Computer Controlled Systems
Lecture 9

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics
e-mail: szederkenyi@itk.ppke.hu

PPKE-ITK, Nov. 22, 2018

Outline

LQR: problem statement

Given

- a (MIMO) LTI state space model

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$
$$y(t) = Cx(t)$$

- a functional (*control goal*)

$$J(x,u) = \frac{1}{2}\int_0^T [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

where $Q^T = Q,\ Q > 0$ és $R^T = R,\ R > 0$.

To be computed: input: $\{u(t)\ ,\quad t \in [0,T]\}$, for which J is minimal along the solutions of the state space model (constraints)

Variational calculus – 1

Problem:
Find u which minimizes:

$$J(x,u) = \int_0^T F(x,u,t)dt$$

constraint: $\dot{x} = f(x,u,t)$.
Solution: using (vector) Lagrange multipliers $\lambda(t) \in \mathbb{R}^n$, $\forall t \geq 0$

$$J(x,\dot{x},u) = \int_0^T [F(x,u,t) + \lambda^T(t)(f(x,u,t) - \dot{x})]dt$$

Hamilton-function: $H(x,u,t) = F(x,u,t) + \lambda^T(t) f$

$$J(x,u,t) = \int_0^T [H(x,u,t) - \lambda^T(t)\dot{x}(t)]dt$$

Variational calculus – 2

$\dot{x}$ can be eliminated through partial integration

$$[\lambda^T x]_0^T = \int_0^T \dot{\lambda}^T x + \int_0^T \lambda^T \dot{x}$$

Then, from $J = \int_0^T [H - \lambda^T \dot{x}]dt$ we obtain:

$$J = -[\lambda^T x]_0^T + \int_0^T [H + \dot{\lambda}^T x]dt$$

variation of x and u:

$$\begin{aligned} x(t) &\longrightarrow x(\alpha, t) = x(t) + \alpha\eta(t) \\ u(t) &\longrightarrow u(\beta, t) = u(t) + \beta\gamma(t), \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$

Euler-Lagrange equations – 1

Objective function:

$$I(\alpha,\beta) = -[\lambda^T(t)x(\alpha,t)]_0^T + \\ + \int_0^T [H(x(\alpha,t),u(\beta,t),t) + \dot{\lambda}^T(t)x(\alpha,t)]dt$$

Necessary condition for extremum within a set of varied x and u:

$$\frac{\partial I}{\partial \alpha} = 0, \quad \frac{\partial I}{\partial \beta} = 0$$

$$\frac{\partial I}{\partial \alpha} = \int_0^T \left[\frac{\partial H}{\partial x} + \dot{\lambda}^T(t)\right]\eta(t)dt = 0$$

$$\frac{\partial I}{\partial \beta} = \int_0^T \frac{\partial H}{\partial u}\gamma(t)dt$$

Euler-Lagrange equations – 2

Euler-Lagrange equations

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0$$

$$\frac{\partial H}{\partial u} = 0$$

LQR Euler-Lagrange equations

Euler-Lagrange equations with the Hamilton function $H = F + \lambda^T f$:

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0 \quad , \quad \frac{\partial H}{\partial u} = 0$$

for LTI systems:

$$\begin{aligned} f &= Ax + Bu \\ F &= \tfrac{1}{2}(x^T Q x + u^T R u) \\ H &= \tfrac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu) \end{aligned}$$

LQR Euler-Lagrange equations: $\frac{\partial}{\partial x}(x^T Q x) = 2x^T Q$

$$\dot{\lambda}^T + x^T Q + \lambda^T A = 0 \quad , \quad \lambda^T(T) = 0$$

$$u^T R + \lambda^T B = 0$$

Dynamics of states and co-states

Re-arranged Euler-Lagrange equations:

$$\dot{\lambda} + Qx + A^T\lambda = 0$$
$$u = -R^{-1}B^T\lambda$$

State equation:

$$\dot{x} = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$

In matrix form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \quad , \quad \begin{matrix} x(0) = x_0 \\ \lambda(T) = 0 \end{matrix}$$

System dynamics + Hammerstein co-state diff. eq.

LQR for LTI systems

Lemma: $\lambda(t)$ can be written as

$$\lambda(t) = K(t)x(t) \quad , \quad K(t) \in \mathcal{R}^{n \times n}$$

Modified state and co-state equations

$$\begin{aligned} \dot{\lambda} + Qx + A^T\lambda = 0 \quad &\Rightarrow \quad \dot{K}x + K\dot{x} = -A^T Kx - Qx \\ u = -R^{-1}B^T\lambda \quad &\Rightarrow \quad u = -R^{-1}B^T Kx \\ \dot{x} = Ax + Bu \quad &\Rightarrow \quad \dot{x} = Ax - BR^{-1}B^T Kx \end{aligned}$$

$$\dot{K}x + K[A - BR^{-1}B^T K]x + A^T Kx + Qx = 0$$

$\forall\ x(t) \quad \Rightarrow$ *Matrix Riccati differential equation for* $K(t)$:

$$\dot{K} + KA + A^T K - KBR^{-1}B^T K + Q = 0$$

Stationary case

Special case: stationary solution $T \to \infty$

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

$$\lim_{t \to \infty} K(t) = K \quad \text{i.e.} \quad \dot{K} = 0$$

Control Algebraic Riccati Equation (CARE)

$$KA + A^T K - KBR^{-1}B^T K + Q = 0$$

Theorem: (R. Kalman) If (A, B) is controllable, then CARE has a unique symmetric solution (K).

solution in Matlab: `care`

The LQR and its properties

Solution: *linear static full state feedback*

$$u^0(t) = -R^{-1}B^T Kx(t) = -Gx(t)$$

where $G = R^{-1}B^T K$.
Closed loop dynamics:

$$\dot{x} = Ax - BR^{-1}B^T Kx = (A - BG)x \quad , \quad x(0) = x_0$$

Properties of the controlled system

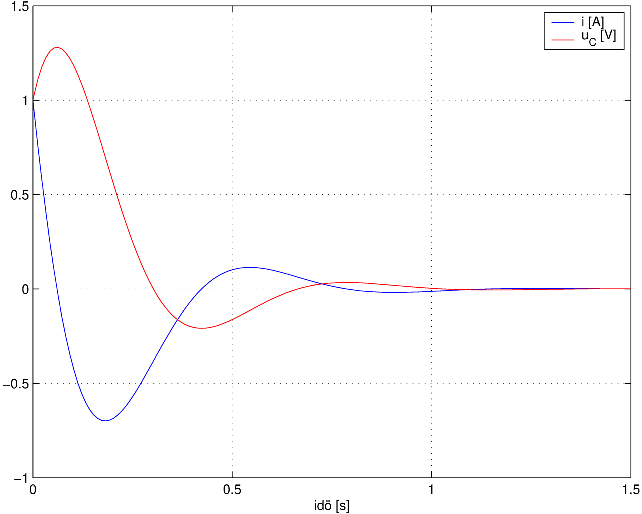
- the closed loop system is asymptotically stable independently of the values of A, B, C, R, Q, i.e.

$$Re\ \lambda_i(A - BG) < 0 \quad , \quad i = 1, 2, ..., n$$

- the poles of the closed loop depend on the choice of Q and R

Example 1: control of the RLC circuit

System: RLC circuit. Response of the open loop system ($u = 0V$) for the initial condition $x(0) = [1\ 1]^T$. (Poles: $-5 \pm 8.6603i$)



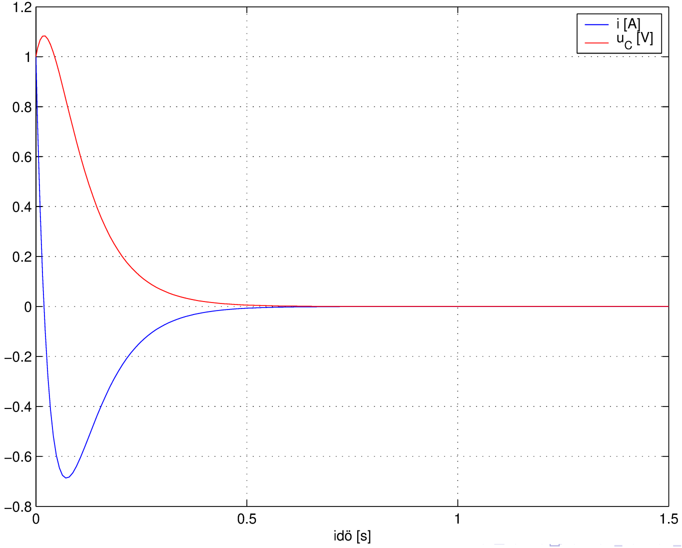
Example 1: control of the RLC circuit

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.1$$

Feedback gain: $G = [2.9539, \quad 2.3166]$
Poles of the closed loop system $(A - BG)$: $\lambda_1 = -27.4616$, $\lambda_2 = -12.0773$

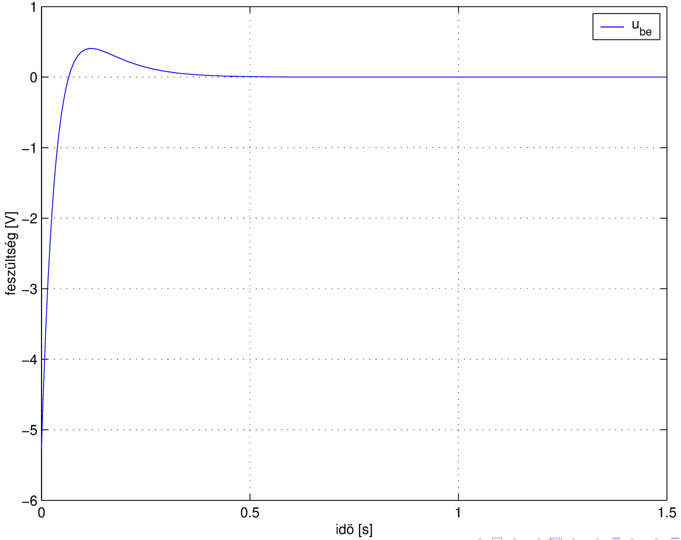
Example 1: control of the RLC circuit

Operation of the closed loop system



Example 1: control of the RLC circuit

Input generated by the controller



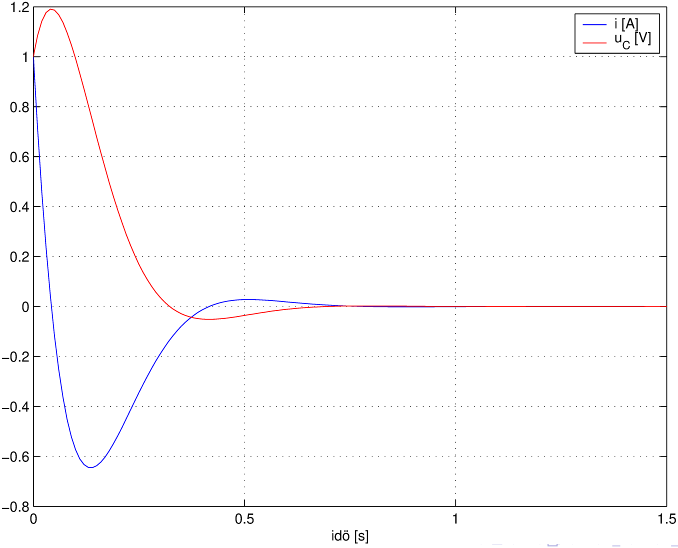
Example 1: control of the RLC circuit

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1$$

Feedback gain: $G = [0.6818, \quad 0.4142]$
Poles of the closed loop system $(A - BG)$: $\lambda_{1,2} = -8.409 \pm 8.409i$

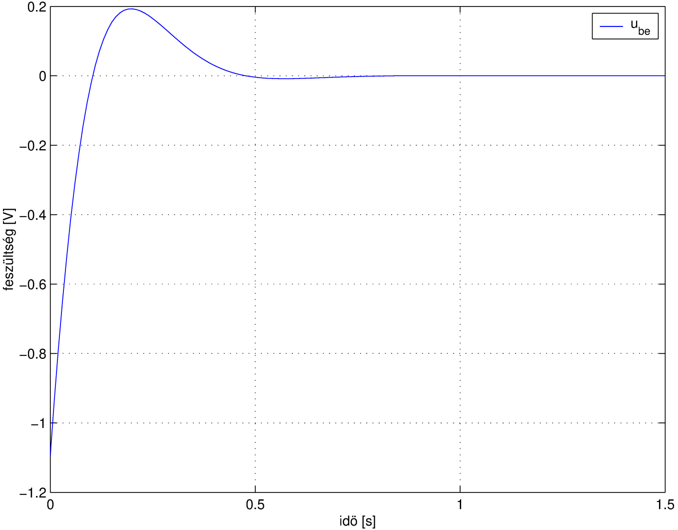
Example 1: control of the RLC circuit

Operation of the closed loop system



Example 1: control of the RLC circuit

Input generared by the controller



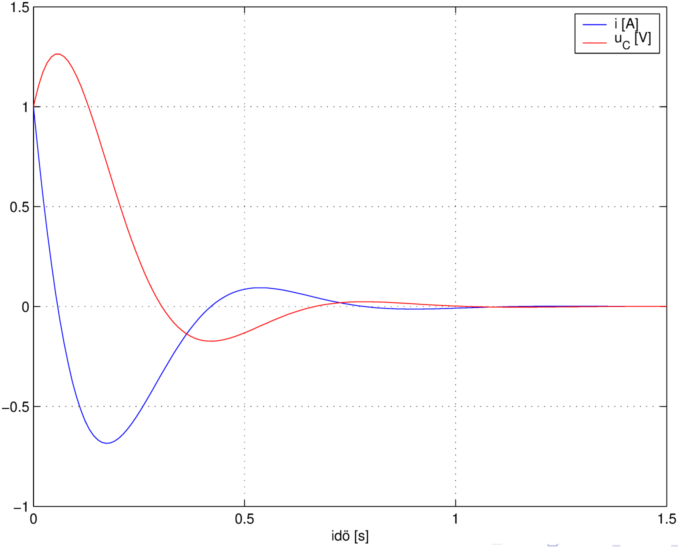
Example 1: control of the RLC circuit

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 10$$

Feedback gain: $G = [0.0944, \quad 0.0488]$
Poles of the closed loop system $(A - BG)$: $\lambda_{1,2} = -5.4718 \pm 8.6568i$

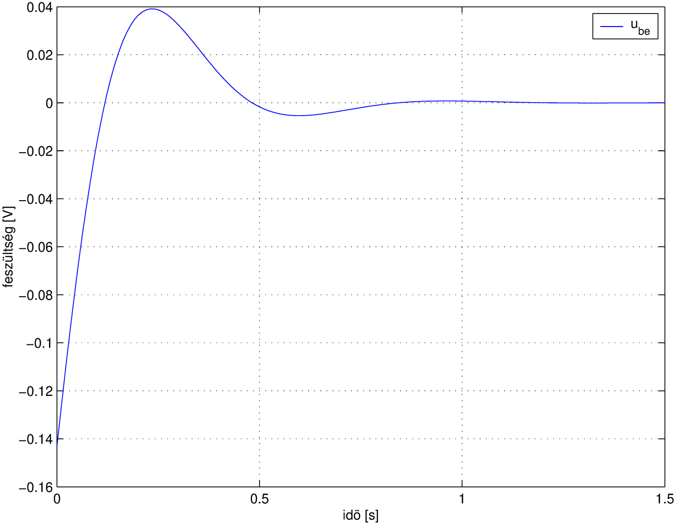
Example 1: control of the RLC circuit

Operaton of the closed loop system



Example 1: control of the RLC circuit

Input generated by the controller



Example 2 - application of the separation principle

System to be controlled: DC motor

Parameters:

J	moment of inertia	0.01 kg m^2/s^2
b	damping coefficient	0.1 Nm s
K	electormotive force coefficient	0.1127 Nm/A
R	resistance	1 ohm
L	inductance	0.5 H

state variables, input, output:

$x_1 = \dot{\theta}$ angular velocity [rad/s]

$x_2 = i$ current [A]

u input voltage [V]

$y = x_1$

Example 2 - application of the separation principle

State space model:

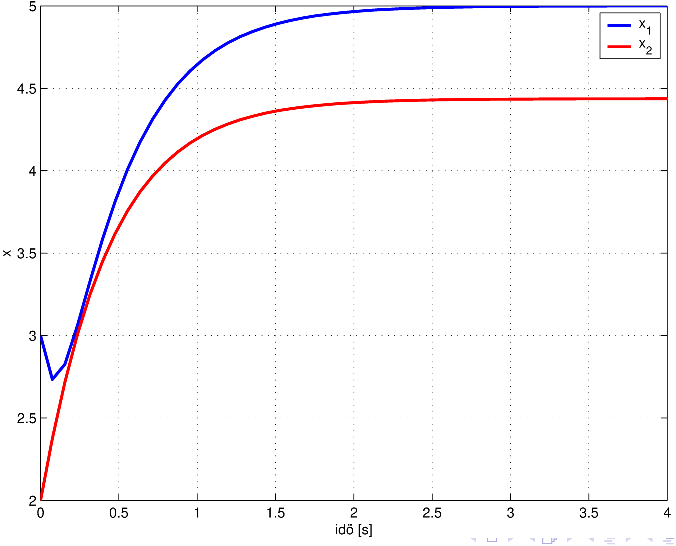
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{b}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Poles: -9.669, -2.331

Example 2 - application of the separation principle

Operation of the open loop system for the input $u(t) = 5$:



State observer design

Prescribed poles of the observer: -15, -16

("faster" than the poles of the original system)

Values of the L matrix:

$$L = \begin{bmatrix} 19 \\ 15.923 \end{bmatrix}$$

Example 2 - application of the separation principle

Stabilizing state feedback design

Parameters of the designed LQR controller:

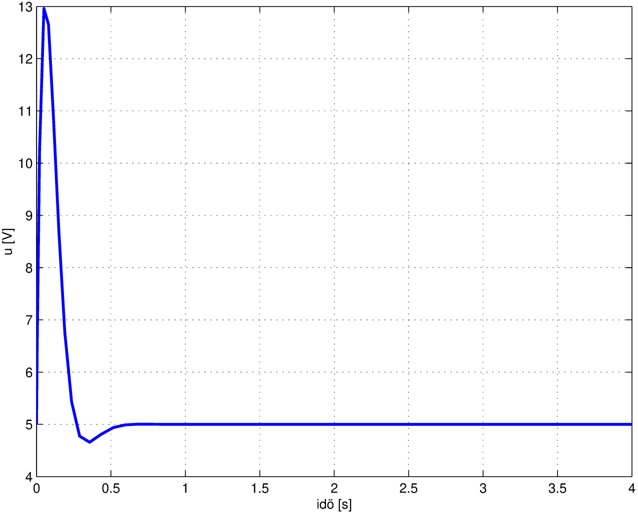
$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 1$$

The obtained feedback gain:

$$G = \begin{bmatrix} 3.807 & 6.342 \end{bmatrix}$$

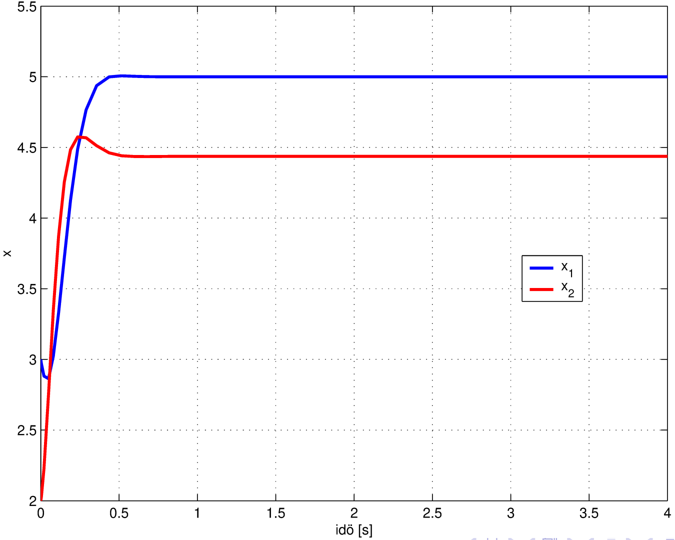
Example 2 - application of the separation principle

Operation of the stabilizing feedback combined with the state observer
Input voltage generated by the controller:



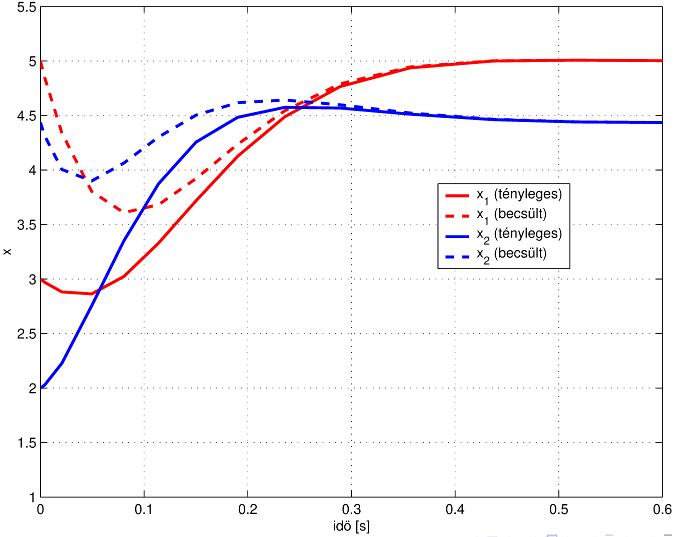
Example 2 - application of the separation principle

State variables of the closed loop system



Example 2 - application of the separation principle

Operation of the state observer



Example 3 - control of the inverted pendulum

Weighting matrices (design parameters):

$$Q = I^{4\times 4}, \quad R = 1$$

The computed feedback gain:

$$G = \begin{bmatrix} -1 & -23.227878 & -2.1084534 & -7.8899369 \end{bmatrix}$$

Eigenvalues of the closed loop system:

$$\lambda = \begin{bmatrix} -13.169677 \\ -1.0463076 + 0.3589175i \\ -1.0463076 - 0.3589175i \\ -3.1028591 \end{bmatrix}$$

Example 3 - control of the inverted pendulum

Operation of the controller:
ipend_lq_1.avi

Summary

- goal of optimal control: to minimize a functional by an appropriate input
- LQR case: system is LTI, functional is quadratic (combines performance and 'input energy/price' terms)
- solution principle: constrained minimization using time-dependent Lagrange multipliers (co-states)
- explicit solution is obtained assuming an infinite time horizon ($T \to \infty$)
- solution of a quadratic matrix equation (CARE) is required (easy with computer)
- result: linear full state feedback (always stabilizing if appropriate conditions are fulfilled)