1. Csörgő 10. hét

1.1. feladat

1.
$$F(x,y) = \left(x^2 + 2xy - y^2, x^2 - 2xy - y^2\right)$$
A vektormező potenciálos, potenciálja $f(x,y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + x^2y - xy^2$.

2.
$$F(x,y) = \left(\frac{y}{3x^2 - 2xy + 3y^2}, -\frac{x}{3x^2 - 2xy + 3y^2}\right)$$
A vektormező potenciálos, potenciálja $f(x,y) = \frac{\operatorname{arctg} \frac{3x-y}{2\sqrt{2}y}}{2\sqrt{2}}$.

3.
$$F(x,y) = \left(1 - \frac{1}{y} + \frac{y}{z}, \frac{x}{z} + \frac{x}{y^2}, -\frac{xy}{z^2}\right)$$
A vektormező potenciálos, potenciálja $f(x,y) = x - \frac{x}{y} + \frac{xy}{z}$.

1.2. feladat

$$F(x,y) = (x, y+2) \qquad \Gamma = \left\{\gamma(t) = (t - \sin t, 1 - \cos t)\middle| t \in [0, 2\pi]\right\}$$

Vegyük észre, hogy F egy potenciális vektormező, melynek potenciálja $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 2y$. Ekkor

$$\int_\Gamma F(\mathbf{r})d\mathbf{r} = f\big(\gamma(2\pi)\big) - f\big(\gamma(0)\big) = 2\pi^2 + \frac{5}{2}.$$

1.3. feladat

$$f(x) = \begin{cases} 1 - |x|, & \text{ha } |x| < 1 \\ 0, & \text{egyébként} \end{cases}$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - |x|)e^{-ixs}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} (1+x)e^{-ixs}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}} \int_{0}^{1} (1-x)e^{-ixs}\,\mathrm{d}x =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(1+x)e^{-ixs}}{-is}\bigg|_{-1}^{0} + \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} \frac{e^{-ixs}}{is}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}} \frac{(1-x)e^{-ixs}}{-is}\bigg|_{0}^{1} - \frac{1}{\sqrt{2\pi}} \int_{0}^{1} \frac{e^{-ixs}}{is}\,\mathrm{d}x =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-ixs}}{s^2}\bigg|_{-1}^{0} - \frac{1}{\sqrt{2\pi}} \frac{e^{-ixs}}{s^2}\bigg|_{0}^{1} = \frac{2 + e^{is} - e^{-is}}{s^2} = \operatorname{sinc}^2 \frac{s}{2}$$

1.4. feladat

$$f(x) = e^{-|x|} \cos x$$

Mivel a függvény páros

$$\hat{f}(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \cos t \cos(st)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t}\big(\cos(s+1)t + \cos(s-1)t\big)\,\mathrm{d}t =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-t} \left(\frac{(s+1)\sin(s+1)t - \cos(s+1)t}{(s+1)^2 + 1} + \frac{(s-1)\sin(s-1)t - \cos(s-1)t}{(s-1)^2 + 1} \right)\Bigg|_0^\infty =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{s^2 + 2}{s^4 + 4}.$$

1.5. feladat

$$f(x) = \begin{cases} -1, & \text{ha } x \in [-1,0] \\ 1, & \text{ha } x \in [0,1] \\ 0, & \text{egyébként} \end{cases}$$

Mivel a függvény páratlan

$$\hat{f}(s) = -i\sqrt{\frac{2}{\pi}} \int_0^1 \sin{(st)}\,\mathrm{d}t = i\sqrt{\frac{2}{\pi}} \frac{\cos s - 1}{s}.$$

1.6. feladat

$$f(x) = \begin{cases} 1, & \text{ha } |x| < 1 \\ 0, & \text{egyébként} \end{cases}$$

Mivel a függvény páros

$$\hat{f}(s) = \sqrt{\frac{2}{\pi}} \int_0^1 \cos{(st)}\,\mathrm{d}t = \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}.$$

Ekkor

$$f(x) = \begin{cases} 1, & \text{ha } x \in [0,2] \\ 0, & \text{egyébként} \end{cases}$$

esetén

$$\hat{f}(s) = e^{-is}\sqrt{\frac{2}{\pi}} \frac{\sin s}{s}.$$

1.7. feladat

$$f(x) = e^{-\frac{(2x+1)^2}{2}}$$

$$\mathcal{F}\left(e^{-\frac{(2x+1)^2}{2}}, s\right) = e^{\frac{1}{2}is}\mathcal{F}\left(e^{-\frac{(2x)^2}{2}}, s\right) = \frac{1}{2}e^{\frac{1}{2}is}\mathcal{F}\left(e^{-\frac{x^2}{2}}, \frac{s}{2}\right) = \frac{1}{2}e^{\frac{1}{2}is}e^{-\frac{s^2}{8}} = \frac{1}{2}e^{-\frac{s^2}{8}+\frac{1}{2}is}$$

1.8. feladat

$$f(x) = e^{-2(x+2)^2}$$

$$\mathcal{F}\left(e^{-2(x+2)^2}, s\right) = \mathcal{F}\left(e^{-\frac{(2x+4)^2}{2}}, s\right) = e^{2is}\mathcal{F}\left(e^{-\frac{(2x)^2}{2}}, s\right) = \frac{1}{2}e^{2is}\mathcal{F}\left(e^{-\frac{x^2}{2}}, \frac{s}{2}\right) =$$

$$= \frac{1}{2}e^{2is}e^{-\frac{s^2}{8}} = \frac{1}{2}e^{-\frac{s^2}{8}+2is}$$

1.9. feladat

$$f(x) = e^{-|2x-6|}$$

$$\mathcal{F}\left(e^{-|2x-6|}, s\right) = e^{-3is}\mathcal{F}\left(e^{-|2x|}, s\right) = \sqrt{\frac{2}{\pi}}e^{-3is}\frac{2}{4+s^2}$$

1.10. feladat

$$f(x) = e^{-|3x+3|}$$

$$\mathcal{F}\left(e^{-|3x+3|}, s\right) = e^{is}\mathcal{F}\left(e^{-|3x|}, s\right) = \sqrt{\frac{2}{\pi}}e^{is}\frac{3}{9+s^2}$$

1.11. feladat

$$f(x) = x^2 e^{-|x|}$$

$$\mathcal{F}\Big(x^2 e^{-|x|}, s\Big) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}\Big(e^{-|x|}, s\Big) = -\sqrt{\frac{2}{\pi}}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{1}{1+s^2} = \sqrt{\frac{2}{\pi}}\frac{2-6s^2}{\big(1+s^2\big)^3}$$

1.12. feladat

$$f(x) = \big(x^2 - x + 2\big)e^{-|x|}$$

$$\mathcal{F}\Big(\big(x^2 - x + 2\big)e^{-|x|}, s\Big) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}\Big(e^{-|x|}, s\Big) + i\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}\Big(e^{-|x|}, s\Big) + 2\mathcal{F}\Big(e^{-|x|}, s\Big) =$$

$$= -\sqrt{\frac{2}{\pi}}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{1}{1+s^2} + i\sqrt{\frac{2}{\pi}}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{1+s^2} + 2\sqrt{\frac{2}{\pi}}\frac{1}{1+s^2} = 2\sqrt{\frac{2}{\pi}}\frac{1-3s^2-is\big(1+s^2\big)+\big(1+s^2\big)^2}{\big(1+s^2\big)^3}$$

1.13. feladat

$$f(x) = xe^{-\frac{x^2}{2}}$$

$$\mathcal{F}\left(xe^{-\frac{x^2}{2}}, s\right) = -i\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}\left(e^{-\frac{x^2}{2}}, s\right) = -i\frac{\mathrm{d}}{\mathrm{d}s}e^{-\frac{s^2}{2}} = ise^{-\frac{s^2}{2}}$$

1.14. feladat

$$f(x) = x^2 e^{-\frac{x^2}{2}}$$

$$\mathcal{F}\left(x^2 e^{-\frac{x^2}{2}}, s\right) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}\left(e^{-\frac{x^2}{2}}, s\right) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}e^{-\frac{s^2}{2}} = -e^{-\frac{s^2}{2}}\big(s^2-1\big)$$

1.15. feladat

$$f(x) = \frac{1}{1+(x+5)^2}$$

$$\mathcal{F}\left(\frac{1}{1+(x+5)^2}, s\right) = e^{5is}\mathcal{F}\left(\frac{1}{1+x^2}, s\right) = \sqrt{\frac{\pi}{2}}e^{5is-|s|}$$

1.16. feladat

$$f(x) = \frac{1}{1+(3x)^2}$$

$$\mathcal{F}\left(\frac{1}{1+(3x)^2}, s\right) = \frac{1}{3}\mathcal{F}\left(\frac{1}{1+x^2}, \frac{s}{3}\right) = \frac{1}{3}\sqrt{\frac{\pi}{2}}e^{-\frac{1}{3}|s|}$$

1.17. feladat

$$f(x) = \frac{1}{1+(2x-6)^2}$$

$$\mathcal{F}\left(\frac{1}{1+(2x-6)^2}, s\right) = e^{-3is}\mathcal{F}\left(\frac{1}{1+(2x)^2}, s\right) = \frac{1}{2}e^{-3is}\mathcal{F}\left(\frac{1}{1+x^2}, \frac{s}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-3is-|s|}$$