1. Csörgő 8. hét

1.1. feladat

$$f(x,y) = 4x^3 - x^2y - 2xy^2 + 2y^3 + x^2 - y^2 - 2xy + 3x + 2y - 1 \qquad (1,1)$$

$$\frac{\partial f}{\partial x} = 12x^2 - 2xy - 2y^2 + 2x - 2y + 3\Big|_{(1,1)} = 11$$

$$\frac{\partial f}{\partial y} = -x^2 - 4xy + 6y^2 - 2y - 2x + 2\Big|_{(1,1)} = -1$$

$$\frac{\partial^2 f}{\partial x^2} = 24x - 2y + 2\Big|_{(1,1)} = 24$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -2x - 4y - 2\Big|_{(1,1)} = -8$$

$$\frac{\partial^2 f}{\partial y^2} = -4x + 12 - 2\Big|_{(1,1)} = 6$$

$$T_2(x,y) = 5 + 11(x-1) - (y-1) + 12(x-1)^2 - 4(x-1)(y-1) + 3(y-1)^2$$

1.2. feladat

$$f(x,y) = \frac{\cos x}{\cos y} \qquad (0,0)$$

$$\frac{\partial f}{\partial x} = -\frac{\sin x}{\cos y}\Big|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y} = \frac{\cos x \sin y}{\cos^2 y}\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\cos x}{\cos y}\Big|_{(0,0)} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -\frac{\sin x \sin y}{\cos^2 y}\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos x \cos^3 y + 2\cos x \sin^2 y \cos y}{\cos^4 y} = \frac{\cos x \sin^2 y + \cos x}{\cos^3 y}\Big|_{(0,0)} = 1$$

$$T_2(x,y) = 1 + \frac{1}{2}\Big(-x^2 + y^2\Big)$$

1.3. feladat

$$f(x,y)=\sqrt{1-x^2-y^2} \qquad (0,0)$$

$$\frac{\partial f}{\partial x}=-\frac{x}{\sqrt{1-x^2-y^2}}\Bigg|_{(0,0)}=0$$

$$\frac{\partial f}{\partial y}=-\frac{y}{\sqrt{1-x^2-y^2}}\Bigg|_{(0,0)}=0$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{y^2-1}{\left(1-x^2-y^2\right)^{\frac{3}{2}}}\Bigg|_{(0,0)}=-1$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}=-\frac{xy}{\sqrt{1-x^2-y^2}}\Bigg|_{(0,0)}=0$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{x^2-1}{\left(1-x^2-y^2\right)^{\frac{3}{2}}}=\Bigg|_{(0,0)}=-1$$

$$T_2(x,y)=1-\frac{1}{2}\Big(x^2+y^2\Big)$$

1.4. feladat

$$f(x,y)=3(x-2)^2-4(y+3)^2+2(x-2)(y+3)+8(x-2)+25(y+3)-24=$$
$$=3x^2+2xy-4y^2+2x-3y-1$$

1.5. feladat

$$\iint_R \left(2x^2+3xy+4y^2\right)\mathrm{d}R=\int_0^3\int_1^2\left(2x^2+3xy+4y^2\right)\mathrm{d}x\,\mathrm{d}y=$$
$$=\int_0^3 \frac{2}{3}x^3+\frac{3}{2}x^2y+4xy^2\Bigg|_1^2\mathrm{d}y=\int_0^3\left(\frac{14}{3}+\frac{9}{2}y+4y^2\right)\mathrm{d}y=\frac{14}{3}y+\frac{9}{4}y^2+\frac{4}{3}y^3\Bigg|_0^3=\frac{281}{4}$$

$$\iint_R \left(2x^2+3xy+4y^2\right)\mathrm{d}R=\int_1^2\int_0^3\left(2x^2+3xy+4y^2\right)\mathrm{d}y\,\mathrm{d}x=$$
$$=\int_1^2 x^2y+\frac{3}{2}xy^2+\frac{4}{3}y\Bigg|_0^3\mathrm{d}x=\int_1^2\left(6x^2+\frac{27}{2}x+36\right)\mathrm{d}x=2x^3+\frac{27}{4}x^2+36x\Bigg|_1^2=\frac{281}{4}$$

1.6. feladat

$$\iint_R xy\sin\left(x^2+y^2\right)\mathrm{d}R=\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{\sqrt{\frac{\pi}{2}}} xy\sin\left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y=$$
$$=\int_0^{\sqrt{\frac{\pi}{2}}} -\frac{1}{2}y\cos\left(x^2+y^2\right)\Bigg|_0^{\sqrt{\frac{\pi}{2}}}\mathrm{d}y=\int_0^{\sqrt{\frac{\pi}{2}}}-\frac{1}{2}y\left(\cos\left(\frac{\pi}{2}+y^2\right)-\cos y^2\right)\mathrm{d}y=$$
$$=-\frac{1}{4}\left(\sin\left(\frac{\pi}{2}+y^2\right)-\sin y^2\right)\Bigg|_0^{\sqrt{\frac{\pi}{2}}}=\frac{1}{2}$$

$$\iint_R xy\sin\left(x^2+y^2\right)\mathrm{d}R=\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{\sqrt{\frac{\pi}{2}}} xy\sin\left(x^2+y^2\right)\mathrm{d}y\,\mathrm{d}x=$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} -\frac{1}{2}x\cos\left(x^2+y^2\right)\Big|_0^{\sqrt{\frac{\pi}{2}}} \mathrm{d}x = \int_0^{\sqrt{\frac{\pi}{2}}} -\frac{1}{2}x\left(\cos\left(x^2+\frac{\pi}{2}\right)-\cos x^2\right)\mathrm{d}x =$$

$$= -\frac{1}{4}\left(\sin\left(x^2+\frac{\pi}{2}\right)-\sin x^2\right)\Bigg|_0^{\sqrt{\frac{\pi}{2}}} = \frac{1}{2}$$

1.7. feladat

$$\iint_R e^{x+y}\,\mathrm{d}R = \int_1^2\int_1^4 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 e^{x+y}\Big|_1^4 \mathrm{d}y = \int_1^2 \left(e^{4+y}-e^{1+y}\right)\mathrm{d}y =$$

$$= e^{4+y}-e^{1+y}\Big|_1^2 = e^6-e^5-e^3+e^2$$

$$\iint_R e^{x+y}\,\mathrm{d}R = \int_1^2\int_1^4 e^{x+y}\,\mathrm{d}y\,\mathrm{d}x = \int_1^4 e^{x+y}\Big|_1^2 \mathrm{d}x = \int_1^4 \left(e^{x+2}-e^{x+1}\right)\mathrm{d}x =$$

$$= e^{x+2}-e^{x+1}\Big|_1^4 = e^6-e^5-e^3+e^2$$

$$\iint_R e^{x+y}\,\mathrm{d}R = \int_1^4 e^x\,\mathrm{d}x\int_1^2 e^y\,\mathrm{d}y = e^x\Big|_1^4 e^y\Big|_1^2 = \left(e^4-e\right)\left(e^2-e\right) = e^6-e^5-e^3+e^2$$

1.8. feladat

Könnyen látható, hogy az R tartományra

$$R = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x\in[0,1], y\in[1-x,2+2x]\right\}.$$

Ekkor

$$\iint_R \left(x^2+2xy\right)\mathrm{d}R = \int_0^1\int_{1-x}^{2+2x}\left(x^2+2xy\right)\mathrm{d}y\,\mathrm{d}x = \int_0^1 x^2y+xy^2\Big|_{1-x}^{2+2x}\mathrm{d}x =$$

$$= \int_0^1 \left(x^2(2+2x-1+x)+x\left((2+2x)^2-(1-x)^2\right)\right)\mathrm{d}x =$$

$$= \int_0^1 \left(6x^3+11x^2+3x\right)\mathrm{d}x = \frac{3}{2}x^4+\frac{11}{3}x^3+\frac{3}{2}x^2\Big|_0^1 = \frac{20}{3}.$$

1.9. feladat

$$f(x,y) = x^2+y$$

$$R = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, y\in[0,1], x\in\left[y^2,\sqrt{y}\right]\right\} = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x\in[0,1], y\in\left[x^2,\sqrt{x}\right]\right\}$$

$$\iint_R \left(x^2+y\right)\mathrm{d}R = \int_0^1\int_{y^2}^{\sqrt{y}}\left(x^2+y\right)\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{1}{3}x^3+xy\Big|_{y^2}^{\sqrt{y}}\mathrm{d}y =$$

$$\int_0^1 \left(\frac{1}{3}y^{\frac{3}{2}}-\frac{1}{3}y^6+y^{\frac{3}{2}}-y^3\right)\mathrm{d}y = \frac{8}{15}y^{\frac{5}{2}}-\frac{1}{21}y^7-\frac{1}{4}y^4\Big|_0^1 = \frac{33}{140}$$

$$\iint_R \left(x^2+y\right) \mathrm{d}R = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2+y\right) \mathrm{d}y\,\mathrm{d}x = \int_0^1 x^2y + \frac{1}{2}y^2 \bigg|_{x^2}^{\sqrt{x}} \mathrm{d}x =$$

$$= \int_0^1 \left(x^{\frac{5}{2}} - x^4 + \frac{1}{2}x - \frac{1}{2}x^4\right) \mathrm{d}x = \frac{2}{7}x^{\frac{7}{2}} - \frac{1}{5}x^5 + \frac{1}{4}x^2 - \frac{1}{10}x^5 \bigg|_0^1 = \frac{33}{140}$$

1.10. feladat

$$f(x,y) = x^2 + y^2$$

$$R = \left\{(x,y) \in \mathbb{R}^2 \middle| x \in [0,1], y \in \left[x^2, \sqrt{x}\right]\right\} = \left\{(x,y) \in \mathbb{R}^2 \middle| y \in [0,1], x \in \left[y^2, \sqrt{y}\right]\right\}$$

$$\iint_R \left(x^2+y^2\right) \mathrm{d}R = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2+y^2\right) \mathrm{d}y\,\mathrm{d}x = \int_0^1 x^2y + \frac{1}{3}y^3 \bigg|_{x^2}^{\sqrt{x}} \mathrm{d}x =$$

$$= \int_0^1 \left(x^{\frac{5}{2}} - x^5 + \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{3}x^6\right) \mathrm{d}x = \frac{2}{7}x^{\frac{7}{2}} - \frac{1}{6}x^6 + \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{21}x^7 \bigg|_0^1 = \frac{43}{210}$$

$$\iint_R \left(x^2+y^2\right) \mathrm{d}R = \int_0^1 \int_{y^2}^{\sqrt{y}} \left(x^2+y^2\right) \mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{1}{3}x^3 + xy^2 \bigg|_{y^2}^{\sqrt{y}} \mathrm{d}y =$$

$$= \int_0^1 \left(\frac{1}{3}y^{\frac{3}{2}} - \frac{1}{3}y^6 + y^{\frac{5}{2}} - y^5\right) \mathrm{d}y = \frac{2}{15}y^{\frac{5}{2}} - \frac{1}{21}y^7 + \frac{2}{7}y^{\frac{7}{2}} - \frac{1}{6}y^6 \bigg|_0^1 = \frac{43}{210}$$

1.11. feladat

$$f(x,y) = 2y + x + 2 \qquad R = \left\{(x,y) \in \mathbb{R}^2 \middle| x \in [1,3], y \in \left[0, \frac{1}{x}\right]\right\}$$

$$\iint_R (2y+x+2) \,\mathrm{d}R = \int_1^3 \int_0^{\frac{1}{x}} (2y+x+2) \,\mathrm{d}y\,\mathrm{d}x = \int_1^3 y^2 + xy + 2y \bigg|_0^{\frac{1}{x}} \mathrm{d}x =$$

$$= \int_1^3 \left(\frac{1}{x^2} + 1 + \frac{2}{x}\right) \mathrm{d}x = -\frac{1}{x} + x + 2\ln x \bigg|_1^3 = \frac{8}{3} + \ln 9$$

1.12. feladat

$$f(x,y) = xe^y$$

$$R = \left\{(x,y) \in \mathbb{R}^2 \middle| x \in [0,1], y \in \left[x^2, x\right]\right\} = \left\{(x,y) \in \mathbb{R}^2 \middle| y \in [0,1], x \in \left[y, \sqrt{y}\right]\right\}$$

$$\iint_R xe^y \,\mathrm{d}R = \int_0^1 \int_{x^2}^{x} xe^y \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 xe^y \bigg|_{x^2}^{x} \mathrm{d}x = \int_0^1 \left(xe^x - xe^{x^2}\right) \mathrm{d}x =$$

$$= e^x(x-1) - \frac{e^{x^2}}{2} \bigg|_0^1 = \frac{3-e}{2}$$

$$\iint_R xe^y \,\mathrm{d}R = \int_0^1 \int_y^{\sqrt{y}} xe^y \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{x^2}{2}e^y \bigg|_y^{\sqrt{y}} = \int_0^1 \left(\frac{y}{2}e^y - \frac{y^2}{2}e^y\right) \mathrm{d}y =$$

$$= -\frac{1}{2}e^y\left(y^2 - 3y + 3\right) \bigg|_0^1 = \frac{3-e}{2}$$

1.13. feladat

$$f(x,y) = x^2 + y^2 \qquad R = \Big\{(x,y) \in \mathbb{R}^2 \Big| y \in [0,1], x \in [y, 3-y]\Big\}$$

$$\iint_R \left(x^2+y^2\right) \mathrm{d}R = \int_0^1 \int_y^{3-y} \left(x^2+y^2\right) \mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{1}{3}x^3 + xy^2 \Bigg|_y^{3-y} \mathrm{d}y =$$

$$= \int_0^1 \left(\frac{1}{3}(3-y)^3 - \frac{1}{3}y^3 + (3-y)y^2 - y^3\right) \mathrm{d}y = \int_0^1 \left(-\frac{8}{3}y^3 + 6y^2 - 9y + 9\right) \mathrm{d}y =$$

$$= -\frac{2}{3}y^4 + 2y^3 - \frac{9}{2}y^2 + 9y \Bigg|_0^1 = \frac{35}{6}$$

1.14. feladat

$$f(x,y) = 4 - y^2 \qquad R = \Big\{(x,y) \in \mathbb{R}^2 \Big| x \in \big[-\sqrt{2}, \sqrt{2}\big], y \in \big[x^2-2, 2-x^2\big]\Big\}$$

$$\iint_R \left(4-y^2\right) \mathrm{d}R = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-2}^{2-x^2} \left(4-y^2\right) \mathrm{d}y\,\mathrm{d}x = \int_{-\sqrt{2}}^{\sqrt{2}} 4y - \frac{1}{3}y^3 \Bigg|_{x^2-2}^{2-x^2} \mathrm{d}x =$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{2}{3}x^6 - 4x^4 + \frac{32}{3}\right) \mathrm{d}x = \frac{2}{21}x^7 - \frac{4}{5}x^5 + \frac{32}{3}x \Bigg|_{-\sqrt{2}}^{\sqrt{2}} = \frac{576\sqrt{2}}{35}$$

1.15. feladat

$$f(x,y) = y^2 - 2x \qquad R = \Big\{(x,y) \in \mathbb{R}^2 \Big| y \in \big[-\sqrt{3}, \sqrt{3}\big], x \in \big[y^2-3, 0\big]\Big\}$$

$$\iint_R \left(y^2-2x\right) \mathrm{d}R = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2-3}^{0} \left(y^2-2x\right) \mathrm{d}x\,\mathrm{d}y = \int_{-\sqrt{3}}^{\sqrt{3}} xy^2 - x^2 \Bigg|_{y^2-3}^{0} \mathrm{d}y =$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\left(y^2-3\right)^2 - \left(y^2-3\right)y^2\right) \mathrm{d}y = \int_{-\sqrt{3}}^{\sqrt{3}} \left(9-3y^2\right) \mathrm{d}y = 9y - y^3 \Bigg|_{-\sqrt{3}}^{\sqrt{3}} = 12\sqrt{3}$$

1.16. feladat

$$f(x,y) = xy \qquad R = \left\{(x,y) \in \mathbb{R}^2 \middle| x \in [0,2], y \in \left[\frac{1}{2}x, 2 - \frac{1}{2}x\right]\right\}$$

$$\iint_R xy \,\mathrm{d}R = \int_0^2 \int_{\frac{1}{2}x}^{2-\frac{1}{2}x} xy \,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \frac{1}{2}xy^2 \Bigg|_{\frac{1}{2}x}^{2-\frac{1}{2}x} \mathrm{d}x =$$

$$= \int_0^2 \left(4x - 2x^2\right) \mathrm{d}x = 2x^2 - \frac{2}{3}x^3 \Bigg|_0^2 = \frac{8}{3}$$