

# Computer Controlled Systems

## Homework 4.

Submission deadline: **13th of December**, at 13:00 (approx. 2 weeks)

*All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs (e.g. Matlab) can be used for self-verification, but all problems have to contain the detailed steps of solutions*

## Problems

1. Design an optimal LQR controller  $u(t)$  for the following system:

$$\dot{x} = Ax + Bu, \text{ where } A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}, B = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

that minimizes the cost function

$$J(x, u) = \frac{1}{2} \int_0^\infty x^T(t) Q x(t) + u^T(t) R u(t) dt. \quad (1)$$

- (a) The continuous-time algebraic Riccati equation (CARE) may have multiple solutions. Which of them will result in a stable closed loop dynamics?
- (b) In order to check your solution, you can use function `care` of Matlab's Control System Toolbox.

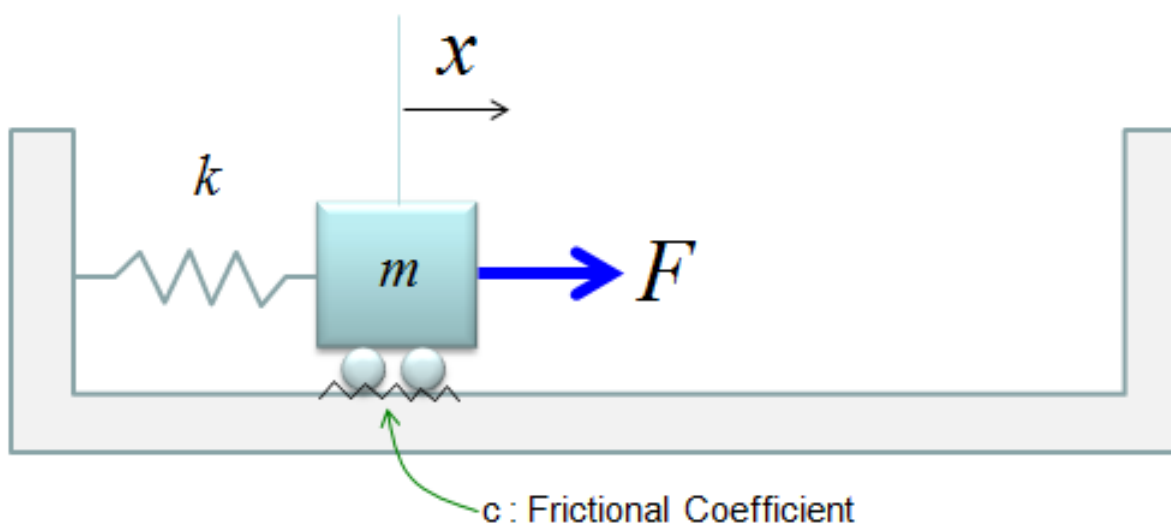


Figure 1: Spring-mass system with friction.

2. Consider a simple mass-spring-dumper system, illustrated by the picture above. We can describe the system with one differential equation by using Newton's second law  $\sum_{i=1}^n F_i = m \frac{dv}{dt}$ . Moreover we know that the velocity is the first order derivative of the position. Applying these, we get the following system

$$\begin{cases} \frac{dx}{dt} = v \\ m \frac{dv}{dt} = -kx - cv + F \end{cases} \quad (2)$$

where  $k$  is the spring constant and  $c$  is the frictional coefficient. Let us choose  $x$  and  $v$  as state variables, and rename them as  $x_1$  and  $x_2$ . The external force  $F$  is the input of the system. Now the equations are the followings

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{-kx_1}{m} - \frac{cx_2}{m} + \frac{F}{m} \end{cases} \quad (3)$$

We would like to measure the position of the system  $y = x$ . Now we can rewrite the system in matrix-vector form

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F \\ y = (1 \quad 0) x \end{cases} \quad (4)$$

- (a) Rewrite the system in matrix-vector form with the following parameters  $m = 0.1$ ,  $k = 0.4$ ,  $c = 0$ .
- (b) Compute the model matrices ( $\Phi$  and  $\Gamma$ ) of the discrete-time (DT) state space model of the mass-spring-dumper if the sampling time is  $h = \frac{\pi}{12}$ !
- (c) Compute the eigenvalues of DT state transition matrix ( $\Phi$ ) and determine whether the system is stable or not?
- (d) Determine the value of the state vector of the DT system in the following sampling points:  $x(2)$ , if the input is  $u(k) = 2$  for all  $k = 0, 1, \dots$  and the initial state is  $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

3. Given the following DT-LTI system

$$\begin{cases} x(k+1) = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \\ y(k) = (1 \quad -1) x(k) \end{cases} \quad (5)$$

- (a) Give the transfer operator  $H(q)$  of the system!
- (b) Is the system controllable and reachable?
- (c) Is the system observable?