1 Homework

1.1 problem

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{4} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

We will choose $Q = R = I$, i.e. the functional to be minimized is

$$\mathcal{J}(x,u) = \frac{1}{2}\int_0^\infty \sum_{i=1}^{2} \left(x_i^2 + u_i^2\right) \mathrm{d}t\,.$$

The Control Algebraic Riccati Equation (CARE) is

$$KA + A^{\mathrm{T}}K - KBR^{-1}B^{\mathrm{T}}K + Q = KA + A^{\mathrm{T}}K - KBB^{\mathrm{T}}K + I = 0.$$

1.1.1 (a)

By definition we should choose the positive definite solution, since the solution of CARE corresponds to a full-state negative feedback (thus the negative definite solution would be an unstable positive feedback).

1.1.2 (b)

The solution is

$$K = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

thus we get optimal control by the feedback

$$u = -R^{-1}B^{\mathrm{T}}Kx = -\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} x.$$

1.2 problem

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

1.2.1 (a)

With $m = 0.1,\ k = 0.4$ and $c = 0$ we get

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

1.2.2 (b)

With $h = \frac{\pi}{12}$ sampling time

$$\Phi = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{4} \\ -1 & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \Gamma = \begin{bmatrix} \frac{5}{2} - \frac{5\sqrt{3}}{4} \\ \frac{5}{2} \end{bmatrix}.$$

1.2.3 (c)

The eigenvalues are $\frac{\sqrt{3}}{2} \pm \frac{1}{2}j$, thus the system is not stable (since the eigenvalues must be in the unit circle on the complex plane).

1.2.4 (d)

$$\begin{aligned}
x(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
x(1) = \Phi x(0) + \Gamma u(0) &= \begin{bmatrix} 5 - 2\sqrt{3} \\ 4 \end{bmatrix} \\
x(2) = \Phi x(1) + \Gamma u(1) &= \begin{bmatrix} 3 \\ 4\sqrt{3} \end{bmatrix}
\end{aligned}$$

1.3 problem

$$\Phi = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

1.3.1 (a)

$$H(q) = C(qI - \Phi)^{-1}\Gamma = -\frac{1}{q-3}$$

1.3.2 (b)

Since Φ is invertible controllability is equivalent to reachability. The DT controllability matrix

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$$

which is not of full rank, thus the system is not controllable and not reachable.

1.3.3 (c)

The system is observable since the DT observability matrix

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}$$

is of full rank.