Computer Controlled Systems
Lecture 10

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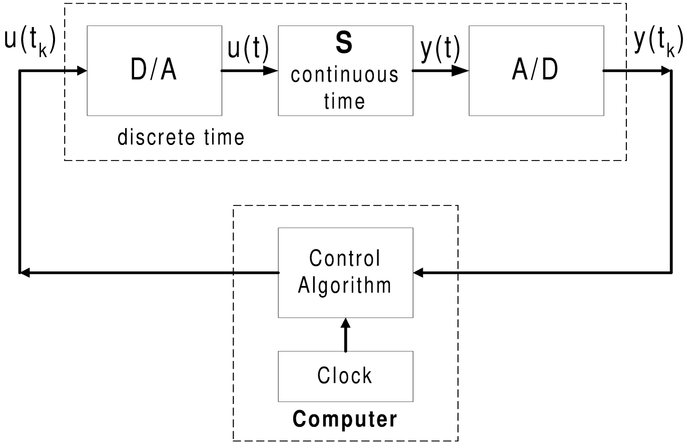
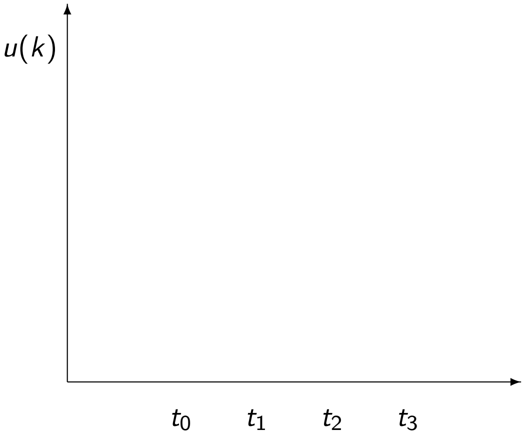
Sampling



u(t_k)
D/A
u(t)
S
continuous
time
y(t)
A/D
y(t_k)
discrete time
Control
Algorithm
Clock
Computer

Zero order hold sampling

Transforming a continuous function into a piecewise constant signal



Sampling of CT-LTI systems

Given:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

sampling of u using zero order hold

$$u(\tau) = u(t_k) = u(k) \quad , \quad t_k \leq \tau < t_{k+1}$$

Uniform (equidistant) **sampling**: $t_{k+1} - t_k = h = const$

To be computed:
state space model of the sampled (discretized) system

Discrete time state equations - 1

Solution of the continuous time state equation

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Substitution: $t = t_{k+1}$ and $t_0 = t_k$

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau$$

uniform sampling and $\theta = \tau - t_k,\ t_{k+1} - \tau = h - \theta$

$$x(k+1) = e^{Ah}x(k) + \int_0^h e^{A(h-\theta)}Bu(k)d\theta =$$
$$x(k+1) = e^{Ah}x(k) + e^{Ah}\int_0^h e^{-A\theta}d\theta Bu(k)$$

Discrete time state equations - 2

$$x(k+1) = e^{Ah}x(k) + e^{Ah}\int_0^h e^{-A\theta}d\theta Bu(k)$$

and

$$\int_0^h e^{-A\theta}d\theta = [-A^{-1}e^{-A\theta}]_0^h = A^{-1}(I - e^{Ah})$$

Discrete time state equations

$$x(k+1) = e^{Ah}x(k) + A^{-1}(e^{Ah} - I)Bu(k)$$

DT-LTI state equations for **sampled systems**

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$\Phi = e^{Ah} = I + Ah + \ldots \quad , \quad \Gamma = A^{-1}(e^{Ah} - I)B = (Ih + \frac{Ah^2}{2!} + \ldots)B$$

DT-LTI state space models

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) && \text{state equation} \\
y(k) &= Cx(k) + Du(k) && \text{output equation}
\end{aligned}
$$

with given $x(0)$ initial condition and

$$x(k) \in \mathbb{R}^n \ , \ y(k) \in \mathbb{R}^p \ , \ u(k) \in \mathbb{R}^r$$

finite dimensional vectors, and

$$\Phi \in \mathbb{R}^{n \times n} \ , \ \Gamma \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{p \times n} \ , \ D \in \mathbb{R}^{p \times r}$$

matrices

Solution of DT-LTI state equations

$$x(1) = \Phi x(0) + \Gamma u(0)$$
$$x(2) = \Phi x(1) + \Gamma u(1) = \Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma u(1)$$
$$x(3) = \Phi x(2) + \Gamma u(2) = \Phi^3 x(0) + \Phi^2\Gamma u(0) + \Phi\Gamma u(1) + \Gamma u(2)$$
..

..
$$x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1}\Gamma u(j)$$

DT-LTI I/O system models – 1

Impulse response function: I/O model for SISO systems

$$\mathcal{U} = [u(0)\ u(1) \ldots u(N-1)]^T \quad , \quad \mathcal{Y} = [y(0)\ y(1) \ldots y(N-1)]^T$$

General linear model

$$\mathcal{Y} = \overline{H}\mathcal{U} + Y_p$$

where $\overline{H}$ is an $n \times n$ matrix, and Y_p contains the initial conditions.
In case of **causal systems**, $\overline{H}$ is lower triangular

$$y(k) = \sum_{j=0}^{k} \overline{h}(k,j)u(j) + y_p(k)$$

where $\overline{h}(k,j)$ is the **impulse response function**

DT-LTI I/O system models – 2

Impulse response function of LTI models: $\overline{h}(k,j) = h(k-j)$
From the solution of the state equation (with $D = 0$):

$$x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1} \Gamma u(j)$$
$$y(k) = Cx(k) = C\Phi^k x(0) + \sum_{j=0}^{k-1} C\Phi^{k-j-1} \Gamma u(j)$$

$$h(k) = \begin{cases} 0 & k < 1 \\ C\Phi^{k-1}\Gamma & k \geq 1 \end{cases}$$

discrete time analogue of the impulse response function.

Discrete time Markov parameters: $C\Phi^{k-1}\Gamma$

Discrete time signals

$$f = \{f(k), k = 0, 1, ...\}$$

signal norms of *scalar valued discrete time signals*

- *infinity norm*

$$||f||_\infty = \sup_k |f(k)|$$

- *2-norm*

$$||f||_2^2 = \sum_{k=-\infty}^{\infty} f^2(k)$$

Shift operators

Definition: forward shift operator: q
performs the following operation with a DT signal:

$$qf(k) = f(k+1) \tag{1}$$

Definition: backward shift operator (delay): q^{-1}
performs the following operation:

$$q^{-1}f(k) = f(k-1) \tag{2}$$

DT-LTI I/O system models – 3

Discrete difference equations: for SISO systems
Using forward differences

$$y(k+n_a)+a_1y(k+n_a-1)+\ldots+a_{n_a}y(k)=b_0u(k+n_b)+\ldots+b_{n_b}u(k)$$

where $n_a \geq n_b$ (proper). *More compact form:*

$$A(q)y(k)=B(q)u(k)\ ,\ A(q)=q^{n_a}+a_1q^{n_a-1}+\ldots+a_{n_a}\ ,\ B(q)=b_0q^{n_b}+b_1q^{n_b-1}+\ldots+b_{n_b}$$

Using backward differences

$$y(k)+a_1y(k-1)+\ldots+a_{n_a}y(k-n_a)=b_0u(k-d)+\ldots+b_{n_b}u(k-d-n_b)$$

where $d=n_a-n_b>0$ is the *delay*. *More compact form:*

$$A^*(q^{-1})y(k)=B^*(q^{-1})u(k-d)\quad ,\quad A^*(q^{-1})=q^{n_a}A(q^{-1})$$

DT-LTI I/O system models – 4

Pulse transfer operator
Computed from the DT-LTI state space model

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad , \quad y(k) = Cx(k) + Du(k)$$

$$x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k)$$
$$x(k) = (qI - \Phi)^{-1}\Gamma u(k)$$
$$y(k) = Cx(k) + Du(k) = [C(qI - \Phi)^{-1}\Gamma + D]u(k)$$

$H(q)$ **pulse transfer operator** of the state space model (Φ, Γ, C, D):

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D$$

discrete time analogue of the transfer function.

DT-LTI I/O system models – 5

Pulse transfer operator, SISO case:

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)} \quad , \quad deg\ B(q) < deg\ A(q) = n$$

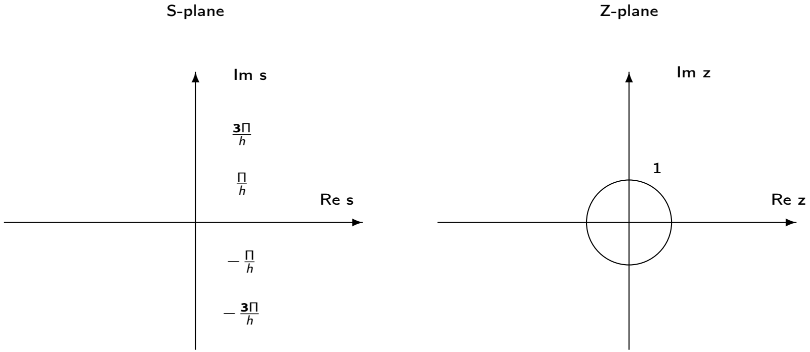
where $A(q)$ is the characteristic polynomial of matrix Φ.
Relation with **discrete difference equations**

$$y(k) + a_1 y(k-1) + ... + a_n y(k-n) = b_1 u(k-1) + ... + b_n u(k-n)$$

Poles of DT-LTI systems – 1

	continuous time	discrete time
state eq.	$\dot{x}(t) = Ax(t) + Bu(t)$	$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$ $\Phi = e^{Ah}$
output eq.	$y(t) = Cx(t)$	$y(kh) = Cx(kh)$
poles	$\lambda_i(A)$	$\lambda_i(\Phi)$ $\lambda_i(\Phi) = e^{\lambda_i(A)h}$

Poles of DT-LTI systems – 2



Summary

- discretization of CT-LTI models: constant sampling time is assumed
- zero order hold: the input is constant between two sampling instants
- state equation can be integrated: LTI difference equation is obtained, output equation remains the same
- state equation can be solved in discrete time
- shift operator: I/O models (filters) can be obtained from state space models
- asymptotic stability: poles are strictly inside the unit circle