingular because matrix $\tilde{I}_n$ is always nonsingular
- $b(A_c)$ is nonsingular iff $det(b(A_c)) \neq 0$
  which depends on the eigenvalues of $b(A_c)$ matrix
- the eigenvalues of the matrix $b(A_c)$ are $b(\lambda_i), \quad i = 1, 2, ..., n$
  λ_i is an eigenvalue of A_c, i.e a root of $a(s) = det(sI - A)$

$$det(b(A_c)) = \prod_{i=1}^{n} b(\lambda_i) \neq 0$$

$$\Updownarrow$$

$a(s)$ and $b(s)$ have no common roots, i.e. they are relative primes

Minimal realization conditions – 1

Theorem (1)

$H(s) = \frac{b(s)}{a(s)}$ (where $a(s)$ is an n-th order polynomial) is irreducible if and only if all of its n-th order realizations are jointly controllable and observable.

Proof: combine Lemma 1. and 2.

- We assume that any nth order realization $H(s)$ is jointly controllable and observable $\Longrightarrow$ A controller form is jointly controllable and observable $\Longrightarrow$ $H(s)$ is irreducible (Lemma 2)
- We assume that $H(s)$ is irreducible $\Longrightarrow$ the controller form realization is jointly controllable and observable (Lemma 2) $\Longrightarrow$ Any nth order realization is jointly controllable and observable (Lemma 1)

Minimal realization conditions – 2

Definition (Minimal realization)

An n-dimensional realization (A, B, C) of the transfer function $H(s)$ is minimal if one cannot find another realization of $H(s)$ with dimension less than n.

Theorem (2)

$H(s) = \frac{b(s)}{a(s)}$ is irreducible iff any of its realization (A, B, C) is minimal where $H(s) = C(sI - A)^{-1}B$

Proof: by contradiction

- We assume that $H(s)$ is irreducible, but there exists an nth order realization, which is not minimal $\Longrightarrow$ there exists an mth $(m < n)$ order realization $(\bar{A}, \bar{B}, \bar{C})$ of $H(s)$ $\Longrightarrow$ from this realization we can obtain the transfer function $\bar{H}(s)$, for which the order of its denominator m, which is a contradiction (since $H(s)$ is reducible).
- We assume that the nth order realization (A, B, C) is minimal, but $H(s) = C(sI - A)^{-1}B$ is reducible $\Longrightarrow$ From the simplified transfer function one can obtain an mth order realization, such that $m < n$, that is a contradiction.

Minimal realization conditions – 3

Theorem (3)

A realization (A, B, C) is minimal iff the system is jointly controllable and observable.

Proof: Combine Theorem 1 and Theorem 2 .

Lemma (3)

Any two minimal realizations can be connected by a unique similarity transformation (which is invertible).

Proof: (Just the idea of it)

$$T = \mathcal{O}^{-1}(C_1, A_1)\mathcal{O}(C_2, A_2) = \mathcal{C}(A_1, B_1)\mathcal{C}^{-1}(A_2, B_2)$$

exists and it is invertible: this is used as a transformation matrix.

Decomposition of uncontrollable systems

We assume that (A, B, C) is not controllable. Then, there exists an invertible transformation T such that the transformed system in the new coordinates system ($\bar{x} = Tx$) will have the form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

and

$$H(s) = C_c(sI - A_c)^{-1}B_c$$

Controllability decomposition – example

Matrices of the state-space :

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Controllability matrix:

$$\mathcal{C}_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Transformation:

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

The transformed model:

$$\bar{A} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Decomposition of unobservable systems

We assume that (A, B, C) is not observable. Then there exists an invertible matrix transformation T, such that the transformed system in the new coordinates system ($\bar{x} = Tx$) will have the form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

and

$$H(s) = C_o(sI - A_o)^{-1} B_o$$

Observability decomposition – example

Matrices of the state-space model:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Observability matrix:

$$\mathcal{O}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Transformation:

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

The transformed model:

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ -4 & 1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

General decomposition theorem

Given an (A, B, C) SSR, it is always possible to transform it to another realization $(\overline{A}, \overline{B}, \overline{C})$ with partitioned state vector and matrices

$$\overline{x} = \left[\begin{array}{cccc} \overline{x}_{co} & \overline{x}_{c\overline{o}} & \overline{x}_{\overline{c}o} & \overline{x}_{\overline{co}} \end{array} \right]^T$$

$$\overline{A} = \left[\begin{array}{cccc} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\overline{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\overline{co}} \end{array} \right] \quad \overline{B} = \left[\begin{array}{c} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \\ 0 \\ 0 \end{array} \right]$$

$$\overline{C} = \left[\begin{array}{cccc} \overline{C}_{co} & 0 & \overline{C}_{\overline{c}o} & 0 \end{array} \right]$$

General decomposition theorem

The partitioning defines subsystems

- *Controllable and observable subsystem*: $(\overline{A}_{co}, \overline{B}_{co}, \overline{C}_{co})$ is minimal, i.e. $\overline{n} \leq n$ and

$$H(s) = \overline{C}_{co}(s\overline{I} - \overline{A}_{co})^{-1}\overline{B}_{co} = C(sI - A)^{-1}B$$

- *Controllable subsystem*

$$\left(\begin{bmatrix} \overline{A}_{co} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} \end{bmatrix}, \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \end{bmatrix}, \begin{bmatrix} \overline{C}_{co} & 0 \end{bmatrix} \right)$$

- *Observable subsystem*

$$\left(\begin{bmatrix} \overline{A}_{co} & \overline{A}_{13} \\ 0 & \overline{A}_{\overline{c}o} \end{bmatrix}, \begin{bmatrix} \overline{B}_{co} \\ 0 \end{bmatrix}, \begin{bmatrix} \overline{C}_{co} & \overline{C}_{\overline{c}o} \end{bmatrix} \right)$$

- *Uncontrollable and unobservable subsystem*

$$([\overline{A}_{\overline{co}}], [0], [0])$$

Introductory example – review

Consider the following SISO CT-LTI system withe realization (A,B,C)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

The model is *observable* but it is *not controllable*.
Its transfer function and its simplified form:

$$H(s) = \frac{2s^2 + 4s}{s^3 + 2s^2 - s} = \frac{2s + 4}{s^2 + 2s - 1}$$

Its minimal state space realization (eq. controller form):

$$\bar{A} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

Summary

- joint controllability and observability of (A, B, C) has important consequences, since it is equivalent to:
  - a state space realization with the minimum number of state variables (minimal realization, i.e., A cannot be smaller)
  - $H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)}$ is irreducible
- non-controllable and/or non-observable state space models can be transformed such that the non-controllable / non-observable states are clearly visible in the new coordinates
- it's easy to determine a minimal realization from a non-controllable/non-observable SS model (simplification of the transfer function, canonical realization)

Computer Controlled Systems
Lecture 5

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, Oct 18, 2018

Overview

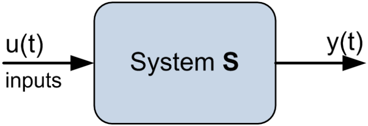
1. Basic notions
2. Bounded input-bounded output (BIBO) stability
3. Stability in the state space
4. Examples
5. Stability region of nonlinear systems

Systems

- System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

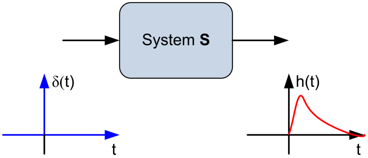
- inputs (u) and outputs (y)



CT-LTI I/O system models

- Time domain: Impulse response function
  is the response of a SISO LTI system to a Dirac-delta input function with zero initial condition.
- The output of **S** can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$



CT-LTI state-space models

- **General form - revisited**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0) \\ y(t) &= Cx(t) \end{aligned}$$

  with
  - signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
  - system parameters: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{p \times n}$ ($D = 0$ by using **centering** the inputs and outputs)
- Dynamic system properties:
  - observability
  - controllability
  - stability

Overview

1. Basic notions
2. Bounded input-bounded output (BIBO) stability
3. Stability in the state space
4. Examples
5. Stability region of nonlinear systems

Signal spaces

- $\mathcal{L}_q$ signal spaces

$$\mathcal{L}_q[0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \int_0^\infty |f(t)|^q \,\mathrm{d}t < 0 \right\}$$

special case

$$\mathcal{L}_\infty[0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \sup_{t\geq 0} |f(t)| < \infty \right\}$$

- Remark: $\mathcal{L}_q$ spaces are Banach spaces with norms

$$\|f\|_q = \left(\int_0^\infty |f(t)|^q \,\mathrm{d}t \right)^{1/q}$$

$$\|f\|_\infty = \sup_{t\geq 0} |f(t)|$$

Vector valued signals

- $\mathcal{L}_q^n$ multidimensional signal spaces

  Let $f(t) \in \mathbb{R}^n$, $\forall t \geq 0$, then

$$\mathcal{L}_q^n[0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R}^n \;\middle|\; f \text{ is measurable, } \int_0^\infty \|f(t)\|_2^q \,\mathrm{d}t < \infty \right\}$$

  where $\|f(t)\| = \sqrt{f^T(t)f(t)}$ is the Euclidean norm in $\mathbb{R}^n$

- $\mathcal{L}_q^n$ is a Banach space equipped with the signal norm

$$\text{norm: } \|f\|_q = \left(\int_0^\infty \|f(t)\|_2^q \,\mathrm{d}t \right)^{1/q}$$

- Remark: The case $\mathcal{L}_2$ is special, because the norm can be originated from an inner product (therefore, $\mathcal{L}_2$ is a Hilbert-space)

BIBO stability – general

Definition (BIBO stability)

A system is *externally or BIBO stable* if for any bounded input it responds with a bounded output

$$\|u\| \leq M_1 < \infty \Rightarrow \|y\| \leq M_2 < \infty$$

where $\|\cdot\|$ is a signal norm.

- This applies to any type of systems.
- Stability is a system property, i.e. it is realization-independent.

- Bounded input-bounded output (BIBO) stability for SISO systems

$$|u(t)| \leq M_1 < \infty, \ \forall t \geq 0 \quad \Rightarrow \quad |y(t)| \leq M_2 < \infty, \ \forall t \geq 0$$

Theorem (BIBO stability)

A SISO LTI system is BIBO stable if and only if

$$\int_0^\infty |h(t)| \mathrm{d}t \leq M < \infty$$

where $M \in \mathbb{R}^+$ and h is the impulse response function.

BIBO stability – 2

Proof:

$\Leftarrow$ Assume $\int_0^\infty |h(t)|dt \leq M < \infty$ and u is bounded, i.e. $|u(t)| \leq M_1 < \infty$, $\forall t \in \mathbb{R}_0^+$. Then

$$|y(t)| \leq |\int_0^\infty h(\tau)u(t-\tau)d\tau| \leq M_1 \int_0^\infty |h(\tau)|d\tau \leq M_1 \cdot M = M_2$$

$\Rightarrow$ (indirect) Assume $\int_0^\infty |h(\tau)|d\tau = \infty$, but the system is BIBO stable. Consider the **bounded** input:

$$u(t-\tau) = \text{sign}\ h(\tau) = \begin{cases} 1 & \text{if} \quad h(\tau) > 0 \\ 0 & \text{if} \quad h(\tau) = 0 \\ -1 & \text{if} \quad h(\tau) < 0 \end{cases}$$

Overview

Stability of nonlinear systems

- Consider the autonomous nonlinear system:

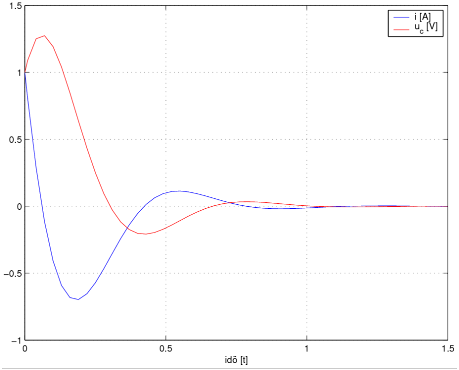
$$\dot{x} = f(x), \quad x \in \mathcal{X} = \mathbb{R}^n, \ f : \mathbb{R}^n \to \mathbb{R}^n$$

with an equilibrium point: $f(x^*) = 0$

  - x^* stable equilibrium point: for any $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for $\|x^* - x(0)\| < \delta$ $\|x^* - x(t)\| < \varepsilon$ holds.
  - x^* asymptotically stable equilibrium pint: x^* stable and $\lim_{t\to\infty} x(t) = x^*$.
  - x^* unstable equilibrium point: not stable
  - x^* locally (asymptotically) stable: there exists a neighborhood U of x^* within which the (asymptotic) stability conditions hold
  - x^* globally (asymptotically) stable: $U = \mathbb{R}^n$

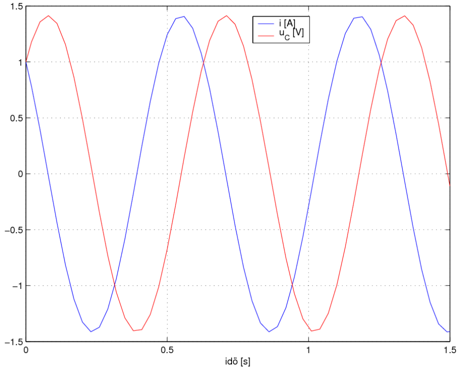
Example: asymptotic stability

RLC circuit, parameters: $R = 1\ \Omega$, $L = 10^{-1}H$, $C = 10^{-1}F$.
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



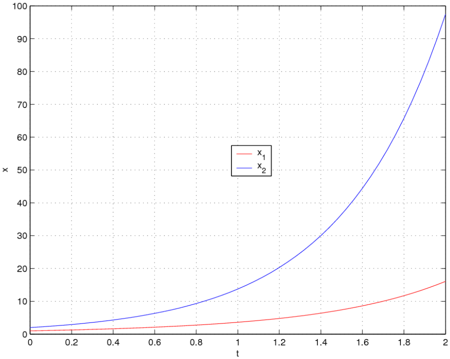
Non-asymptotic stability

(R)LC circuit, parameters: $R = 0\ \Omega(!)$, $L = 10^{-1}H$, $C = 10^{-1}F$.
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



Example: instability

$$\begin{aligned} \dot{x}_1 &= x_1 + 0.1x_2 \\ \dot{x}_2 &= -0.2x_1 + 2x_2 \end{aligned} , \quad x(0) = [12]^T$$



Overview

Stability of CT-LTI systems

- (Truncated) LTI state equation with $(u \equiv 0)$:

$$\dot{x} = A \cdot x, \quad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}, \; x(0) = x_0$$

- Equilibrium pont: $x^* = 0$
- Solution:

$$x(t) = e^{At} \cdot x_0$$

- Recall: A diagonalizable (there exists invertible T, such that

$$T \cdot A \cdot T^{-1}$$

is diagonal) if and only if, A has n linearly independent eigenvectors.

Asymptotic stability of LTI systems – 1

Stability types:

- the real part of every eigenvalue of A is negative (A is a *stability matrix*): **asymptotic stability**
- A has eigenvalues with zero and negative real parts
  - the eigenvectors related to the zero real part eigenvalues are linearly independent: **(non-asymptotic) stability**
  - the eigenvectors related to the zero real part eigenvalues are not linearly independent: **(polynomial) instability**
- A has (at least) an eigenvalue with positive real part: **(exponential) instability**

Asymptotic stability of LTI systems – 2

Theorem

The eigenvalues of a square $A \in \mathcal{R}^{nxn}$ matrix remain unchanged after a similarity transformation on A by a transformation matrix T:

$$A' = TAT^{-1}$$

Proof:

Let us start with the eigenvalue equation for matrix A

$$A\xi = \lambda\xi \ , \ \xi \in \mathcal{R}^n \ , \ \lambda \in \mathbb{C}$$

If we transform it using $\xi' = T\xi$ then we obtain

$$TAT^{-1}T\xi = \lambda T\xi$$

$$A'\xi' = \lambda\xi'$$

Asymptotic stability of LTI systems – 3

Theorem

A CT-LTI system is asymptotically stable iff A is a stability matrix.

Sketch of *Proof*: Assume A is diagonalizable

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

$$\bar{x}(t) = e^{\bar{A}t} \cdot \bar{x}_0 \quad , \quad e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}$$

BIBO and asymptotic stability

Theorem

Asymptotic stability implies BIBO stability for LTI systems.

Proof:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad y(t) = Cx(t)$$

$$\begin{aligned} ||x(t)|| \leq ||e^{At}x(t_0) + M \textstyle\int_0^t e^{A(t-\tau)}Bd\tau|| = \\ = ||e^{At}(x(t_0) + M \textstyle\int_0^t e^{-A\tau}Bd\tau)|| = \\ = ||e^{At}(x(t_0) + M[-A^{-1}e^{-A\tau}B]_0^t)|| = \\ = ||e^{At}[x(t_0) - MA^{-1}e^{-At}B + MA^{-1}B]|| \end{aligned}$$

$$||x(t)|| \leq ||e^{At}(x(t_0) + MA^{-1}B) - MA^{-1}B||$$

Overview

Lyapunov theorem of stability

- Lyapunov-function: $V : \mathcal{X} \to \mathbb{R}$
  - $V > 0$, if $x \neq x^*$, $V(x^*) = 0$
  - V continuously differentiable
  - V non-increasing, i.e. $\frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) \leq 0$

Theorem (Lyapunov stability theorem)

- *If there exists a Lyapunov function to the system $\dot{x} = f(x)$, $f(x^*) = 0$, then x^* is a stable equilibrium point.*
- *If $\frac{d}{dt}V < 0$ then x^* is an asymptotically stable equilibrium point.*
- *If the properties of a Lyapunov function hold only in a neighborhood U of x^*, then x^* is a locally (asymptotically) stable equilibrium point.*

Lyapunov theorem – example

- System:

$$\dot{x} = -(x-1)^3$$

- Equilibrium point: $x^* = 1$
- Lyapunov function: $V(x) = (x-1)^2$

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}\dot{x} = 2(x-1)\cdot(-(x-1)^3) =$$
$$= -2(x-1)^4 < 0$$

- The system is **globally asymptotically stable**

CT-LTI Lyapunov theorem – 1

Basic notions:

- $Q \in \mathbb{R}^{n\times n}$ **symmetric matrix**: $Q = Q^T$, i.e. $[Q]_{ij} = [Q]_{ji}$ (every eigenvalue of Q is real)
- symmetric matrix Q is **positive definite** $(Q > 0)$: $x^T Qx > 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is positive)
- symmetric matrix Q is **negative definite** $Q < 0$: $x^T Qx < 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is negative)

Theorem (Lyapunov criterion for LTI systems)

The state matrix (A) of an LTI system is a stability matrix if and only if there exists a positive definite symmetric matrix P for every given positive definite symmetric matrix Q such that

$$A^T P + PA = -Q$$

CT-LTI Lyapunov theorem – 2

Proof:
$\Leftarrow$ Assume $\forall\ Q > 0\ \exists\ P > 0$ such that $A^TP + PA = -Q$. Let $V(x) = x^TPx$.

$$\frac{d}{dt}V = \dot{x}^TPx + x^TP\dot{x} = x^T(A^TP + PA)x < 0$$

$\Rightarrow$ Assume A is a stability matrix. Then

$$P = \int_0^\infty e^{A^Tt}Qe^{At}dt$$

$$A^TP + PA = \int_{\mathbf{0}}^\infty A^Te^{A^Tt}Qe^{At}dt + \int_{\mathbf{0}}^\infty e^{A^Tt}Qe^{At}Adt = [e^{A^Tt}Qe^{At}]_{\mathbf{0}}^\infty = \mathbf{0} - Q = -Q$$

Example: stability of RLC circuit – 1

Model ($x_1 = i_L,\ x_2 = u_C,\ u_{be} = 0,\ R = 1,\ C = 0.1,\ L = 0.05$):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigenvalues of A (roots of $\equiv b(s)$): $-10 \pm 10i$
$\Rightarrow$ the RLC circuit is asymptotically stable

Example: stability of RLC circuit – 2

Lyapunov function: sum of kinetic and potential energies

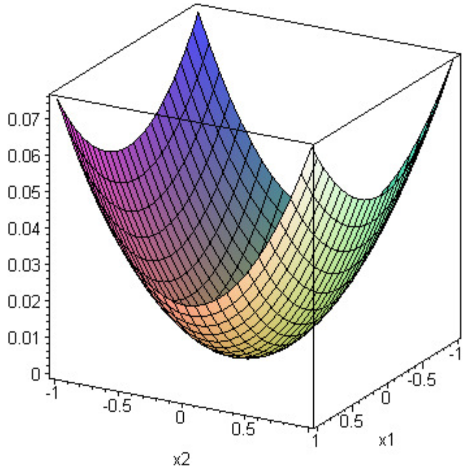
$$V(x) = \frac{1}{2}(Lx_1^2 + Cx_2^2) = \frac{1}{2}x^T \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} x$$

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}\dot{x} = \frac{1}{2}(\dot{x}^T Px + x^T P\dot{x}) = -Rx_1^2$$

the sum of energies is not increasing (decreasing if $x_1 \neq 0$ and $R > 0$) independently of the actual values of the parameters

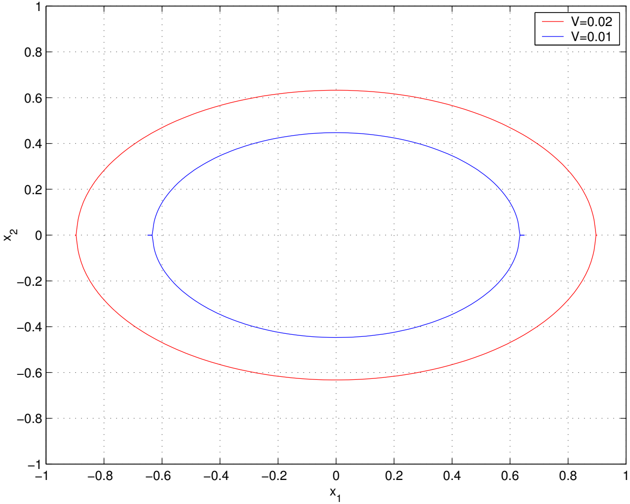
! the electric energy is preserved (is constant: $\frac{d}{dt}V = 0$), if $R = 0$.

Plot of the Lyapunov function:



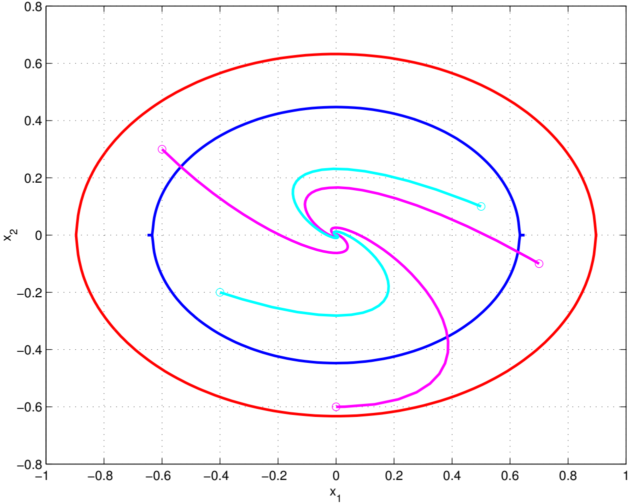
Example: stability of RLC circuit – 4

Level sets of the Lyapunov function (ellipses):



Example: stability of RLC circuit – 5

The solution of the ODE (voltages and currents) in the phase space:



Overview

Quadratic stability region

- Use quadratic Lyapunov function candidate with a given positive definite diagonal weighting matrix Q (tuning parameter!)

$$V\left[x(t)\right] = (x - x^*)^T \cdot Q \cdot (x - x^*)$$

- Dissipativity condition gives a conservative estimate of the stability region

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}\overline{f}(x)$$

  - $\overline{f}(x) = f(x)$ in the open loop case with $u = 0$
  - $\overline{f}(x) = f(x) + g(x) \cdot C(x)$ in the closed-loop case where $C(x)$ is the static state feedback

Quadratic stability region: an example - 1

- Nonlinear system

$$\begin{aligned}\dot{x}_1 &= 0.4x_1x_2 - 1.5x_1\\ \dot{x}_2 &= -0.8x_1x_2 - 1.5x_2 + 1.5u\\ y &= x_2\end{aligned}$$

- Equilibrium point with $u^* = 7.75$

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ 3.75 \end{bmatrix}$$

- Locally linearized system

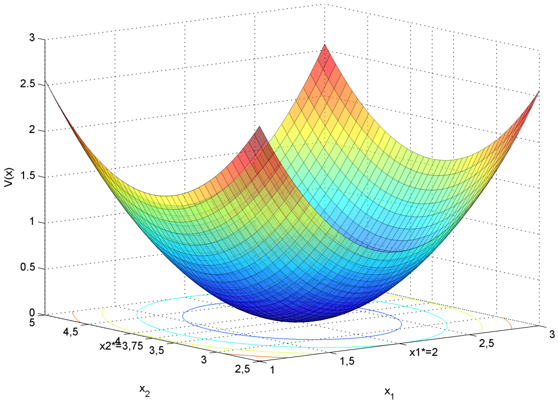
$$\begin{aligned}\dot{\tilde{x}} &= \begin{bmatrix} 0 & 0.8 \\ -3 & -3.1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \tilde{u}\\ \tilde{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{x}\end{aligned}$$

- Eigenvalues of the state matrix are $\lambda_1 = -1.5$ and $\lambda_2 = -1.6$ so equilibrium x^* (and not the whole system!) is locally asymptotically stable.

Quadratic stability region: an example - 2

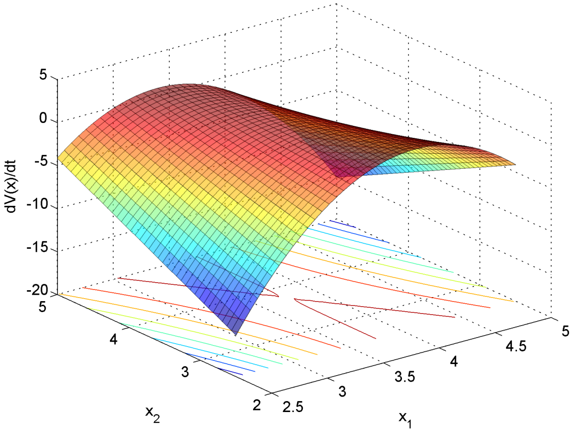
- Quadratic Lyapunov function

$$V(x) = (x - x^*)^T \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot (x - x^*)$$



Quadratic stability region: an example - 3

- Time derivative of the quadratic Lyapunov function



Computer Controlled Systems
Lecture 6

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, Oct. 25, 2018

Transfer functions and stability

SISO case: $H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)} =$

$$\frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{(s-\beta_1)(s-\beta_2)\ldots(s-\beta_m)}{(s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_n)}$$

- Zeros: $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{C}$
- Poles: $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ (identical to the eigenvalues of A)

Asymptotic stability $\Leftrightarrow$ $\mathrm{Re}(\lambda_i) < 0$

Routh's stability criterion – 1

$a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$

$$\begin{array}{ccccc} a_0 & a_2 & a_4 & a_6 & \dots \\ a_1 & a_3 & a_5 & a_7 & \dots \\ \frac{a_1 a_2 - a_0 a_3}{a_1} & \frac{a_1 a_4 - a_0 a_5}{a_1} & \frac{a_1 a_6 - a_0 a_7}{a_1} & \dots & \\ \dots & & & & \\ a_n & \dots & & & \end{array}$$

Routh-coefficients: $R_0 = a_0,\ R_1 = a_1,\ R_2 = \frac{a_1 a_2 - a_0 a_3}{a_1},\ \dots,\ R_n = a_n$.
(elements of the first column)

Routh's stability criterion – 2

number of sign changes in the column of coefficients = number of roots with positive real part (unstable)
necessary and sufficient condition for stability: $R_i > 0,\ i = 0, \ldots, n$.
Example: $a(s) = s^3 + s^2 + 3s + 10$.
$R_0 = 1,\ R_1 = 1,\ R_2 = -7,\ R_3 = 10 \Rightarrow$ 2 roots with positive real parts (unstable system)
Remarks:

- necessary condition for stability (not sufficient for polynomials with degree greater than 2): all coefficients a_i are positive
- in the case of purely imaginary root(s), zero(s) appear among the coefficients

$$W = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & & 0 \\ 0 & a_0 & a_2 & a_4 & a_6 & \dots & 0 \\ \dots & & & & & & 0 \\ 0 & 0 & 0 & \dots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{n-4} & a_{n-2} & a_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

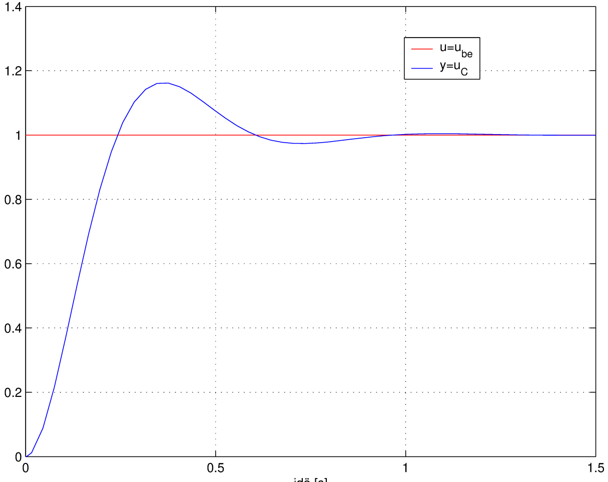
Minors: $H_1, H_2, \dots, H_n$.

Hurwitz's stability criterion – 2

- necessary and sufficient condition for stability: $H_i > 0$, $i = 1, \ldots, n$
- 0 minor: imaginary root
- negative minor: root with positive real part
- relation between Routh- and Hurwitz-coefficients: $R_i = \frac{H_i}{H_{i-1}}$, $H_0 = 1$.

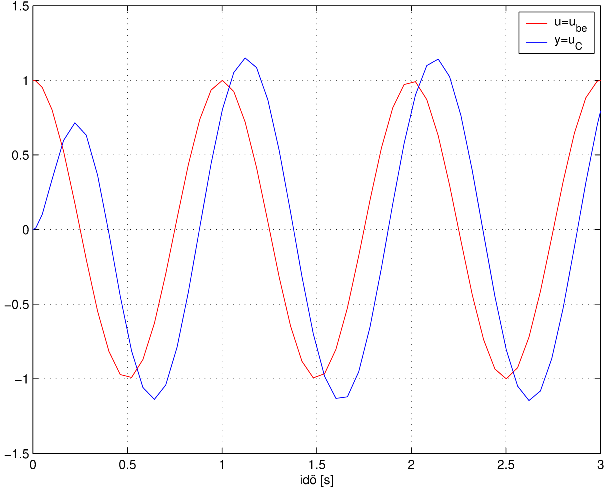
Example: RLC circuit – 1

$R = 1\Omega,\ L = 0.1H,\ C = 0.1F,\ x(0) = [0\ \ 0]^T,\ y = u_C,$
$u(t) = u_{be} = \cos(\omega \cdot t)$
$\omega = 0$ rad/s $= 0$ Hz



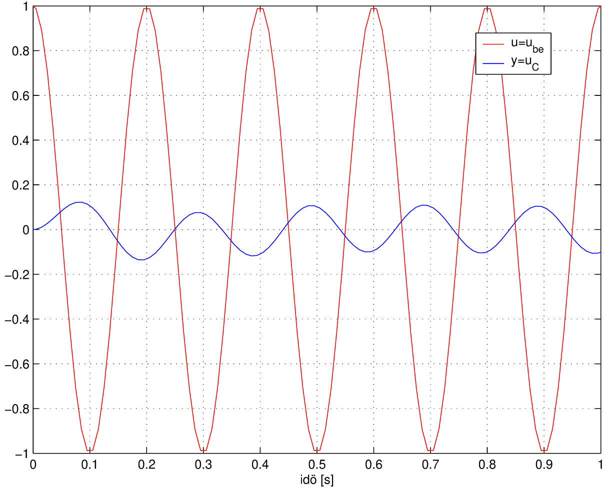
Example: RLC circuit – 2

$\omega = 2\Pi$ rad/s $= 1$ Hz



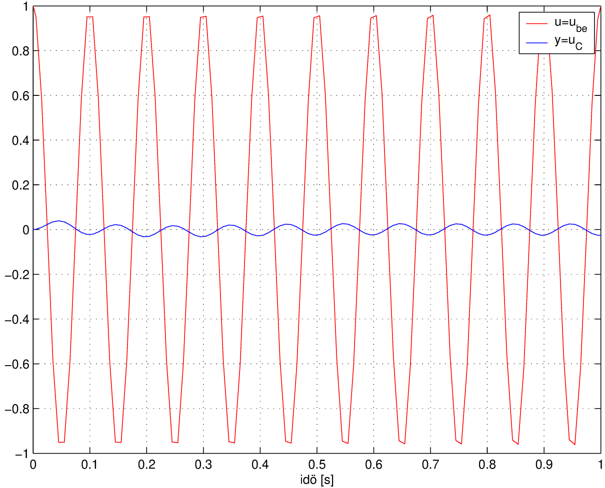
Example: RLC circuit – 3

$\omega = 5 \cdot 2\Pi$ rad/s $= 5$ Hz



Example: RLC circuit – 4

$\omega = 10 \cdot 2\Pi$ rad/s $= 10$ Hz



Fourier- and Laplace-transforms

Revision: $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$
Fourier-transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \quad \omega \in \mathbb{R}$$

Laplace-transform:

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt, \quad s \in \mathbb{C}$$

Assume that s is on the imaginary axis. Then: $s \longleftrightarrow j\omega$

Frequency response function

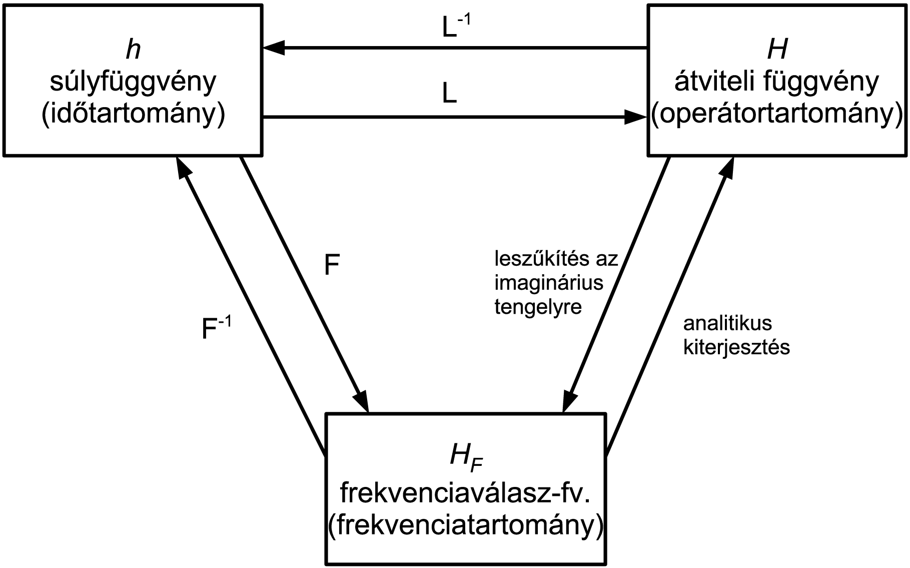
Transfer function: $H(s)$
Definition: $H_F(\omega) = H(j\omega)$ (frequency response function)
Then H_F is the Fourier-transform of the impulse response function (h) since:

$$H_F(\omega) = \int_0^{\infty} h(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

H_F is the restriction of H to the imaginary axis
Question: Can we compute H from the restriction on the complex plane, where the Laplace-transform is defined?
Answer: Using the fact that the transfer function is *analytic*, the computation is the following, if the poles of H are on the left half-plane:

$$H(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H_F(\omega)}{s - i\omega} d\omega$$

Time- operator- and frequency-domains



Response of stable LTI systems to periodic inputs

Theorem: Let $H(s)$ be the transfer function of an asymptotically stable LTI system, and $\omega > 0$. Then the response of the system to the input $u(t) = u_0 \sin(\omega t)$ is of the following form:

$$y(t) = u_0 \mathrm{Re}(H_F(\omega)) \sin(\omega t) + u_0 \mathrm{Im}(H_F(\omega)) \cos(\omega t)$$

(we do not prove)
Remarks:

- It is visible that the output is also periodic with a period $T = \frac{2\pi}{\omega}$ equal to the period of the input.
- The theorem is still valid if the transfer function has purely imaginary poles of the form $i\hat{\omega}$, but $\omega/\hat{\omega} \notin \mathbb{Z}$.

Response of stable LTI systems to periodic inputs

transfer function: $G(j\omega)$, $\quad (G(s))$

$$u(t) = u_0 \sin(\omega t + \alpha)$$

$$y(t) \longrightarrow y_0 \sin(\omega t + \beta)$$

gain: $k = \left|\frac{y_0}{u_0}\right| = |G(j\omega)|$ (frequency dependent!)
phase: $\phi = \beta - \alpha = \angle G(j\omega[\text{rad}])$ (frequency dependent!)
E.g. let $G(j\omega) = a + bj$

$$|G(j\omega)| = \sqrt{(a^2 + b^2)}, \;\; \angle G(j\omega) = \arctan(b/a)$$

Gain in time and frequency domains

$$u(t) = a_0 \sin(\omega t), \quad y(t) = a_1 \sin(\omega t + \phi)$$

$$U(s) = \frac{a_0 \omega}{s^2 + \omega^2}, \quad Y(s) = \frac{a_1(s \sin(\phi) + \omega \cos(\phi))}{s^2 + \omega^2}$$

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|U(j\omega)|} = \left| \frac{a_1(j\omega \sin(\phi) + \omega \cos(\phi))}{a_0 \omega} \right| = \left| \frac{a_1}{a_0} \right|$$

$$\angle G(j\omega) = \arctan\left(\frac{\omega \sin(\phi)}{\omega \cos(\phi)}\right) = \phi$$

Example: RLC circuit – 5

Transfer function: $C(sI - A)^{-1}B = \frac{100}{s^2+10s+100} = \frac{100}{(j\omega)^2+10(j\omega)+100}$

- $f = 0$ Hz, $\omega = 0$ rad/s, $G(j\omega) = 1 + 0j$, $|G(j\omega)| = 1$, $\phi = 0$ rad
- $f = 1$ Hz, $\omega = 6.2832$ rad/s, $G(j\omega) = 0.7952 - 0.8256j$, $|G(j\omega)| = 1.1463$, $\phi = -0.8041$ rad
- $f = 5$ Hz, $\omega = 31.4159$ rad/s, $G(j\omega) = -0.1002 - 0.0355j$, $|G(j\omega)| = 0.1063$, $\phi = 0.3404$ rad
- $f = 10$ Hz, $\omega = 62.8319$ rad/s, $G(j\omega) = -0.0253 - 0.004j$, $|G(j\omega)| = 0.0256$, $\phi = 0.1619$ rad

Gain of transfer functions

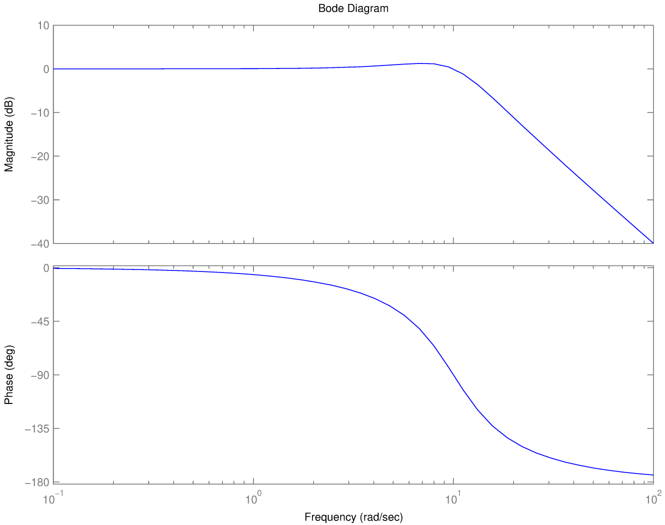
$$A = \left| \frac{y_0}{u_0} \right|$$

in dB:

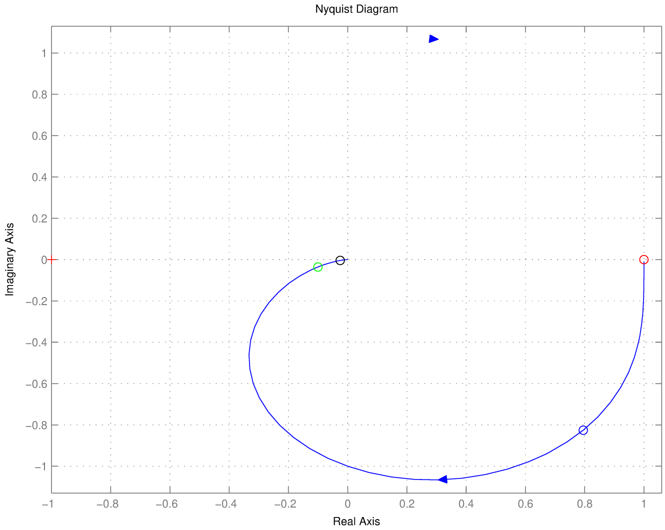
$$A_d = 20 \cdot \log_{10}(A) \text{ [dB]}$$

- $|G(j\omega)| = 1$, $A_d = 0$ dB
- $|G(j\omega)| = 1.1463$, $A_d = 1.1860$ dB
- $|G(j\omega)| = 0.1063$, $A_d = -19.4693$ dB
- $|G(j\omega)| = 0.0256$, $A_d = -31.8352$ dB

Bode-diagram



Nyquist-diagram



Bandwidth of SISO systems

Bandwidth: Frequency, where $|G(j\omega)|$ first crosses the value $1/\sqrt{2}$ (≈ -3 dB) from above

Example: RLC circuit

$y = u_C$, $\omega_c \approx 2.03$ Hz

Transfer function of MIMO systems

$u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$

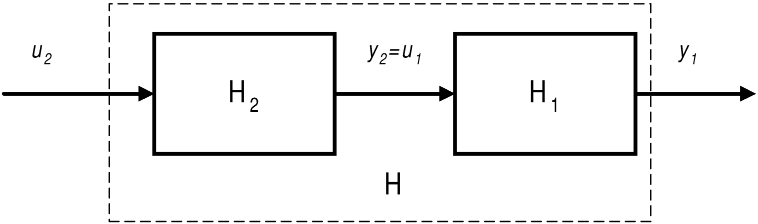
$$Y(s) = H(s)U(s),$$

$$H(s) = \begin{bmatrix} h_{11}(s) & \dots & h_{1m}(s) \\ & & \\ h_{r1}(s) & \dots & h_{rm}(s) \end{bmatrix} \in \mathbb{C}^{r \times m}$$

Pl. RLC-circuit, $u = u_{in}, y = [i \;\; u_C]^T$

$$H(s) = \begin{bmatrix} \frac{10s}{s^2+10s+100} \\ \frac{100}{s^2+10s+100} \end{bmatrix}$$

Serial interconnection of subsystems

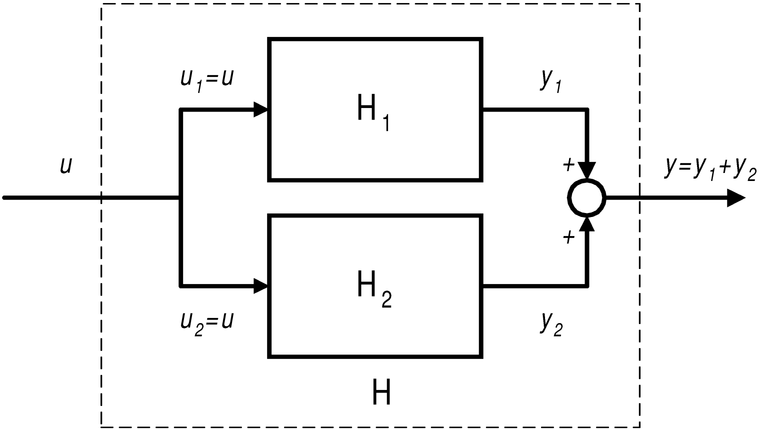


$$H(s) = H_1(s) \cdot H_2(s)$$

i.e.

$$h(t) = (h_1 * h_2)(t)$$

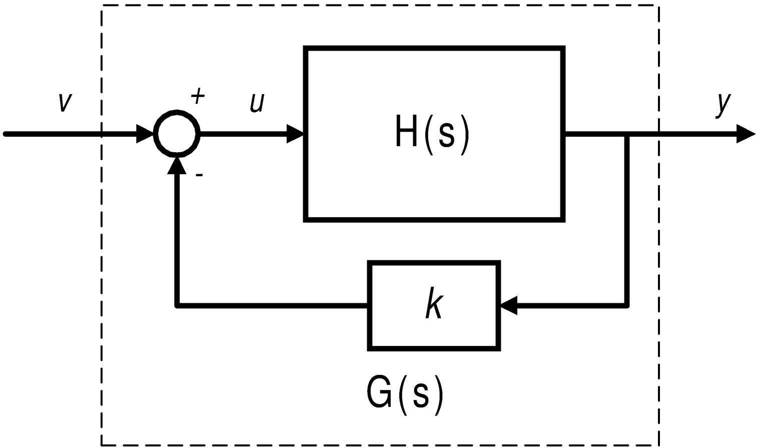
Parallel interconnection of subsystems



$$H(s) = H_1(s) + H_2(s)$$

$$h(t) = h_1(t) + h_2(t)$$

Proportional negative feedback



$$G(s) = \frac{H(s)}{1 + k \cdot H(s)}$$

Negative feedback – example

Original system:

$$H(s) = \frac{1}{s-1}, \quad \text{(unstable)}$$

Feedback system:

$$G(s) = \frac{1}{s+k-1}$$

stable, if $k > 1$

High gain output feedback

$$H(s) = \frac{b(s)}{a(s)}$$

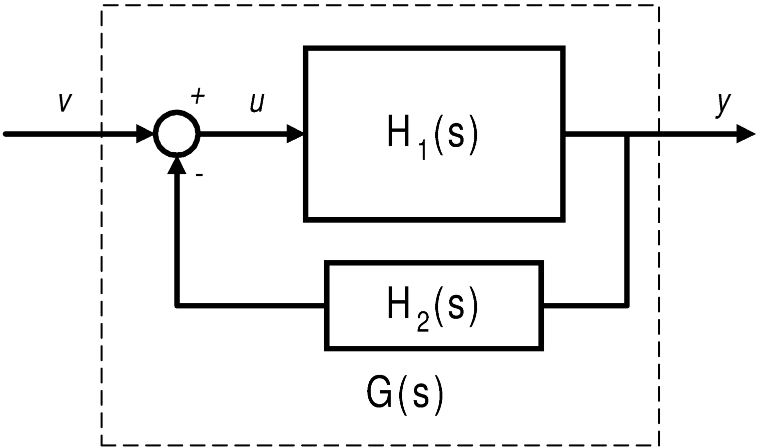
Transfer function of the feedback system:

$$G(s) = \frac{b(s)}{a(s) + k \cdot b(s)} = \frac{n(s)}{d(s)}$$

For $k \to \infty$, $d(s) \to b(s)$, i.e. by increasing the feedback gain, the poles of the feedback system converge to the zeros of the original system.

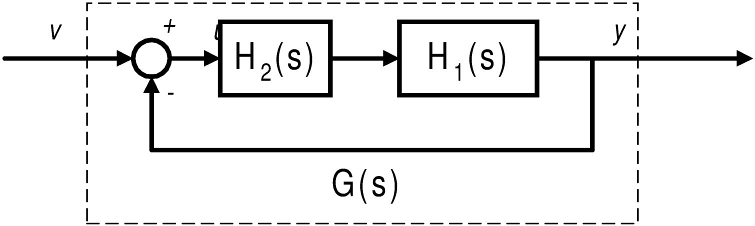
Minimum phase systems: Such systems where the real part of each zero is negative. (They can be stabilized by high gain feedback.)

General negative feedback – 1



$$G(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

General negative feedback – 2



$$G(s) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)}$$

Summary

- SISO transfer functions (TFs) are complex numbers (with absolute value and angle) at any given s
- frequency domain interpretation: assuming periodic (sinusoidal) input, $s = j\omega$
- absolute value of TF: gain (ratio of O/I amplitudes) at a given frequency
- angle of TF: phase shift at a given frequency
- visualization: Bode diagram, Nyquist diagram
- overall transfer functions were computed for different basic interconnection of subsystems

Computer Controlled Systems
Lecture 7

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, Nov. 07, 2018

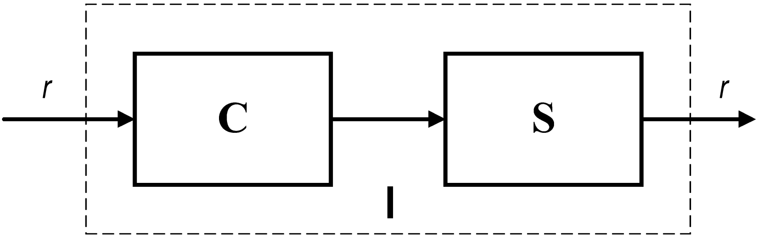
Contents

1. Introduction into the control of (SISO) systems

2. PID-control

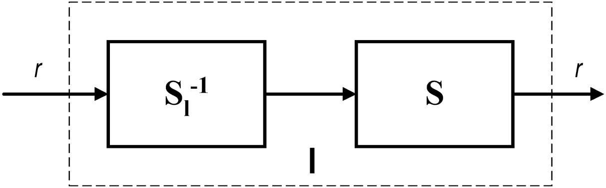
The control goal

Goal: the output of the system is *identical* to the prescribed reference signal. ("Everything is under control")
Straightforward approach: Let us transform the system operator into the identity operator (the output is exactly the same as the input)

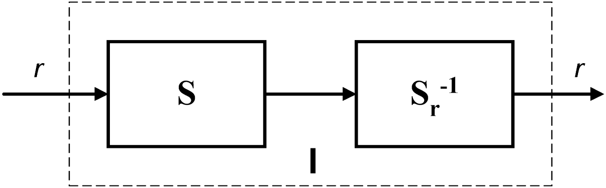


Left and right inverse (MIMO case)

Left inverse:



Right inverse:

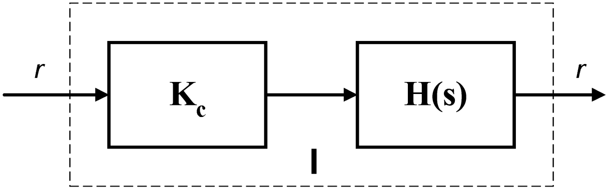


Inversion problems

- The system operator is not invertible
- The system to be controlled is unstable
- The inverse is unstable
- The inverse is not causal (not computable)
- The system operator is uncertain $\rightarrow$ the inverse (might be) even more uncertain
- The system is not isolated in reality (there are external disturbances)

Setting the steady state gain

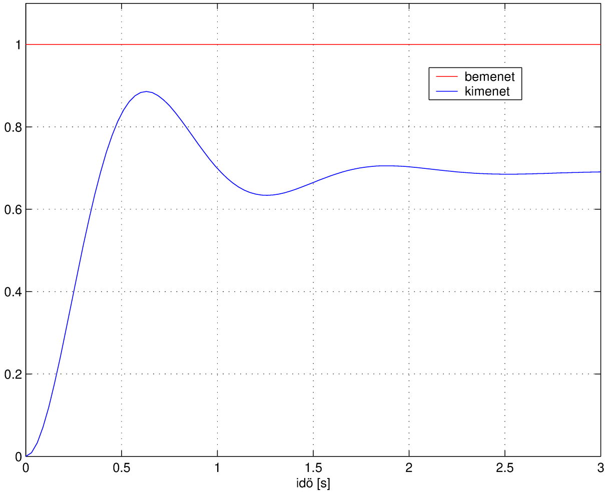
Assumption: a *stable* SISO transfer function is given
Goal: the output of the "controlled" system should asymptotically follow the constant reference signal (the gain should be 1 at frequency 0)



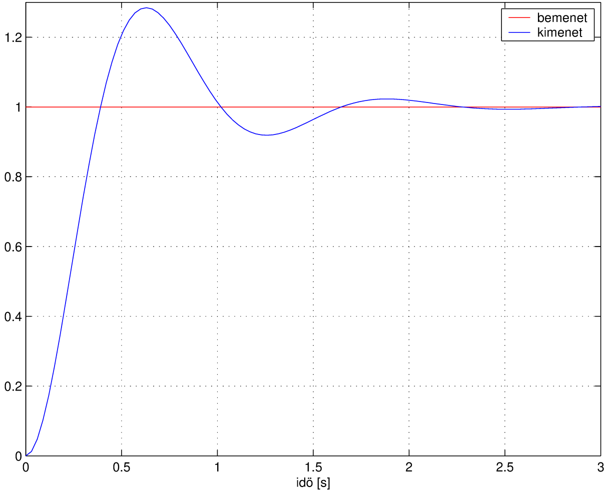
$|H(j \cdot 0)| = k \Rightarrow K_c = 1/k$

Example – 1

$H(s) = \frac{20}{s^2+4s+29}$, $|H(0)| = \frac{20}{29}$



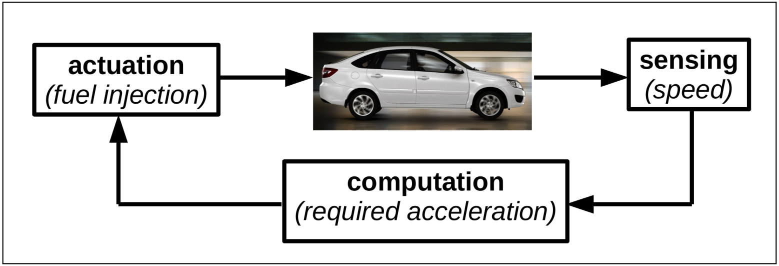
$K_c = \frac{29}{20}$



Feedback – 1

Feedback control:
control goal + sensing + feedback computation + actuation

Example: tracking a (constant) speed reference



may fundamentally change the behaviour (dynamical properties) of the original system

Feedback – 2

Why to apply?

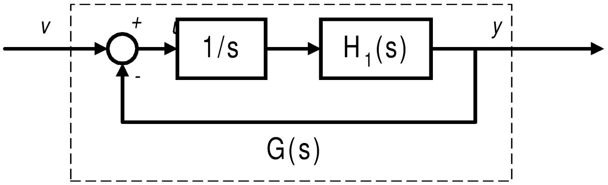
- Often feedback is the only way to stabilize unstable systems
- A well-designed feedback might operate well even with an uncertain system model
- The effect of external disturbances can also be reduced by feedback

Feedback – 3

Types of feedback

- *output feedback*: the input only depends on the outputs of the system, i.e. $u = \mathbf{F}[y]$
- *(full) state feedback*: the input depends on the state variables of the system, i.e. $u = \mathbf{F}[x]$
- *static feedback*: the $\mathbf{F}$ operator is static ($u = F(y)$, $u = F(x)$)
- *dynamic feedback*: the $\mathbf{F}$ operator is dynamic (can be given by a state space model or a transfer function)
- *linear feedback*: the $\mathbf{F}$ operator or the F function is linear

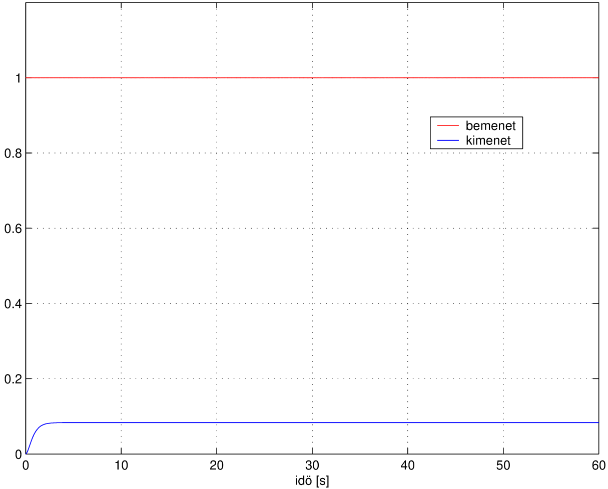
Role of the integrator



$H_1(s) = \frac{b(s)}{a(s)} \Rightarrow G(s) = \frac{k_I \cdot b(s)}{s \cdot a(s) + k_I \cdot b(s)}$
$|G(j \cdot 0)| = 1$
The steady state gain of a stable controlled system containing an integrator is 1.
(The controlled system follows the constant reference signal, if it is asymptotically stable.)

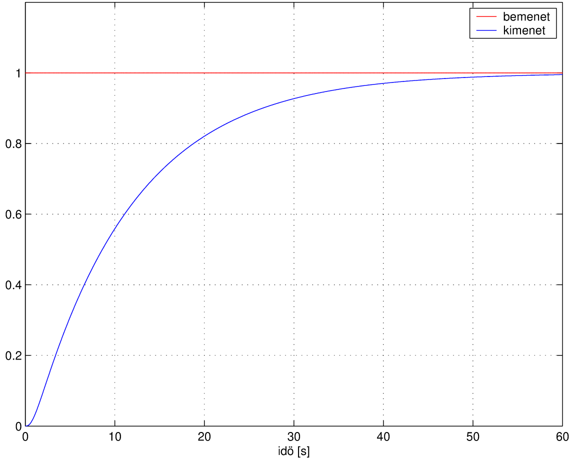
Example – 1

System model: $H(s) = \frac{0.5}{s^2+5s+6}$
Response for a unit step input:



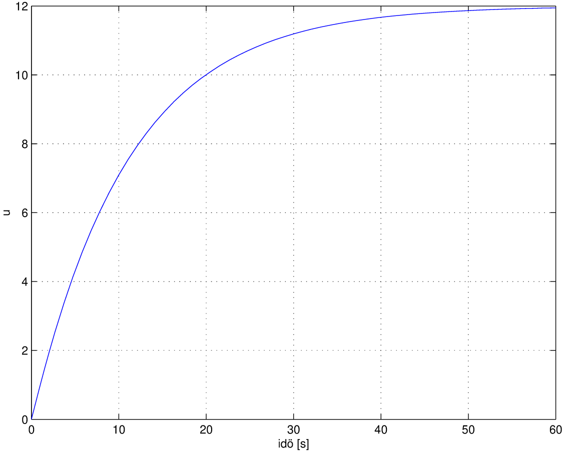
Example – 2

Controlled system containing an integrator ($k_I = 1$): $G(s) = \frac{0.5}{s^3+5s^2+6s+0.5}$
Response to a unit step input:

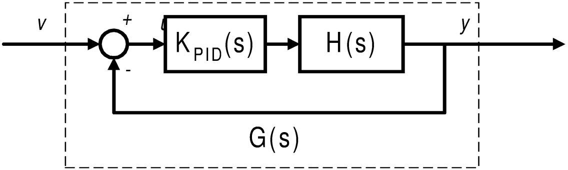


Example – 3

output of the integrator ≡ input of the original system:



Structure of PID controllers – 1

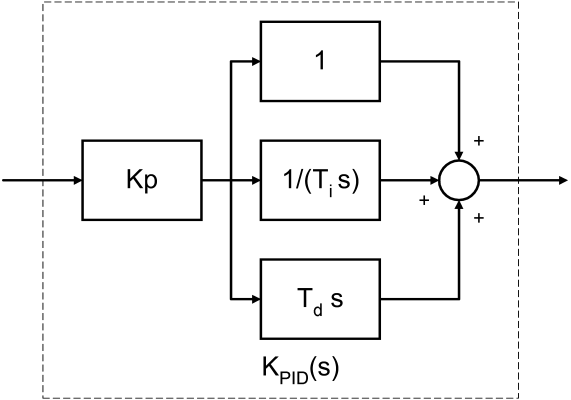


P=Proportional, **I**=Integral, **D**=Derivative

Transfer function:

$$K_{PID}(s) = K_p \left[1 + \frac{1}{T_i \cdot s} + T_d \cdot s\right] = \frac{K_p(T_i \cdot T_d \cdot s^2 + T_i \cdot s + 1)}{T_i \cdot s}$$

Structure of PID controllers – 2

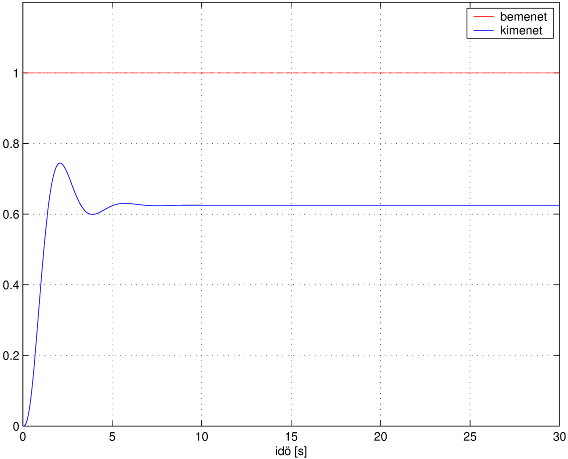


K_p: proportional gain
T_i: integration time constant
T_d: derivation time constant

PID design example – 1

System model: $H(s) = \frac{10}{s^3+6s^2+11s+16}$

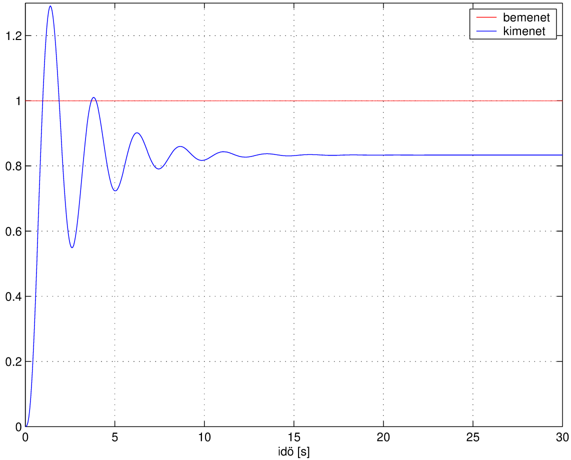
Step response



PID design example – 2

Proportional (P) feedback: $K_p = 3$, $G(s) = \frac{30}{s^3+6s^2+11s+36}$

Unit step response

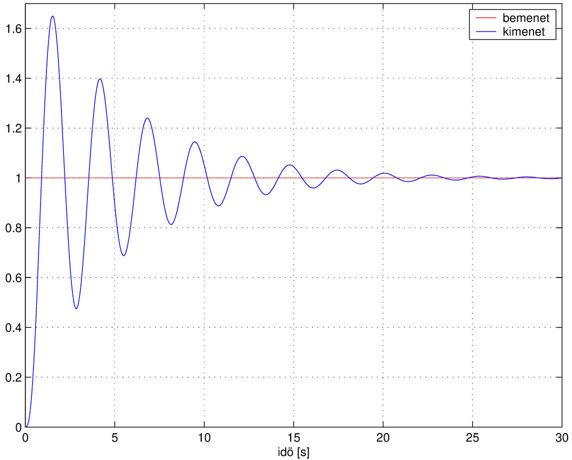


PID design example – 3

Proportional + integrator (PI) feedback: $K_p = 2.7$, $T_i = 1.5$,

$G(s) = \frac{40.5s+27}{1.5s^4+9s^3+16.5s^2+49.5s+27}$

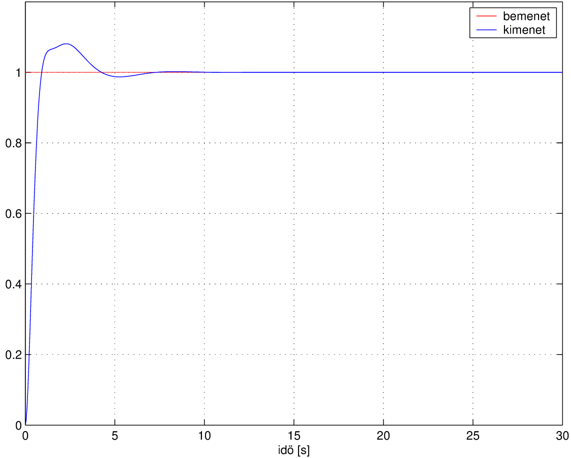
Unit step response



PID design example – 4

Proportional + integrator + derivator (PID) feedback: $K_p = 2$, $T_i = 0.9$, $T_d = 0.6$, $G(s) = \frac{10.8s^2+18s+20}{0.9s^4+5.4s^3+20.7s^2+23.4s+20}$

Unit step response



Tuning of PID controllers – 1

Ziegler-Nichols method

1. Apply a simple proportional feedback
2. Increase the proportional gain (K_p) until the step response becomes an undamped (sinusoidal) oscillation. The critical gain is K_p^*.
3. Measure the period of the oscillation (T_c)

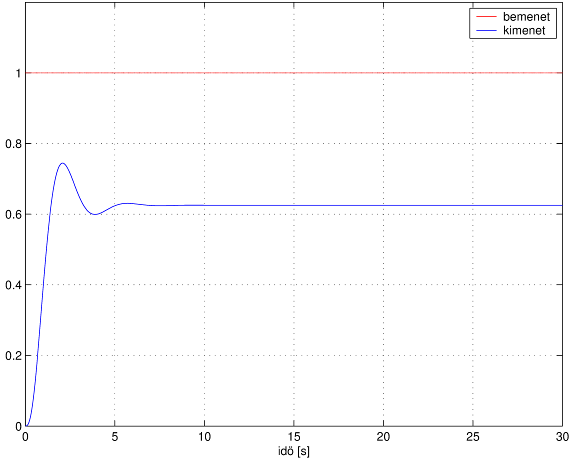
Tuning of PID controllers – 2

Controller tuning:

- **P-controller**: $K_p = 0.5K_p^*$
- **PI-controller**: $K_p = 0.45K_p^*,\ T_i = 0.833T_c$
- **PID-controller (fast)**: $K_p = 0.6K_p^*,\ T_i = 0.5T_c,\ T_d = 0.125T_c$
- **PID-controller (small overshoot)**: $K_p = 0.33K_p^*,\ T_i = 0.5T_c,\ T_d = 0.33T_c$
- **PID-controller (without overshoot)**: $K_p = 0.2K_p^*,\ T_i = 0.3T_c,\ T_d = 0.5T_c$

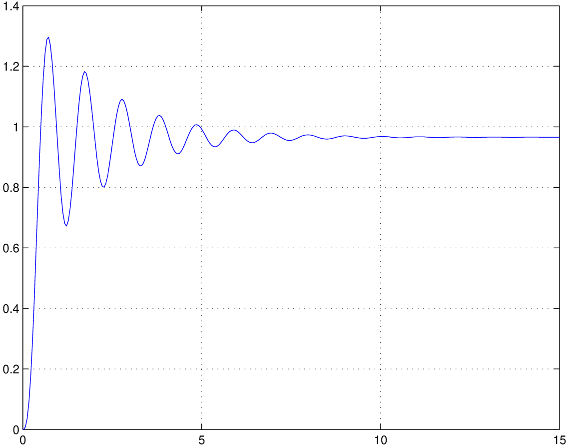
Example – 1

System model: $H(s) = \frac{40}{2s^3+10s^2+82s+10}$
Step response:



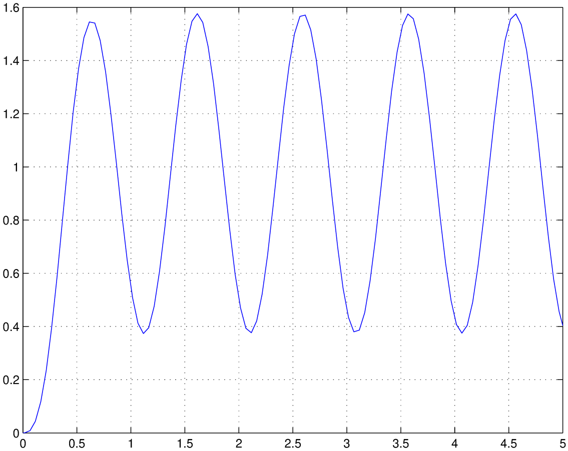
Example – 2

Proportional feedback, $K_p = 7$
Step response:



Example – 3

Proportional feedback, $K_p^* = 10$, $T_c = 1$
Step response:



Example – 4

PID controller parameters: $K_p = 3.3,\ T_i = 0.5,\ T_d = 0.33$
Controller transfer function:

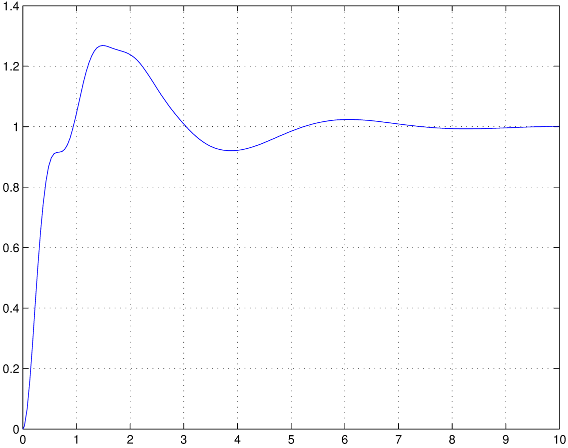
$$K_{PID}(s) = \frac{K_p(T_i \cdot T_d \cdot s^2 + T_i \cdot s + 1)}{T_i \cdot s}$$

Transfer function of the closed loop system:

$$G(s) = \frac{21.78s^2 + 66s + 132}{s4 + 5s^3 + 62.78s^2 + 71s + 132}$$

Example – 5

Step response of the controlled system



Example: DC motor – 1

System equations, parameters and variables:

J	moment of inertia	0.01 kg m^2/s^2
b	damping coefficient	0.1 Nm s
K	electromotive torque coefficient	0.01 Nm/A
R	resistance	1 ohm
L	inductance	0.5 H

state variables, input, output:

$x_1 = \dot{\theta}$ angular velocity [rad/s]

$x_2 = i$ current [A]

u input voltage [V]

$y = x_1$

Example: DC motor – 2

State space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{b}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

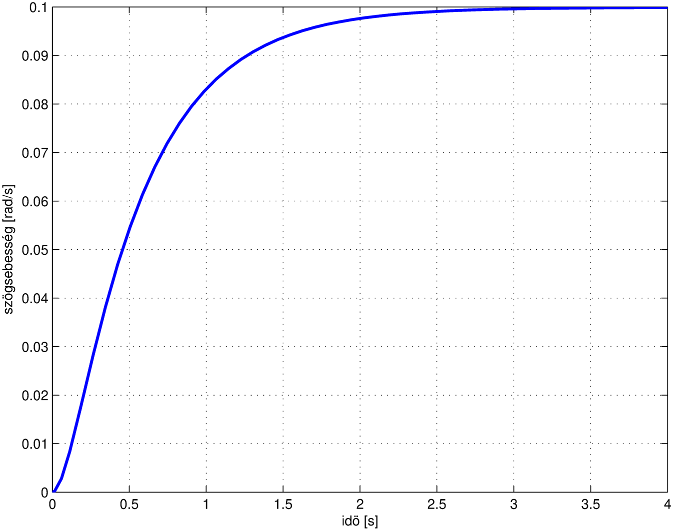
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K}{(Js + b)(Ls + R) + K^2}$$

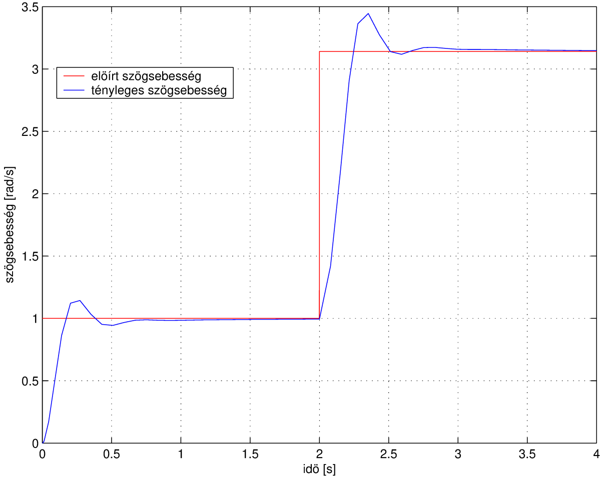
Example: DC motor – 3

Response to $u = 1$V input:



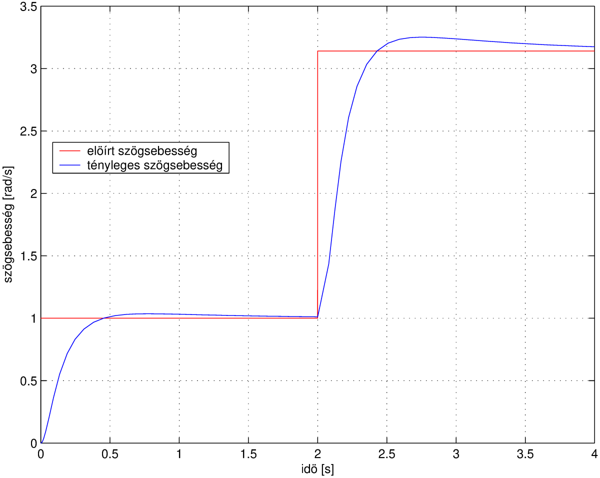
Example: DC motor – 4

PID-parameters: $K_p = 100$, $T_i = 1/100$, $K_d = 1$
Operation of the controlled system:



Example: DC motor – 5

PID-parameters: $K_p = 100$, $T_i = 1/100$, $K_d = 10$
Operation of the controlled system:



Evaluation of SISO control systems – 1

Time domain, unit step response

- e_{max}: maximal overshoot
- t_{max}: time of maximal overshoot
- T_a ($T_{a,50}$): rise time
- T_u: delay
- t_ϵ: settling time

Evaluation of SISO control systems – 2

Time domain, measuring the difference from the reference

- $I_1 = \int_0^\infty e(t)dt$
- $I_2 = \int_0^\infty |e(t)|dt$
- $I_3 = \int_0^\infty e^2(t)dt$
- $I_4 = \int_0^\infty [e^2(t) + \alpha \dot{e}^2(t)]dt$
- $I_5 = \int_0^\infty [e^2(t) + \beta u^2(t)]dt$

Summary

- typical control goals: output reference following (tracking), stabilization, disturbance rejection
- inversion: important theoretical concept, typically not directly implementable
- feedback: helps to achieve several control goals
- classification of feedback types is important
- static output feedback is often not enough even for stabilization
- PID control: frequently used dynamic output feedback with only 3 parameters
- evaluation criteria: help in comparison, acceptance decision

Computer Controlled Systems
Lecture 8

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, Nov. 15, 2018

Outline

General problem statement

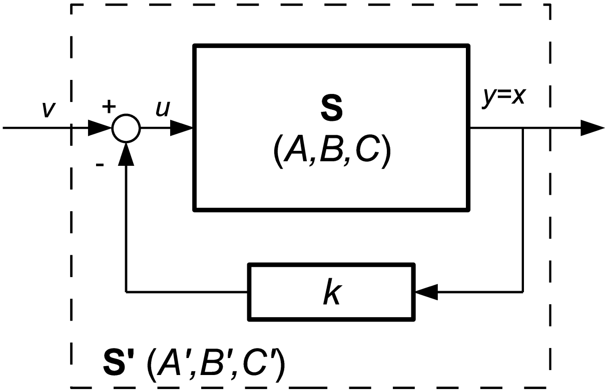
Given:

- a ***SISO LTI*** system with matrices (A, B, C).
  The poles depend on A (on $a(s)$).
- prescribed (expected) poles defined by polynomial $\alpha(s)$, such that
  $\deg\ a(s) = \deg\ \alpha(s) = n$

To be computed:
a ***full state feedback*** such that the poles of the closed loop system will be the roots of $\alpha(s)$.
Subproblem: feedback design, which can stabilize an otherwise unstable system.

Closed loop LTI system – 1

Static linear (full) state feedback:



$$u = -kx + v,$$

where $k \in \mathbb{R}^{r\times n}$, ha $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$

Closed LTI system – 2

The matrices of the SISO system are (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$
$$y(t)\ ,\ u(t) \in \mathbb{R} \quad , \quad x(t) \in \mathbb{R}^n$$
$$A \in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times 1}\ ,\ C \in \mathbb{R}^{1\times n}$$

static linear full state feedback

$$v = u + kx \quad (u = v - kx)$$
$$k = \left[\ k_1 \quad k_2 \quad . \quad . \quad . \quad k_n\ \right]$$
$$k \in \mathbb{R}^{1\times n} \qquad \text{(row vector)}$$

Closed LTI system – 3

Closed loop system

$$\dot{x}(t) = (A - Bk)x(t) + Bv(t)$$
$$y(t) = Cx(t)$$

Namely:

$$A' = A - B \cdot k, \quad B' = B, \quad C' = C$$

Characteristic polynomials
Without feedback (uncontrolled system):

$$a(s) = \det(sI - A)$$

Closed loop system (controlled) system:

$$a_c(s) = \det(sI - A + Bk)$$

Determinant of block matrices

Let us calculate the following determinant

$$\det \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

in two different (but equivalent) ways

$$\det(M_1)\det(M_4 - M_3 M_1^{-1} M_2) = \det(M_4)\det(M_1 - M_2 M_4^{-1} M_3)$$

We apply:

$$\det \begin{bmatrix} sI - A & B \\ -k & 1 \end{bmatrix}$$

then we obtain the following:

$$\det(sI - A)\det(1 + k(sI - A)^{-1}B) = 1 \cdot \det((sI - A) + B \cdot 1^{-1} \cdot k)$$

Resolvent formula

$$a(s) = s^n + a_1 s^{n-1} + \dots + a_n$$

$$(sI - A)^{-1} = \frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \dots)$$

Proof:

$$(sI - A)(sI - A)^{-1} =$$

$$(sI - A)\frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \dots) =$$

$$= \frac{1}{a(s)}\left[s^n I \underbrace{- s^{n-1}A + s^{n-1}A}_{0} + a_1 s^{n-1} I - s^{n-2}A^2 - s^{n-2}a_1 A + \dots\right] =$$

$$\frac{a(s)}{a(s)} I = I$$

Pole placement – 1

$$det(sI - A) \cdot det(1 + k(sI - A)^{-1}B) = 1 \cdot det((sI - A) + B \cdot 1^{-1} \cdot k)$$

$$a(s)(1 + k(sI - A)^{-1}B) = det(sI - A + Bk)$$

$$\alpha(s) = a(s)(1 + k(sI - A)^{-1}B) \quad \Rightarrow \quad \alpha(s) - a(s) = a(s)k(sI - A)^{-1}B$$

Using the ***resolvent formula***

$$(sI - A)^{-1} = \frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \ldots$$

we obtain that

$$\begin{aligned}(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + \ldots(\alpha_n - a_n) &= \\ = kBs^{n-1} + k(A + a_1 I)Bs^{n-2} + \ldots\end{aligned}$$

Pole placement – 2

$$(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + \ldots(\alpha_n - a_n) = kBs^{n-1} + k(A + a_1 I)Bs^{n-2} + \ldots$$

polynomial equation

$$\begin{aligned}
\alpha_1 - a_1 &= kB \\
\alpha_2 - a_2 &= kAB + a_1 kB = a_1 kB + kAB \\
\alpha_3 - a_3 &= kA^2B + a_1 kAB + a_2 kB = a_2 kB + a_1 kAB + kA^2 B \\
&. \\
&.
\end{aligned}$$

$$\underline{\alpha} - \underline{a} = k \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

Pole placement controller

$$\underline{\alpha} - \underline{a} = k \left[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array} \right] \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

$$\underline{\alpha} - \underline{a} = k\mathcal{C}T_\ell^T$$

If **S** is ***controllable*** then

$$k = (\underline{\alpha} - \underline{a})T_\ell^{-T}\mathcal{C}^{-1}$$

Controller form realization

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$
$$y(t) = C_c x(t)$$

where

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix}$$

The polynomials of the transfer function

$a(s) = s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n$ and $b(s) = b_1 s^{n-1} + ... + b_{n-1}s + b_n$

$H(s) = \frac{b(s)}{a(s)}$

Pole placement controller in case of a controller form

$$A_c - B_c k_c = \begin{bmatrix} -(a_1 + k_{c1}) & -(a_2 + k_{c2}) & . & . & . & -(a_n + k_{cn}) \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}$$

the characteristic polynomial of the closed loop system is $\alpha(s)$:

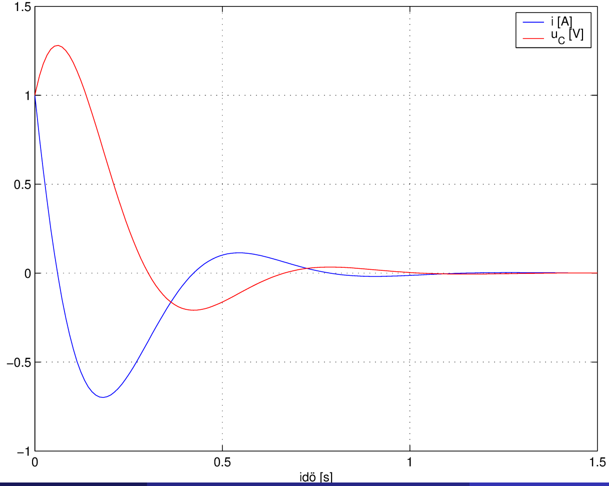
$$\alpha(s) = det(sI - (A_c - B_c k_c)) = s^n + (a_1 + k_{c1})s^{n-1} + ... + (a_n + k_{cn})$$

The coefficients k_c of the state feedback gain is

$$k_c = \underline{\alpha} - \underline{a}$$

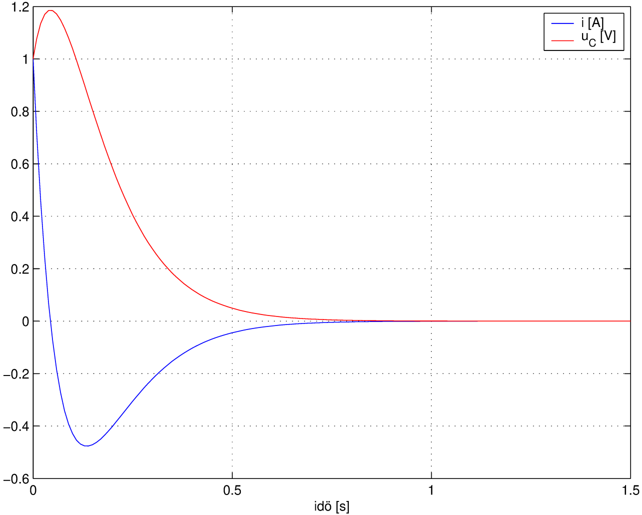
Example – 1

System: RLC circuit. Response of the uncontrolled (open loop) circuit with zero input ($u = 0V$) from initial state $x(0) = [1\ 1]^T$.
(Poles: $-5 \pm 8.6603i$)



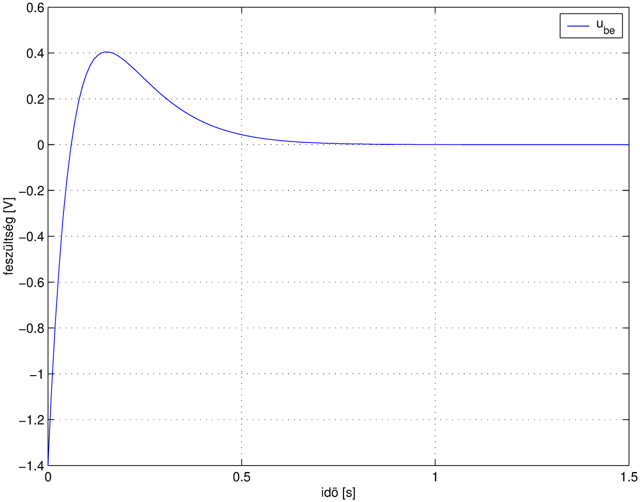
Example – 2

Prescribed poles of the closed loop system: -10, -12. Feedback gain: $k = [1.2 \;\; 0.2]$. Response for $x(0) = [1 \; 1]^T$:

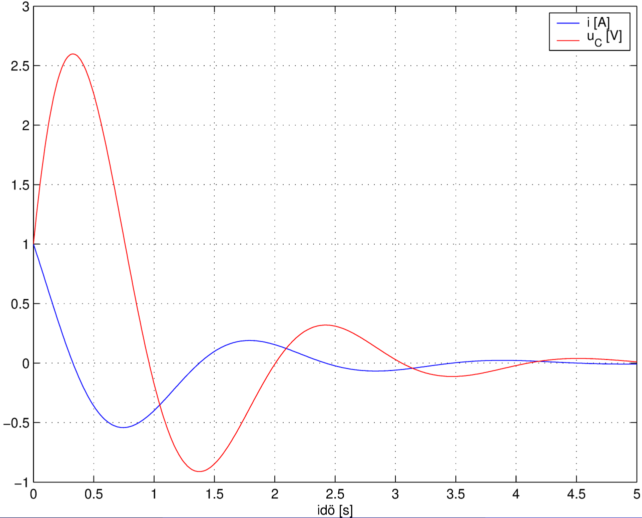


Example – 3

The necessary input for stabilizing control (voltage):

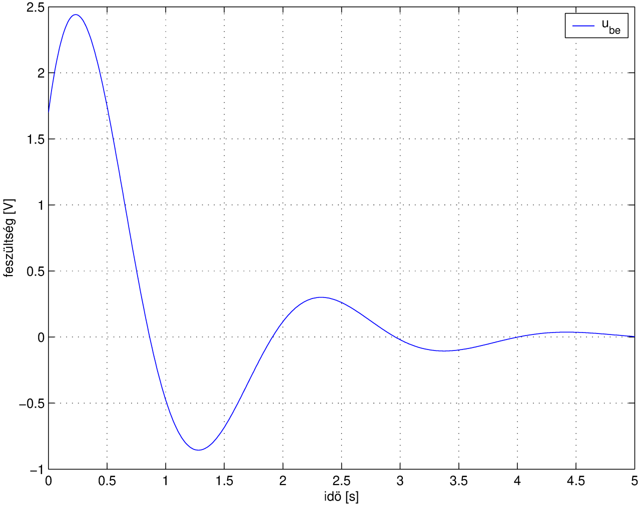


Prescribed poles of the closed loop system: $-1+3i$, $-1-3i$. Feedback gain: $k = [-0.8 \quad -0.9]$. Response:



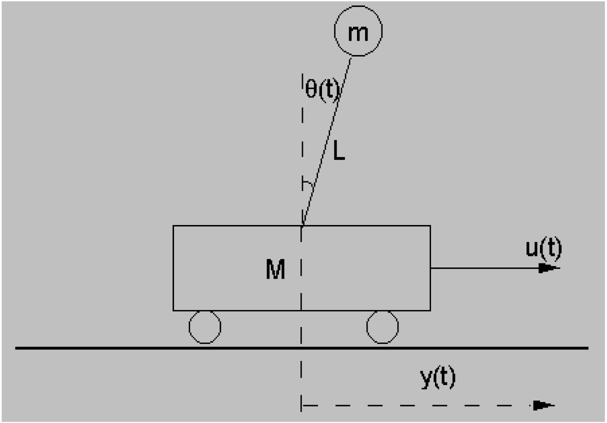
Example – 5

The necessary input for stabilizing control:



Example – 6

System: the inverted pendulum



Example – 7

State vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \qquad (1)$$

Equilibrium point: $x^* = [0\ 0\ 0\ 0]^T$
The linearized state-space model:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{ML} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{ML} \end{bmatrix}, \quad C = I^{4\times 4}$$

Parameters: $m = 0.5$ kg, $M = 0.1$ kg, $L = 1$ m, $g = 10\ \frac{\text{m}}{\text{s}^2}$

Example – 8

The poles of the uncontrolled system: λ_1=0, λ_2=0, $\lambda_3 = 7.746$, $\lambda_4 = -7.746$

Goal: stabilizing controller

Prescribed poles of the closed loop system: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = -1$

The computed feedback gain:

$$k = [-0.01 \quad -6.61 \quad -0.04 \quad -0.44]$$

Example – 9

The operation of the controlled system (simulation: Faludi Gábor)

`ipend_pp-1.avi`

State observer, problem statement

Recall: If a SSM (A, B, C) is observable, then, knowing the input (u) and the output (y), the initial state of the system can be computed, and hence every further state values.

Problems:

- The measurement of the input and the output are (in general) not precise enough, furthermore, we need the 1st, 2nd, ..., $(n-1)$th derivatives of the output in order to compute the initial condition.
- In general, the system model is not perfect

Goal: design such a tool (state observer), for which we do not need the derivatives of the output y, and the estimated state converges to the actual value of the state vector.

Algebraic form of the state observer

State-space model:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$\dot{\hat{x}} = (A - LC)\hat{x} + [B\ L]\begin{bmatrix} u \\ y \end{bmatrix}$$

observation error:

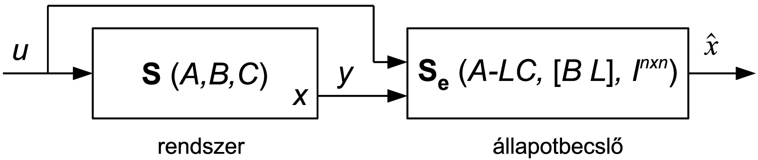
$$e = x - \hat{x}$$

and

$$\dot{e} = (A - LC)e$$

The structure of the state observer

The realization of a state observer (it can be seen from the algebraic equations)



Calculation of the state observer

Reminder: In case of a pole placement controller the system matrices of the closed loop system are $A_c = A - Bk$. (A,B is given, k should be computed, condition: (A,B) is controllable)
System matrix of the state observer: $A_o = A - LC$. (A, C is given, L should be computed, condition: ?)
Solution:

$$A_o^T = A^T - (LC)^T = A^T - C^T L^T$$

In other words, L can be computed using the pole placement algorithm using arbitrary prescribed stable eigenvalues for A_o (i.e. the state observer be stable). Condition: $[C^T \; A^T C^T \; \dots \; (A^{n-1})^T C^T] = \mathcal{O}_n^T$ is a full-rank matrix, namely, the system is observable.

1 Problem statement, full state feedback

2 Pole-placement controller design

3 Examples for controller design

4 Dual problem: state observer design

5 **State observer examples**

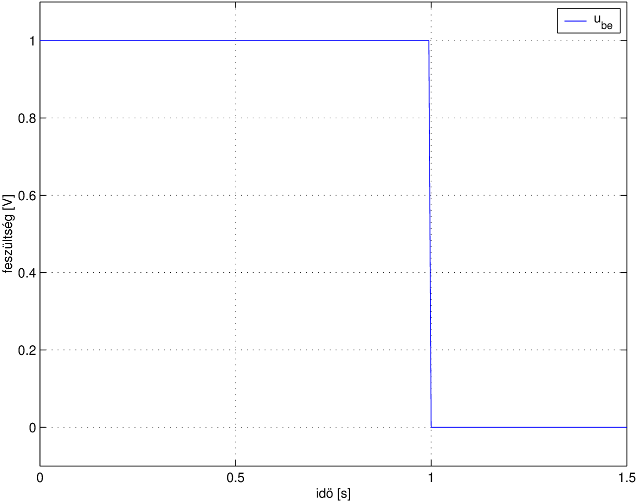
6 The combination of state observer and pole placement controller

Example – 1

RLC circuit, measured output: u_C, namely $C = [0 \;\; 1]$
Prescribed eigenvalues of the state observer: $-10, \; -12$
The computed matrix L of the state observer: $L = [-10 \;\; 12]^T$

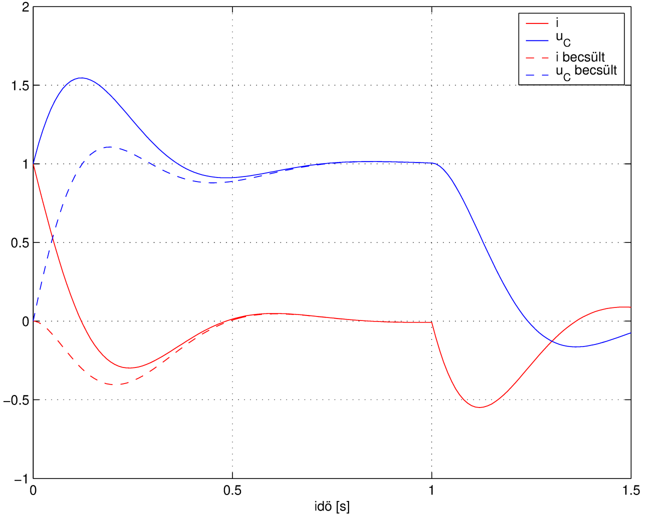
Example – 2

Input of the system:

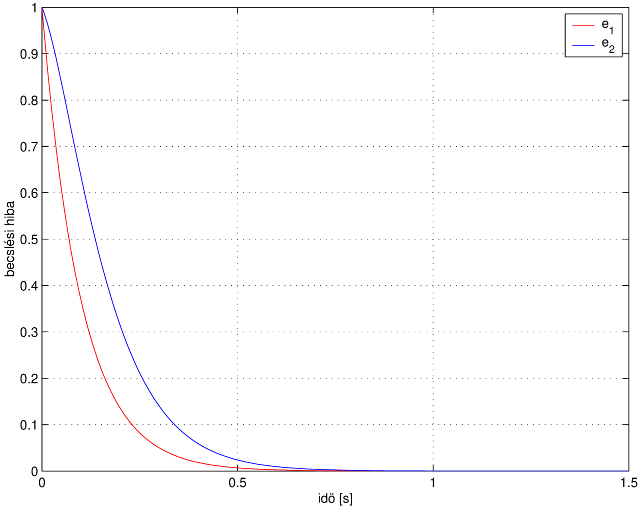


Example – 3

The operation of the state observer:

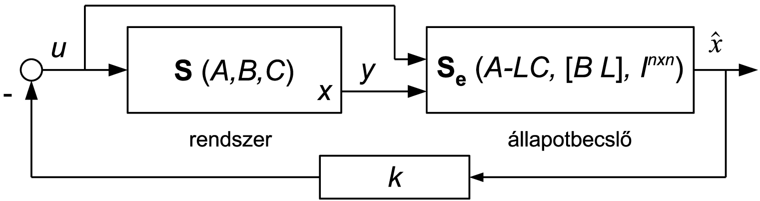


Observation error:



Separation principle

Problem: what happens if the estimated state is fed back by the computed feedback gain k (dynamic output feedback)?



Separation principle: The stabilizing state feedback with the a stable state observer is asymptotically stable, since the dynamics of the closed loop system is the following:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \cdot \begin{bmatrix} x \\ e \end{bmatrix}$$

This means that the stabilizing state feedback (K) and a stable state observer (L) can be designed separately.

Separation principle

Computation:

$$\dot{x} = Ax + Bu,\ \ u = -K\hat{x},\ \text{and: } e = x - \hat{x}$$

From this: $u = -K(x - e) = -Kx + Ke$, and

$$\dot{x} = Ax + B(-Kx + Ke) = (A - BK)x + BKe \quad (2)$$
$$\dot{e} = (A - LC)e \quad (3)$$

Formula for the eigenvalues:

$$\lambda_i \left(\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right) = \lambda_j\left(A - BK\right) \cup \lambda_k\left(A - LC\right),$$

and we know that $A - BK$ ill. $A - LC$ are stability matrices.

Summary

- goal of pole placement: move the poles (eigenvalues) of the controlled system to arbitrary places on the complex plane
- feedback form: full state feedback (requires the knowledge of each state variable)
- condition for computation: controllability
- goal of state observer: asymptotically compute the state variables from the input and the output
- observer gain computation: can be traced back to pole placement (dual problem)
- separation principle: separately designed stabilizing feedback and stable observer results in a stable combined system

Computer Controlled Systems
Lecture 9

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, Nov. 22, 2018

Outline

1. Optimal control: problem statement
2. Basics of variational calculus
3. Solution of the LQR problem
4. Examples

LQR: problem statement

Given

- a (MIMO) LTI state space model

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$
$$y(t) = Cx(t)$$

- a functional (*control goal*)

$$J(x,u) = \frac{1}{2}\int_0^T [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

where $Q^T = Q,\ Q > 0$ és $R^T = R,\ R > 0$.

To be computed: input: $\{u(t)\ ,\quad t \in [0,T]\}$, for which J is minimal along the solutions of the state space model (constraints)

Variational calculus – 1

Problem:
Find u which minimizes:

$$J(x,u) = \int_0^T F(x,u,t)dt$$

constraint: $\dot{x} = f(x,u,t)$.
Solution: using (vector) Lagrange multipliers $\lambda(t) \in \mathbb{R}^n$, $\forall t \geq 0$

$$J(x,\dot{x},u) = \int_0^T [F(x,u,t) + \lambda^T(t)(f(x,u,t) - \dot{x})]dt$$

Hamilton-function: $H(x,u,t) = F(x,u,t) + \lambda^T(t)f$

$$J(x,u,t) = \int_0^T [H(x,u,t) - \lambda^T(t)\dot{x}(t)]dt$$

Variational calculus – 2

$\dot{x}$ can be eliminated through partial integration

$$[\lambda^T x]_0^T = \int_0^T \dot{\lambda}^T x + \int_0^T \lambda^T \dot{x}$$

Then, from $J = \int_0^T [H - \lambda^T \dot{x}]dt$ we obtain:

$$J = -[\lambda^T x]_0^T + \int_0^T [H + \dot{\lambda}^T x]dt$$

variation of x and u:

$$x(t) \longrightarrow x(\alpha, t) = x(t) + \alpha\eta(t)$$
$$u(t) \longrightarrow u(\beta, t) = u(t) + \beta\gamma(t),$$

where $\alpha, \beta \in \mathbb{R}$

Euler-Lagrange equations – 1

Objective function:

$$I(\alpha,\beta) = -[\lambda^T(t)x(\alpha,t)]_0^T + \\ + \int_0^T [H(x(\alpha,t),u(\beta,t),t) + \dot{\lambda}^T(t)x(\alpha,t)]dt$$

Necessary condition for extremum within a set of varied x and u:

$$\frac{\partial I}{\partial \alpha} = 0, \quad \frac{\partial I}{\partial \beta} = 0$$

$$\frac{\partial I}{\partial \alpha} = \int_0^T \left[\frac{\partial H}{\partial x} + \dot{\lambda}^T(t)\right]\eta(t)dt = 0$$

$$\frac{\partial I}{\partial \beta} = \int_0^T \frac{\partial H}{\partial u}\gamma(t)dt$$

Euler-Lagrange equations – 2

Euler-Lagrange equations

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0$$
$$\frac{\partial H}{\partial u} = 0$$

LQR Euler-Lagrange equations

Euler-Lagrange equations with the Hamilton function $H = F + \lambda^T f$:

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0 \quad , \quad \frac{\partial H}{\partial u} = 0$$

for LTI systems:

$$\begin{aligned} f &= Ax + Bu \\ F &= \tfrac{1}{2}(x^T Q x + u^T R u) \\ H &= \tfrac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu) \end{aligned}$$

LQR Euler-Lagrange equations: $\frac{\partial}{\partial x}(x^T Q x) = 2x^T Q$

$$\dot{\lambda}^T + x^T Q + \lambda^T A = 0 \quad , \quad \lambda^T(T) = 0$$

$$u^T R + \lambda^T B = 0$$

Dynamics of states and co-states

Re-arranged Euler-Lagrange equations:

$$\dot{\lambda} + Qx + A^T\lambda = 0$$
$$u = -R^{-1}B^T\lambda$$

State equation:

$$\dot{x} = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$

In matrix form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \quad , \quad \begin{matrix} x(0) = x_0 \\ \lambda(T) = 0 \end{matrix}$$

System dynamics + Hammerstein co-state diff. eq.

LQR for LTI systems

Lemma: $\lambda(t)$ can be written as

$$\lambda(t) = K(t)x(t) \quad , \quad K(t) \in \mathcal{R}^{n\times n}$$

Modified state and co-state equations

$$\begin{aligned} \dot{\lambda} + Qx + A^T\lambda = 0 \quad &\Rightarrow \quad \dot{K}x + K\dot{x} = -A^TKx - Qx \\ u = -R^{-1}B^T\lambda \quad &\Rightarrow \quad u = -R^{-1}B^TKx \\ \dot{x} = Ax + Bu \quad &\Rightarrow \quad \dot{x} = Ax - BR^{-1}B^TKx \end{aligned}$$

$$\dot{K}x + K[A - BR^{-1}B^TK]x + A^TKx + Qx = 0$$

$\forall\ x(t) \quad \Rightarrow$ *Matrix Riccati differential equation for* $K(t)$:

$$\dot{K} + KA + A^TK - KBR^{-1}B^TK + Q = 0$$

Stationary case

Special case: stationary solution $T \to \infty$

$$J = \int_0^\infty (x^T Q x + u^T R u) dt$$

$$\lim_{t\to\infty} K(t) = K \quad \text{i.e.} \quad \dot{K} = 0$$

Control Algebraic Riccati Equation (CARE)

$$KA + A^T K - KBR^{-1}B^T K + Q = 0$$

Theorem: (R. Kalman) If (A, B) is controllable, then CARE has a unique symmetric solution (K).

solution in Matlab: `care`

The LQR and its properties

Solution: *linear static full state feedback*

$$u^0(t) = -R^{-1}B^T K x(t) = -Gx(t)$$

where $G = R^{-1}B^T K$.
Closed loop dynamics:

$$\dot{x} = Ax - BR^{-1}B^T Kx = (A - BG)x \quad , \quad x(0) = x_0$$

Properties of the controlled system

- the closed loop system is asymptotically stable independently of the values of A, B, C, R, Q, i.e.

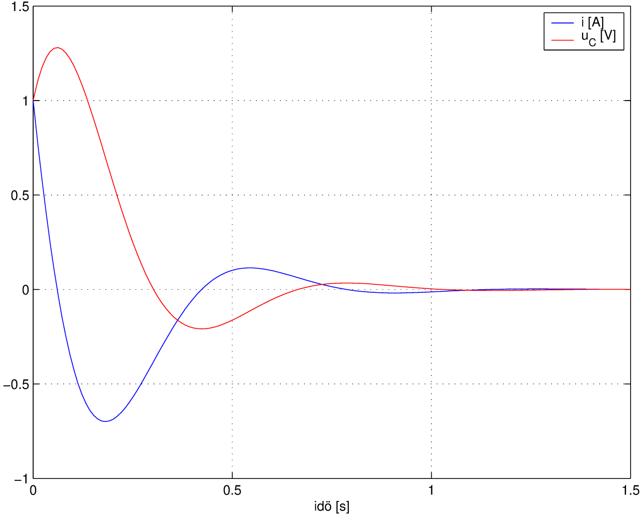
$$Re\ \lambda_i(A - BG) < 0 \quad , \quad i = 1, 2, ..., n$$

- the poles of the closed loop depend on the choice of Q and R

1. Optimal control: problem statement
2. Basics of variational calculus
3. Solution of the LQR problem
4. **Examples**

Example 1: control of the RLC circuit

System: RLC circuit. Response of the open loop system ($u = 0V$) for the initial condition $x(0) = [1\ 1]^T$. (Poles: $-5 \pm 8.6603i$)

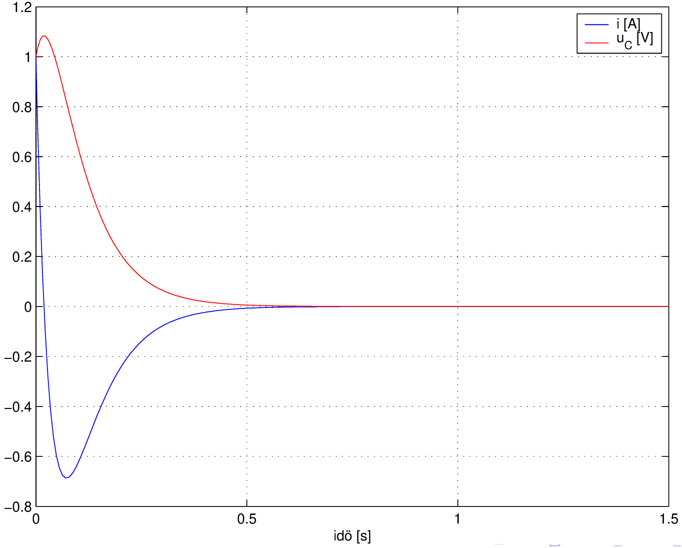


Example 1: control of the RLC circuit

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.1$$

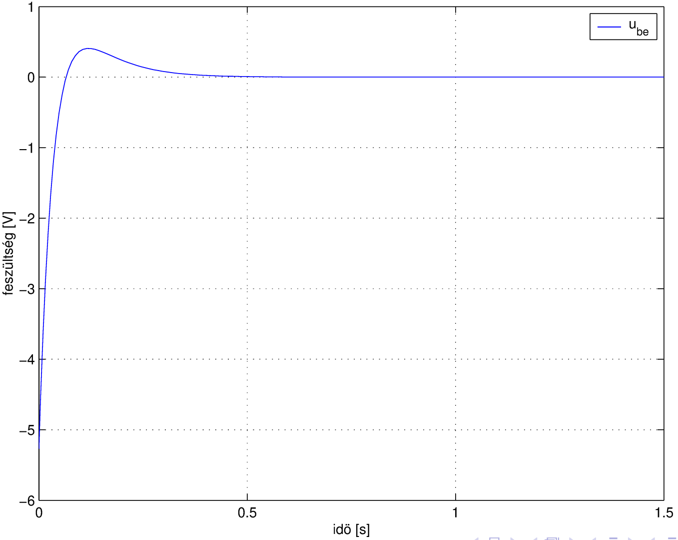
Feedback gain: $G = [2.9539, \quad 2.3166]$
Poles of the closed loop system $(A - BG)$: $\lambda_1 = -27.4616$, $\lambda_2 = -12.0773$

Operation of the closed loop system



Example 1: control of the RLC circuit

Input generated by the controller



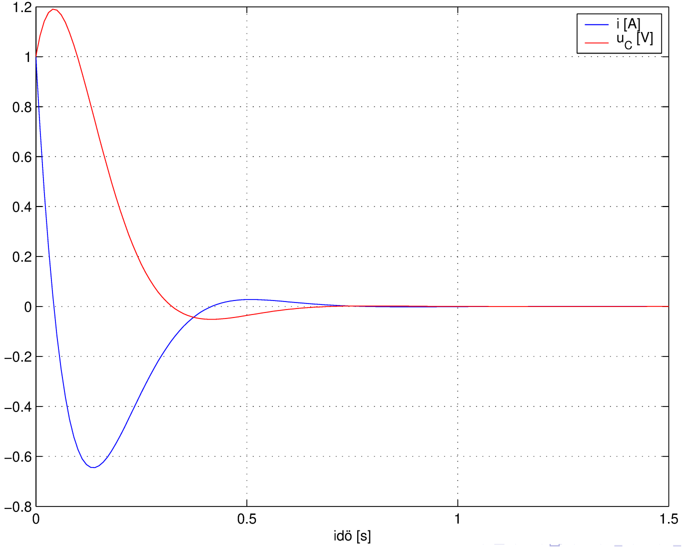
Example 1: control of the RLC circuit

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1$$

Feedback gain: $G = [0.6818, \quad 0.4142]$
Poles of the closed loop system $(A - BG)$: $\lambda_{1,2} = -8.409 \pm 8.409i$

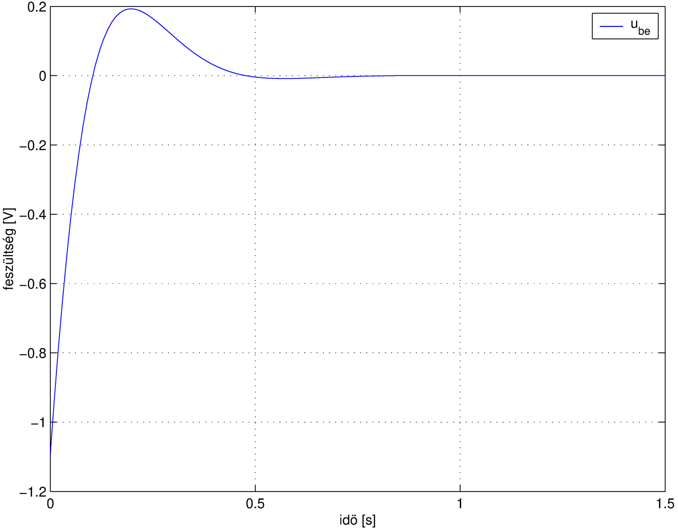
Example 1: control of the RLC circuit

Operation of the closed loop system



Example 1: control of the RLC circuit

Input generared by the controller



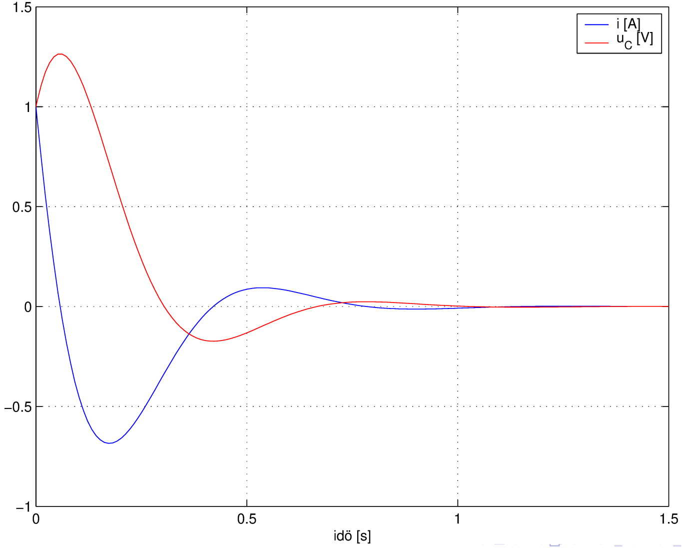
Example 1: control of the RLC circuit

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 10$$

Feedback gain: $G = [0.0944, \quad 0.0488]$
Poles of the closed loop system $(A - BG)$: $\lambda_{1,2} = -5.4718 \pm 8.6568i$

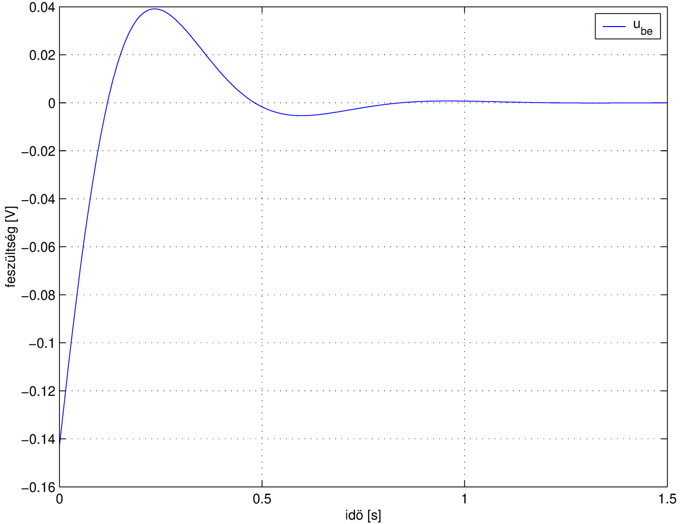
Example 1: control of the RLC circuit

Operaton of the closed loop system



Example 1: control of the RLC circuit

Input generated by the controller



Example 2 - application of the separation principle

System to be controlled: DC motor

Parameters:

J	moment of inertia	0.01 kg m^2/s^2
b	damping coefficient	0.1 Nm s
K	electormotive force coefficient	0.1127 Nm/A
R	resistance	1 ohm
L	inductance	0.5 H

state variables, input, output:

$x_1 = \dot{\theta}$ angular velocity [rad/s]

$x_2 = i$ current [A]

u input voltage [V]

$y = x_1$

State space model:

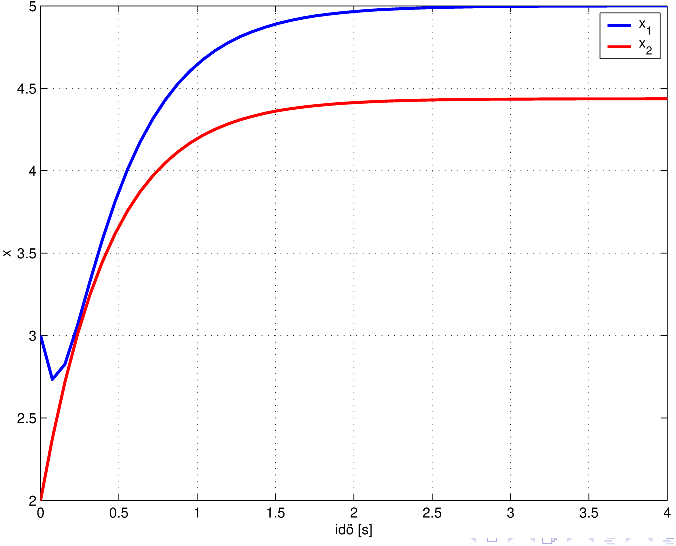
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{b}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Poles: -9.669, -2.331

Example 2 - application of the separation principle

Operation of the open loop system for the input $u(t) = 5$:



State observer design

Prescribed poles of the observer: -15, -16

("faster" than the poles of the original system)

Values of the L matrix:

$$L = \begin{bmatrix} 19 \\ 15.923 \end{bmatrix}$$

Example 2 - application of the separation principle

Stabilizing state feedback design

Parameters of the designed LQR controller:

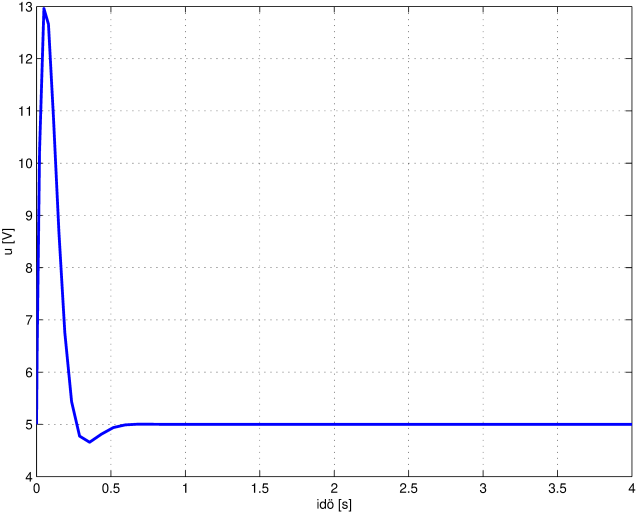
$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 1$$

The obtained feedback gain:

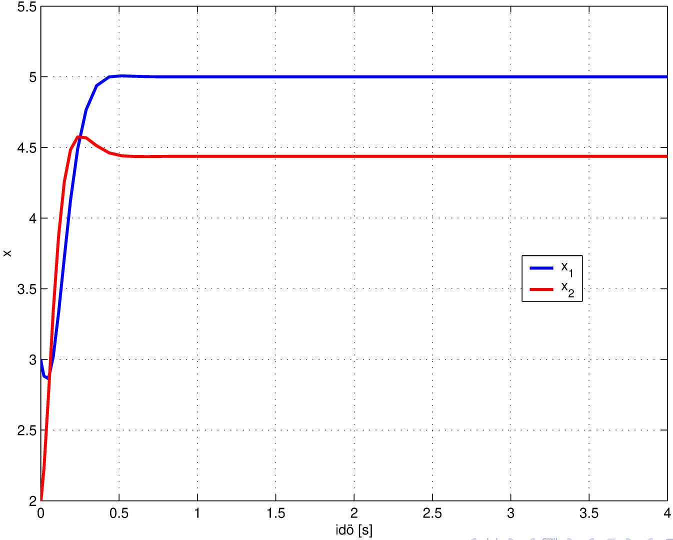
$$G = \begin{bmatrix} 3.807 & 6.342 \end{bmatrix}$$

Example 2 - application of the separation principle

Operation of the stabilizing feedback combined with the state observer
Input voltage generated by the controller:

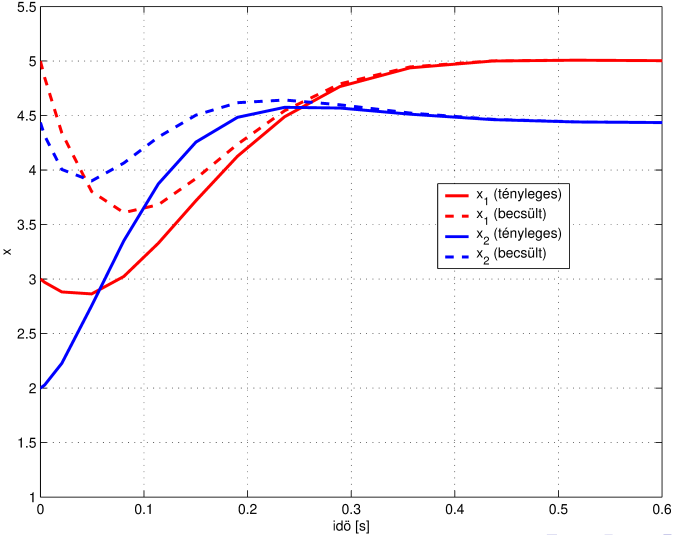


State variables of the closed loop system



Example 2 - application of the separation principle

Operation of the state observer



Example 3 - control of the inverted pendulum

Weighting matrices (design parameters):

$$Q = I^{4\times 4}, \quad R = 1$$

The computed feedback gain:

$$G = \begin{bmatrix} -1 & -23.227878 & -2.1084534 & -7.8899369 \end{bmatrix}$$

Eigenvalues of the closed loop system:

$$\lambda = \begin{bmatrix} -13.169677 \\ -1.0463076 + 0.3589175i \\ -1.0463076 - 0.3589175i \\ -3.1028591 \end{bmatrix}$$

Example 3 - control of the inverted pendulum

Operation of the controller:

`ipend_lq_1.avi`

Summary

- goal of optimal control: to minimize a functional by an appropriate input
- LQR case: system is LTI, functional is quadratic (combines performance and 'input energy/price' terms)
- solution principle: constrained minimization using time-dependent Lagrange multipliers (co-states)
- explicit solution is obtained assuming an infinite time horizon ($T \to \infty$)
- solution of a quadratic matrix equation (CARE) is required (easy with computer)
- result: linear full state feedback (always stabilizing if appropriate conditions are fulfilled)

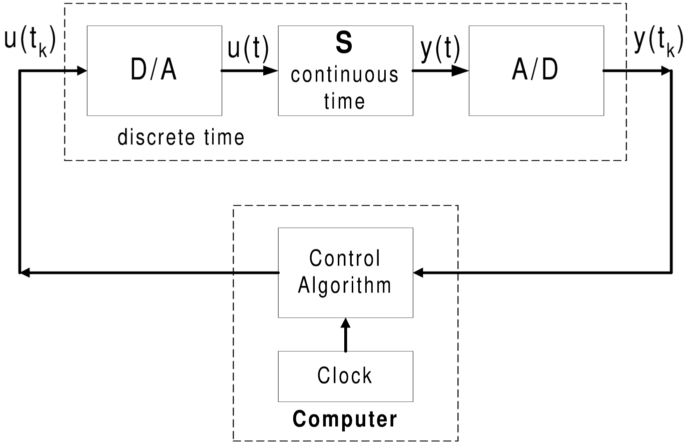
Computer Controlled Systems
Lecture 10

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

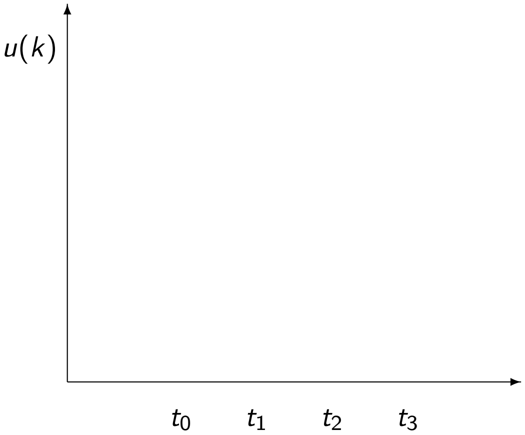
PPKE-ITK, Nov. 29, 2018

Sampling


u(t_k)
D/A
u(t)
S
continuous
time
y(t)
A/D
y(t_k)
discrete time
Control
Algorithm
Clock
Computer

Zero order hold sampling

Transforming a continuous function into a piecewise constant signal



Sampling of CT-LTI systems

Given:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

sampling of u using zero order hold

$$u(\tau) = u(t_k) = u(k) \quad , \quad t_k \leq \tau < t_{k+1}$$

Uniform (equidistant) **sampling**: $t_{k+1} - t_k = h = const$

To be computed:

state space model of the sampled (discretized) system

Discrete time state equations - 1

Solution of the continuous time state equation

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Substitution: $t = t_{k+1}$ and $t_0 = t_k$

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau$$

uniform sampling and $\theta = \tau - t_k,\ t_{k+1} - \tau = h - \theta$

$$x(k+1) = e^{Ah}x(k) + \int_0^h e^{A(h-\theta)}Bu(k)d\theta =$$
$$x(k+1) = e^{Ah}x(k) + e^{Ah}\int_0^h e^{-A\theta}d\theta Bu(k)$$

Discrete time state equations - 2

$$x(k+1) = e^{Ah}x(k) + e^{Ah}\int_0^h e^{-A\theta}d\theta Bu(k)$$

and

$$\int_0^h e^{-A\theta}d\theta = [-A^{-1}e^{-A\theta}]_0^h = A^{-1}(I - e^{Ah})$$

Discrete time state equations

$$x(k+1) = e^{Ah}x(k) + A^{-1}(e^{Ah} - I)Bu(k)$$

DT-LTI state equations for **sampled systems**

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$\Phi = e^{Ah} = I + Ah + \ldots \quad , \quad \Gamma = A^{-1}(e^{Ah} - I)B = (Ih + \frac{Ah^2}{2!} + \ldots)B$$

DT-LTI state space models

$$x(k+1) = \Phi x(k) + \Gamma u(k) \qquad \text{state equation}$$
$$y(k) = Cx(k) + Du(k) \qquad \text{output equation}$$

with given $x(0)$ initial condition and

$$x(k) \in \mathbb{R}^n\ ,\ y(k) \in \mathbb{R}^p\ ,\ u(k) \in \mathbb{R}^r$$

finite dimensional vectors, and

$$\Phi \in \mathbb{R}^{n\times n}\ ,\ \Gamma \in \mathbb{R}^{n\times r}\ ,\ C \in \mathbb{R}^{p\times n}\ ,\ D \in \mathbb{R}^{p\times r}$$

matrices

Solution of DT-LTI state equations

$$x(1) = \Phi x(0) + \Gamma u(0)$$
$$x(2) = \Phi x(1) + \Gamma u(1) = \Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma u(1)$$
$$x(3) = \Phi x(2) + \Gamma u(2) = \Phi^3 x(0) + \Phi^2\Gamma u(0) + \Phi\Gamma u(1) + \Gamma u(2)$$

..

..

$$x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1}\Gamma u(j)$$

DT-LTI I/O system models – 1

Impulse response function: I/O model for SISO systems

$$\mathcal{U} = [u(0)\ u(1) \dots u(N-1)]^T \quad , \quad \mathcal{Y} = [y(0)\ y(1) \dots y(N-1)]^T$$

General linear model

$$\mathcal{Y} = \overline{H}\mathcal{U} + Y_p$$

where $\overline{H}$ is an $n \times n$ matrix, and Y_p contains the initial conditions.
In case of **causal systems**, $\overline{H}$ is lower triangular

$$y(k) = \sum_{j=0}^{k} \overline{h}(k,j)u(j) + y_p(k)$$

where $\overline{h}(k,j)$ is the **impulse response function**

DT-LTI I/O system models – 2

Impulse response function of LTI models: $\overline{h}(k,j) = h(k-j)$
From the solution of the state equation (with $D = 0$):

$$x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1}\Gamma u(j)$$
$$y(k) = Cx(k) = C\Phi^k x(0) + \sum_{j=0}^{k-1} C\Phi^{k-j-1}\Gamma u(j)$$

$$h(k) = \begin{cases} 0 & k < 1 \\ C\Phi^{k-1}\Gamma & k \geq 1 \end{cases}$$

discrete time analogue of the impulse response function.

Discrete time Markov parameters: $C\Phi^{k-1}\Gamma$

Discrete time signals

$$f = \{f(k), k = 0, 1, ...\}$$

signal norms of *scalar valued discrete time signals*

- *infinity norm*

$$||f||_\infty = \sup_k |f(k)|$$

- *2-norm*

$$||f||_2^2 = \sum_{k=-\infty}^{\infty} f^2(k)$$

Shift operators

Definition: forward shift operator: q
performs the following operation with a DT signal:

$$qf(k) = f(k+1) \tag{1}$$

Definition: backward shift operator (delay): q^{-1}
performs the following operation:

$$q^{-1}f(k) = f(k-1) \tag{2}$$

DT-LTI I/O system models – 3

Discrete difference equations: for SISO systems
Using forward differences

$$y(k+n_a) + a_1 y(k+n_a-1) + \dots + a_{n_a} y(k) = b_0 u(k+n_b) + \dots + b_{n_b} u(k)$$

where $n_a \geq n_b$ (proper). *More compact form:*

$$A(q)y(k) = B(q)u(k)\ ,\ A(q) = q^{n_a} + a_1 q^{n_a-1} + \dots + a_{n_a}\ ,\ B(q) = b_0 q^{n_b} + b_1 q^{n_b-1} + \dots + b_{n_b}$$

Using backward differences

$$y(k) + a_1 y(k-1) + \dots + a_{n_a} y(k-n_a) = b_0 u(k-d) + \dots + b_{n_b} u(k-d-n_b)$$

where $d = n_a - n_b > 0$ is the *delay*. *More compact form:*

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)\quad ,\quad A^*(q^{-1}) = q^{n_a} A(q^{-1})$$

DT-LTI I/O system models – 4

Pulse transfer operator
Computed from the DT-LTI state space model

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad , \quad y(k) = Cx(k) + Du(k)$$

$$\begin{aligned} x(k+1) &= qx(k) = \Phi x(k) + \Gamma u(k) \\ x(k) &= (qI - \Phi)^{-1}\Gamma u(k) \\ y(k) &= Cx(k) + Du(k) = [C(qI - \Phi)^{-1}\Gamma + D]u(k) \end{aligned}$$

$H(q)$ **pulse transfer operator** of the state space model (Φ, Γ, C, D):

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D$$

discrete time analogue of the transfer function.

DT-LTI I/O system models – 5

Pulse transfer operator, SISO case:

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)} \quad , \quad deg\ B(q) < deg\ A(q) = n$$

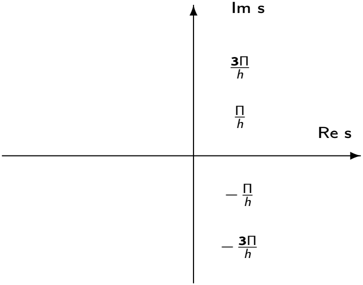
where $A(q)$ is the characteristic polynomial of matrix Φ.
Relation with **discrete difference equations**

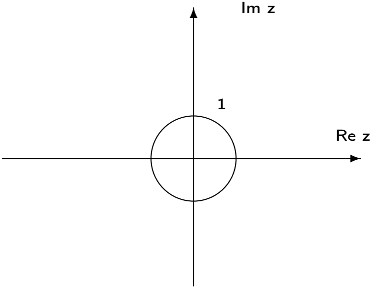
$$y(k) + a_1 y(k-1) + ... + a_n y(k-n) = b_1 u(k-1) + ... + b_n u(k-n)$$

Poles of DT-LTI systems – 1

	continuous time	discrete time
state eq.	$\dot{x}(t) = Ax(t) + Bu(t)$	$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$ $\Phi = e^{Ah}$
output eq.	$y(t) = Cx(t)$	$y(kh) = Cx(kh)$
poles	$\lambda_i(A)$	$\lambda_i(\Phi)$ $\lambda_i(\Phi) = e^{\lambda_i(A)h}$

Poles of DT-LTI systems – 2





Summary

- discretization of CT-LTI models: constant sampling time is assumed
- zero order hold: the input is constant between two sampling instants
- state equation can be integrated: LTI difference equation is obtained, output equation remains the same
- state equation can be solved in discrete time
- shift operator: I/O models (filters) can be obtained from state space models
- asymptotic stability: poles are strictly inside the unit circle