# 1. Órai feladatok 11. hét

## 1.1. feladat

$$\int_1^2 x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3}\Big|_1^2 = \frac{1}{3}\left(e^8 - e\right)$$

## 1.2. feladat

$$\int_{e^{-\frac{\pi}{2}}}^1 \frac{1}{x}\sin\ln x dx = -\cos\ln x\Big|_{e^{-\frac{\pi}{2}}}^1 = -1$$

## 1.3. feladat

$$\int_0^{\frac{\pi}{2}} \sin(3x)dx = \frac{-\cos(3x)}{3}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

## 1.4. feladat

$$\int_{-1}^1 \frac{1}{2x-4}dx = \frac{1}{2}\ln|x-2|\Big|_{-1}^1 = -\frac{\ln 3}{2}$$

## 1.5. feladat

$$\int_0^{\pi} \sin^3 x\cos x dx = \frac{\sin^4 x}{4}\Big|_0^{\pi} = 0$$

## 1.6. feladat

Vegyük észre, hogy $\forall x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ esetén

$$-\operatorname{ctg}(\pi - x) = \operatorname{ctg}(\pi + x).$$

Így láthatjuk, hogy

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \operatorname{ctg}(x)dx = \int_{\frac{\pi}{2}}^{\pi} \operatorname{ctg}(x)dx + \int_{\pi}^{\frac{3\pi}{2}} \operatorname{ctg}(x)dx = \int_{\frac{\pi}{2}}^{\pi} \operatorname{ctg}(x)dx - \int_{\frac{\pi}{2}}^{\pi} \operatorname{ctg}(x)dx = 0.$$

## 1.7. feladat

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\sin x dx = -\frac{\cos^3 x}{3}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$

## 1.8. feladat

$$\int_2^3 \frac{1}{(2x-1)^3}dx = -\frac{1}{4(2x-1)^2}\Big|_2^3 = \frac{4}{225}$$

## 1.9. feladat

$$\int_0^1 \frac{1}{x+1}dx = \ln|x+1|\Big|_0^1 = \ln 2$$

## 1.10. feladat

$$\int_0^1 xe^{\frac{x^2}{2}} dx = e^{\frac{x^2}{2}}\bigg|_0^1 = \sqrt{e} - 1$$

## 1.11. feladat

$$\int_0^4 \sqrt{3x+4}dx = \frac{2}{9}(3x+4)^{\frac{3}{2}}\bigg|_0^4 = \frac{112}{9}$$

## 1.12. feladat

$$\int_0^{\frac{\pi}{2}} e^{3x}\cos x dx = e^{3x}\sin x\bigg|_0^{\frac{\pi}{2}} - 3\int_0^{\frac{\pi}{2}} e^{3x}\sin x dx = e^{3x}\sin x\bigg|_0^{\frac{\pi}{2}} + 3e^{3x}\cos x\bigg|_0^{\frac{\pi}{2}} - 9\int_0^{\frac{\pi}{2}} e^{3x}\cos x dx$$

Átrendezve

$$\int_0^{\frac{\pi}{2}} e^{3x}\cos x dx = \frac{e^{3x}(\sin x + 3\cos x)}{10}\bigg|_0^{\frac{\pi}{2}} = \frac{e^{\frac{3\pi}{2}} - 3}{10}.$$

## 1.13. feladat

$$\int_0^2 xe^x dx = xe^x\bigg|_0^2 - \int_0^2 e^x dx = (x-1)e^x\bigg|_0^2 = e^2 + 1$$

## 1.14. feladat

$$\int_0^1 \sin(2x)e^{-x}dx = -\sin(2x)e^{-x}\bigg|_0^1 + 2\int_0^1 \cos(2x)e^{-x}dx =$$

$$= -\sin(2x)e^{-x}\bigg|_0^1 - 2\cos(2x)e^{-x}\bigg|_0^1 - 4\int_0^1 \sin(2x)e^{-x}dx$$

Átrendezve

$$\int_0^1 \sin(2x)e^{-x}dx = -\frac{1}{5}e^{-x}\big(\sin(2x) + 2\cos(2x)\big)\bigg|_0^1 = \frac{2e - \sin 2 - 2\cos 2}{5e}.$$

## 1.15. feladat

$$\int_{-\pi}^{\pi} \cos^4 x \sin^2 x dx$$

Mivel $\sin(2x) = 2\sin x\cos x$

$$\int_{-\pi}^{\pi} \cos^4 x\sin^2 x dx = \frac{1}{4}\int_{-\pi}^{\pi} \cos^2 x \sin^2(2x)dx.$$

Továbbá mivel $\cos^2 x = \frac{1+\cos(2x)}{2}$ így

$$\frac{1}{4}\int_{-\pi}^{\pi} \cos^2 x\sin^2(2x)dx = \frac{1}{8}\int_{-\pi}^{\pi} \sin^2(2x)dx + \frac{1}{8}\int_{-\pi}^{\pi} \sin^2(2x)\cos(2x)dx.$$

Mivel $\sin^2(2x) = \frac{1-\cos(4x)}{2}$ így az integrál

$$\frac{1}{16}\int_{-\pi}^{\pi} \big(1-\cos(4x)\big)dx + \frac{1}{48}\int_{-\pi}^{\pi} 3\sin^2(2x)\cdot 2\cos(2x)dx = \frac{x}{16} - \frac{\sin(4x)}{64} + \frac{\sin^3(2x)}{48}\bigg|_{-\pi}^{\pi} = \frac{\pi}{8}.$$

## 1.16. feladat

$$\int_0^\pi \sin(2x)\cos x dx = 2\int_0^\pi \sin x \cos^2 x dx = -\frac{2}{3}\cos^3 x\Big|_0^\pi = \frac{4}{3}$$

## 1.17. feladat

$$\int_0^1 x^n e^{-ax} dx$$

Legyen $t = ax$, ekkor $dx = \frac{dt}{a}$.

$$\int_0^1 x^n e^{-ax} dx = \int_0^a \left(\frac{t}{a}\right)^n e^{-t} \cdot \frac{dt}{a} = \frac{1}{a^{n+1}} \int_0^a t^n e^{-t} dt$$

Ekkor legyen

$$\gamma(n+1, a) = \int_0^a t^n e^{-t} dt.$$

Parciális integrálással láthatjuk, hogy

$$\gamma(n+1, a) = \int_0^a t^n e^{-t} dt = -t^n e^{-t}\Big|_0^a + n\int_0^a t^{n-1} e^{-t} dt = n \cdot \gamma(n, a) - a^n e^{-a}.$$

Ezt a módszert folytatva azt kapjuk, hogy

$$\gamma(n+1, a) = n! \cdot \gamma(1, a) - e^{-a}\sum_{k=1}^{n} \frac{n!}{k!} a^k$$

továbbá

$$\gamma(1, a) = \int_0^a e^{-t} dt = -e^{-t}\Big|_0^a = 1 - e^{-a}.$$

Ekkor

$$\int_0^1 x^n e^{-ax} dx = \frac{1}{a^{n+1}}\left(n! \cdot \gamma(1, a) - e^{-a}\sum_{k=1}^{n} \frac{n!}{k!} a^k\right) = \frac{1}{a^{n+1}}\left(n! - e^{-a}\sum_{k=0}^{n} \frac{n!}{k!} a^k\right).$$

## 1.18. feladat

$$\int_0^{\frac{\pi}{4}} x \sin x dx = -x\cos x\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x dx = -x\cos x + \sin x\Big|_0^{\frac{\pi}{4}} = \frac{\pi - 4}{4\sqrt{2}}$$

## 1.19. feladat

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x dx = e^x \cos x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x dx = e^x\cos x + e^x \sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x dx$$

Amiből

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}e^x(\cos x + \sin x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2} = \operatorname{ch}\frac{\pi}{2}.$$