# 1 Homework

The LPV system is

$$\begin{aligned} \dot{x} &= A(\varrho)x + B(\varrho)u \\ y &= Cx \end{aligned}$$

where

$$\begin{aligned} A(\varrho) &= A_0 + A_1\varrho_1 + A_2\varrho_2 \\ B(\varrho) &= B_0 + B_1\varrho_1 + B_2\varrho_2 \end{aligned}$$

and

$$A_0 = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -5 & 0 & 52 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 21 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0.5 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} -6 \\ 3 \\ 3 \\ -6 \\ 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 0 & -2 & -4 & 7 \end{bmatrix}.$$

## 1.1 problem

Let $\varrho_1 = 0$ and $\varrho_2 = 0.5$. In this case the system is an LTI system.

### 1.1.1 (a)

Using the `ctrbf` function we can get the controllability staircase form of the system. We get that the system has four controllable modes and one uncontrollable modes, thus the system is not fully controllable. The uncontrollable mode is stable, thus the system is stabilisable.
Using the `obsvf` function we can get the observability staircase form of the system. We get that the system has four observable modes and one unobservable modes, thus the system is not fully observable. The unobservable mode is stable, thus the system is detectable.

### 1.1.2 (b)

The desired Lyapunov function is $V(x) = x^{\mathrm{T}}Px$. We need

$$\begin{aligned} &V(x) > 0 \implies P = P^{\mathrm{T}} \succ 0 \\ &\dot{V}(x) < 0. \end{aligned}$$

From the first condition we get the LMI constraint the $P$ should be a positive definite symmetric matrix. We have to work a little bit on the second condition

$$\begin{aligned} \dot{V}(x) = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = x^{\mathrm{T}}\Big(\big(A(\varrho) - B(\varrho)K\big)^{\mathrm{T}}P + P\big(A(\varrho) - B(\varrho)K\big)\Big)x < 0 \implies \\ \implies \big(A(\varrho) - B(\varrho)K\big)^{\mathrm{T}}P + P\big(A(\varrho) - B(\varrho)K\big) \prec 0. \end{aligned}$$

This is not a linear constraint (it is bilinear). Introducing $Q = P^{-1}$ and multiplying with it from both sides

$$QA^{\mathrm{T}}(\varrho) + A(\varrho)Q - QK^{\mathrm{T}}B^{\mathrm{T}}(\varrho) - B(\varrho)KQ \prec 0.$$

Introducing $N = KQ$

$$QA^{\mathrm{T}}(\varrho) + A(\varrho)Q - N^{\mathrm{T}}B^{\mathrm{T}}(\varrho) - B(\varrho)N \prec 0.$$

Furthermore, since we have an LTI system, the LMI above reduces to

$$QA^{\mathrm{T}} + AQ - N^{\mathrm{T}}B^{\mathrm{T}} - BN \prec 0.$$

Using the YALMIP SeDuMi solver the computed full-state static feedback gain $K$ is

$$K = \begin{bmatrix} 2.8558 & -4.0407 & -1.5439 & 2.6476 & 92.6737 \end{bmatrix}.$$

#### 1.1.3 (c)

The desired Lyapunov function is $V(x) = x^{\mathrm{T}} P x$. We need

$$\begin{aligned} V(x) > 0 &\implies P = P^{\mathrm{T}} \succ 0 \\ \dot{V}(x) < 0. & \end{aligned}$$

From the first condition we get the LMI constraint the $P$ should be a positive definite symmetric matrix. We have to work a little bit on the second condition

$$\begin{aligned} &\dot{V}(x) = \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} = x^{\mathrm{T}} \Big( \big(A(\varrho) - LC\big)^{\mathrm{T}} P + P\big(A(\varrho) - LC\big) \Big) x < 0 \implies \\ &\implies \big(A(\varrho) - LC\big)^{\mathrm{T}} P + P\big(A(\varrho) - LC\big) \prec 0. \end{aligned}$$

This is not a linear constraint (it is bilinear). Introducing $M = PL$

$$A^{\mathrm{T}}(\varrho) P + P A(\varrho) - C^{\mathrm{T}} L^{\mathrm{T}} P - PLC = A^{\mathrm{T}}(\varrho) P + P A(\varrho) - C^{\mathrm{T}} M^{\mathrm{T}} - MC \prec 0.$$

Furthermore, since we have an LTI system, the LMI above reduces to

$$A^{\mathrm{T}} P + PA - C^{\mathrm{T}} M^{\mathrm{T}} - MC \prec 0.$$

Using the YALMIP SeDuMi solver the computed observer gain $L$ is

$$L = \begin{bmatrix} -2.8458 \\ 3.2599 \\ 58.1481 \\ -3.2106 \\ 22.3426 \end{bmatrix}.$$

### 1.2 problem

Let $\varrho_1 \in [-1, 1]$ and $\varrho_2 \in [-0.5, 0.5]$.

#### 1.2.1 (a)

The theoretical background was presented in the previous problem. To reiterate, we need to find a $K$ full-state static feedback gain that satisfies the following LMI contraints

$$\begin{aligned} Q = Q^{\mathrm{T}} &\succ 0 \\ QA^{\mathrm{T}}(\varrho) + A(\varrho)Q - N^{\mathrm{T}} B^{\mathrm{T}}(\varrho) - B(\varrho) N &\prec 0. \end{aligned}$$

We only have to ensure that the LMIs are satisfied in the extreme points of the domain we are working with, i.e. the constraints are

$$\begin{aligned} Q = Q^{\mathrm{T}} &\succ 0 \\ QA^{\mathrm{T}}(-1, -0.5) + A(-1, -0.5)Q - N^{\mathrm{T}} B^{\mathrm{T}}(-1, -0.5) - B(-1, -0.5)N &\prec 0 \\ QA^{\mathrm{T}}(-1, 0.5) + A(-1, 0.5)Q - N^{\mathrm{T}} B^{\mathrm{T}}(-1, 0.5) - B(-1, 0.5)N &\prec 0 \\ QA^{\mathrm{T}}(1, -0.5) + A(1, -0.5)Q - N^{\mathrm{T}} B^{\mathrm{T}}(1, -0.5) - B(1, -0.5)N &\prec 0 \\ QA^{\mathrm{T}}(1, 0.5) + A(1, 0.5)Q - N^{\mathrm{T}} B^{\mathrm{T}}(1, 0.5) - B(1, 0.5)N &\prec 0. \end{aligned}$$

Using the YALMIP SeDuMi solver the computed full-state static feedback gain $K$ is

$$K = \begin{bmatrix} 2.4488 & -8.9370 & 0.2312 & 2.6468 & 208.2975 \end{bmatrix}.$$

#### 1.2.2 (b)

With the $K$ feedback gain given above we can estimate the $\mathcal{L}_2$ norm of the system. Using a supply $s(u,v) = \gamma^2 u^\mathrm{T} u - y^\mathrm{T} y = \gamma^2 \|u\|^2 - \|y\|^2$ the dissipativity inequality is

$$\int_{t_0}^{t_1} \dot{V}(x)\,\mathrm{d}t \le \int_{t_0}^{t_1} s(u,v)\,\mathrm{d}t$$

i.e.

$$V\big(x(t_1)\big) - V\big(x(t_0)\big) \le \gamma^2 \int_{t_0}^{t_1} \|u\|^2\,\mathrm{d}t - \int_{t_0}^{t_1} \|u\|^2\,\mathrm{d}t\,.$$

Starting the system from an arbitrary $x(t_0) = x_0$ state with zero energy and $t_1 = \infty$

$$0 \le V\big(x(\infty)\big) - V\big(x(t_0)\big) \le \gamma^2 \|u\|_{\mathcal{L}_2}^2 - \|y\|_{\mathcal{L}_2}^2$$

i.e.

$$\frac{\|y\|_{\mathcal{L}^2}}{\|u\|_{\mathcal{L}^2}} = \frac{\|\mathcal{S}u\|_{\mathcal{L}^2}}{\|u\|_{\mathcal{L}^2}} \le \gamma \implies \|S\|_{\mathcal{L}^2} = \sup_{u\in\mathcal{L}^2} \frac{\|\mathcal{S}u\|_{\mathcal{L}^2}}{\|u\|_{\mathcal{L}^2}} \le \gamma.$$

Expressing the supply with the given matrices

$$\begin{aligned} s(u,v) &= \gamma^2 u^\mathrm{T} u - y^\mathrm{T} y = \gamma^2 u^\mathrm{T} u - (Cx+Du)^\mathrm{T}(Cx+Du) = \\ &= \gamma^2 u^\mathrm{T} u - x^\mathrm{T} C^\mathrm{T} C x - x^\mathrm{T} C^\mathrm{T} D u - u^\mathrm{T} D^\mathrm{T} C x - u^\mathrm{T} D^\mathrm{T} D u = \begin{bmatrix} x^\mathrm{T} & u^\mathrm{T} \end{bmatrix} \begin{bmatrix} -C^\mathrm{T} C & -C^\mathrm{T} D \\ -D^\mathrm{T} C & \gamma^2 I - D^\mathrm{T} D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \end{aligned}$$

Expressing the energy with the given matrices

$$\begin{aligned} V(x) &= x^\mathrm{T} P x \\ \dot{V}(x) &= \dot{x}^\mathrm{T} P x + x^\mathrm{T} P \dot{x} = \big(A(\varrho)x + B(\varrho)u\big)^\mathrm{T} P x + x^\mathrm{T} P \big(A(\varrho)x + B(\varrho)u\big) = \\ &= x^\mathrm{T} A^\mathrm{T}(\varrho) P x + u^\mathrm{T} B^\mathrm{T}(\varrho) P x + x^\mathrm{T} P A(\varrho) x + x^\mathrm{T} P B(\varrho) u = \begin{bmatrix} x^\mathrm{T} & u^\mathrm{T} \end{bmatrix} \begin{bmatrix} A^\mathrm{T}(\varrho) T + P A(\varrho) & P B(\varrho) \\ B^\mathrm{T}(\varrho) P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \end{aligned}$$

From the dissipativity inequality $\dot{V}(x) \le s(u,y)$ we get

$$\begin{bmatrix} x^\mathrm{T} & u^\mathrm{T} \end{bmatrix} \begin{bmatrix} A^\mathrm{T}(\varrho) T + P A(\varrho) & P B(\varrho) \\ B^\mathrm{T}(\varrho) P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \le \begin{bmatrix} x^\mathrm{T} & u^\mathrm{T} \end{bmatrix} \begin{bmatrix} -C^\mathrm{T} C & -C^\mathrm{T} D \\ -D^\mathrm{T} C & \gamma^2 I - D^\mathrm{T} D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

i.e.

$$\begin{bmatrix} A^\mathrm{T}(\varrho) P + P A(\varrho) + C^\mathrm{T} C & P B(\varrho) + C^\mathrm{T} D \\ B^\mathrm{T}(\varrho) P + D^\mathrm{T} C & -\gamma^2 I + D^\mathrm{T} D \end{bmatrix} \preceq 0.$$

Since we have $D = 0$ the optimization constraints for the original systems are

$$\begin{gathered} P = P^\mathrm{T} \succ 0 \\ \begin{bmatrix} A^\mathrm{T}(\varrho) P + P A(\varrho) + C^\mathrm{T} C & P B(\varrho) \\ B^\mathrm{T}(\varrho) P & -\gamma^2 I \end{bmatrix} \preceq 0. \end{gathered}$$

and for the closed loop are (using the same matrices $Q, N$ as before)

$$\begin{gathered} Q = Q^\mathrm{T} \succ 0 \\ \begin{bmatrix} \big(A(\varrho) - B(\varrho)K\big)^\mathrm{T} P + P\big(A(\varrho) - B(\varrho)K\big) + C^\mathrm{T} C & P B(\varrho) \\ B^\mathrm{T}(\varrho) P & -\gamma^2 I \end{bmatrix} \preceq 0. \end{gathered}$$

Using the YALMIP SeDuMi solver we an infeasible solution for the original system, since it is unstable (and it does not have an $\mathcal{L}^2$ gain). For the closed loop I got numerical problems and the constraints were not fulfilled.

#### 1.2.3 (c)

Using the Schur complement lemma and the previously introduces $Q$ and $N$ matrices, we get the following LMI constraint for the optimal feedback gain $K$

$$\begin{bmatrix} QA^{\mathrm{T}}(\varrho) + A(\varrho)Q - N^{\mathrm{T}}B^{\mathrm{T}}(\varrho) - B(\varrho)N & B(\varrho) & QC^{\mathrm{T}} \\ B^{\mathrm{T}}(\varrho) & -\gamma^2 I & 0 \\ CQ & 0 & -I \end{bmatrix} \preceq 0.$$

Using the YALMIP SeDuMi solver the computed full-state static feedback gain $K$ is

$$K = \begin{bmatrix} 8413.3 & -2254.1 & -2395.9 & -8413.1 & 20026 \end{bmatrix}$$

and the $\mathcal{L}_2$ gain is

$$\|\mathcal{S}\|_{\mathcal{L}^2} \leq 0.3312.$$